§ Stable vector bundles

Let \( X \) be a k3 surface and fix an ample line bundle \( H \) (a "polarization"). For a coherent sheaf \( F \) define the slope:

\[
\mu(F) := \frac{\deg_H(F)}{\text{rk}(F)} \quad \text{where } \deg_H(F) = (H \cdot F)
\]

Def: Let \( F \) be torsion-free. Then \( F \) is said to be stable if, for any \( E \subset F \) with \( 0 \subset E \subset \text{Coker}(F) \), we have

\[
\mu(E) < \mu(F)
\]

E.g. \( H \Theta \Theta \Theta X \) is not stable

\[
\deg_H(\Theta \Theta \Theta X) = (H \cdot H) / \mu(\Theta \Theta \Theta X) = \frac{(H^2)}{2}
\]

but \( \mu(H) = (H^2) \).
Let $F$ be an $A$-stable vector bundle.

Let $\varphi: F \to F$ be a non-zero endomorphism. We claim that $\varphi$ is an isomorphism.

Suppose $\varphi$ were not an isomorphism. Then $\ker \varphi \neq 0$. Indeed, otherwise we would have $0 \to F \to E \to G \to 0$ so $\text{rk}(G) = 0$, $c_1(G) = 0$, $c_2(G) = 0 \implies G$ is zero

$\implies \varphi$ is an isomorphism.

So $\ker \varphi \neq 0$. But $\ker \varphi$ is torsion-free.

So $\rk \ker \varphi \geq 1 \implies \rk (\text{Im} \varphi) \leq \rk (F)$.

So $\text{Im}(\varphi) \neq F \implies \mu(\text{Im}(\varphi)) < \mu(F)$. 

But further

$$0 \rightarrow \ker \varphi \rightarrow E \rightarrow \text{Im} \varphi \rightarrow 0$$

$$\deg E = \deg (\ker \varphi) + \deg (\text{Im} \varphi)$$

$$\text{rk}(E) = \text{rk} (\ker \varphi) + \text{rk} (\text{Im} \varphi)$$

$$\Rightarrow \mu(E) - \mu(\ker \varphi) = \frac{\text{rk} (\ker \varphi)}{\text{rk} (\ker \varphi) - \mu(E)}$$

As we have seen $$\mu(E) < 0$$

But $$\ker \varphi \notin E$$, $$E$$ should so

$$\mu(E) > 0$$

Thus any non-zero endomorphism of a stable vector bundle is an isomorphism.

Interlude: The Picard group of a $${K3}$$ surface.
Recall $X$ a smooth projective variety. A cycle of codimension $r$ on $X$ is an element $Z$ of the free abelian group $\mathbb{Z}^r(X)$ generated by closed irreducible subvarieties of $X$ of codimension $r$.

Two cycles $Z_1, Z_2$ are rationally equivalent if there is a cycle $V$ on $X \times \mathbb{P}^1$, which is flat over $\mathbb{P}^1$ and $\mathbb{A}^1$. $X \times \mathbb{P}^1$ which is flat over $\mathbb{P}^1$ and $\mathbb{A}^1$.

$V \cap X \times \mathbb{A}^1 = Z_1$, $V \cap X \times \mathbb{P}^1 = Z_2$.

The Chow group is $A^r(X) := \mathbb{Z}^r(X)/\text{rad.equiv}$. and $\text{Pic}(X) \cong A^1(X)$.

There is also a coarser relation called algebraic equivalence. We say the cycles $Z_1, Z_2$ are algebraically equivalent if the same holds as above but with $\mathbb{P}^1$ replaced by an arbitrary...
Smooth curve $C$, i.e. $Z_2 \times Z_2$ if $T \subset C$ and a cycle $V$ on $T \times C$ flat over $C$ at.

$V \cap X_{\text{alg}} = Z_2$

we define $\text{NS}(X) := \mathbb{Z}^+ / \sim_{\text{alg}}$

The Néron–Severi group

The Néron–Severi group can be identified with the group of components of the Picard scheme of $X$.

The classical Néron–Severi group (NSC) is a finitely generated abelian group.

Now let $X$ be a smooth surface we can define another group $\text{Num}(X)$. 
which is a quotient of \( \text{NS}(X) \) as such: we say two line bundles \( L_1, L_2 \) are numerically equivalent if

\[(L_1 \cdot M) = (L_2 \cdot M) \quad \forall M \in \text{Pic}(X)\]

\[\text{Num}(X) := \text{Pic}(X) / \sim_{\text{num}}\]

\[L_1 \sim_{\text{num}} L_2 \implies \exists n \in \mathbb{N} : L_1^n \sim_{\text{num}} L_2^n\]

Note that \( \text{Num}(X) \) is torsion-free.

\[L^n \sim_0 X \implies L \sim_{\text{num}} X\]

Prop. Let \( X \) be a \( k^2 \) surface.

Then \( \text{Pic}(X) \to \text{NS}(X) \to \text{Num}(X) \).

In particular, \( \text{Pic}(X) \) is a finite, free abelian group of rank \( \text{Pic}^0(X) \).
Let $L$ be numerically trivial. Then for any ample line bundle $H$, $(L \cdot H) = 0$. Assume $L \cong \mathcal{O}_X$.

Makai-Matsuzawa criterion for any (even)

(see Hartshorne, Appendix) $\Rightarrow$ better $L$ acr.

$g^1$ are effective.

Riemann-Roch:

$$X(L) = \frac{\ell(L) + \text{Pic}^0(L)}{2} + \chi(O_X)$$

$-h^1(L)$

$\Rightarrow \frac{1}{2} \ell(L)^2 + 2 \leq 0 \Rightarrow (L \cdot L) \leq 0$.

Then $L \cong \mathcal{O}_X \Rightarrow L \cong \mathcal{O}_X$

$\Rightarrow \text{Pic}(X) \cong \text{Num}(X)$. \qed
Stability of Lazarsfeld-Mukai Bundles

$X \in \mathcal{L}$, $C \subseteq X$ smooth curve

At $Pic(C)$ b.p.f.,

$0 \to F_\mathcal{L} \to H^0(C) \otimes \mathcal{O}_X \to \mathcal{O}_A \to 0$

Prop. Assume $P(x) = 1$ and $Pic(C) = \mathbb{Z} [C]$.

Then $F_\mathcal{L}$ is $H^0(C)$ stable.

Pf. For any vector bundle $V \subseteq \mathcal{O}_X$ and any $1 \leq s \leq rk(V)$, consider $\wedge^s F_\mathcal{L}$.

We have $\wedge^s H \subseteq \wedge^s \mathcal{O}_X \otimes \mathcal{O}_C^s \overset{(s)}{\cong} \mathcal{O}_C^s \\
\text{So } End_{\mathcal{O}_X} (\wedge^s H) \cong \wedge^s H \otimes \wedge^s H^* \subseteq (\wedge^s H^*)^\otimes$
id \in H^0(X, \End \, \mathcal{O}_X)
\Rightarrow h^0(\mathcal{N}^\ast) \geq 1.

Let \mathcal{E} be a subsheaf \mathcal{C} = r \mathcal{E}
\subseteq r \mathcal{K}(\mathcal{E})
and
by the above \ h^0(\det \mathcal{E}^\ast) \geq 1.
Since \ \Pic(X) = \mathbb{Z}
\det(\mathcal{E}) = r \mathcal{H}^\ast, \ r \geq 0.

If \ r = 0 \ : \ \mathcal{E} = \Lambda^{e-1} \otimes \det(\mathcal{E})
\Rightarrow h^0(\mathcal{E}) \geq 1 \Rightarrow h^0(\mathcal{F}_A) \geq 1
(h^0(\mathcal{F}_A) = 0)

so \ r > 0.

But \ \det(\mathcal{F}_A) = \mathcal{H}^\ast \ (c_1(\mathcal{F}_A) = -c_1(\mathcal{A})
= -c_1(\mathcal{i} \mathcal{O}_X)
\Rightarrow \text{A trivial \ \mathcal{O}_X \ bundle}
\Rightarrow \text{cohomology 2}
\Rightarrow h^0(\mathcal{F}_A) = 0 \Rightarrow c_1(\mathcal{F}_A) \Rightarrow
\Rightarrow c_1(\mathcal{F}_A) = 0 \Rightarrow c_1(\mathcal{F}_A) = 0 \Rightarrow
\Rightarrow c_1(\mathcal{F}_A) = 0 \Rightarrow \text{A trivial \ \mathcal{O}_X \ bundle}
\Rightarrow \text{cohomology 2}
\Rightarrow h^0(\mathcal{F}_A) = 0 \Rightarrow c_1(\mathcal{F}_A) = 0 \Rightarrow
As \( r(\mathcal{E}_A) \subset r(\mathcal{F}_A) \) we automatically have 

\[
\frac{\deg(\mathcal{E}_A)}{\deg(\mathcal{F}_A)} \leq \frac{\deg(\mathcal{E}_A)}{\deg(\mathcal{G}_A)} (\leq 0)
\]

So \( \mathcal{E}_A \) is stable.