Lazarsfeld's Brill-Noether theorem

Fix rid \( \mathfrak{r} \), let \( X \) be a \( \mathfrak{r} \)-

surface. Assume \( \text{Pic}(X) = \mathbb{Z} \). Fix \( \nu \subset C^{\mathfrak{r}+1} \)

Define \( \mathfrak{P} \) as the scheme parameterizing

tuple \( (C_{\mathfrak{r}}, A_{\mathfrak{r}}, \lambda) \) s.t.

(i) \( C \in \mathcal{L} \) is smooth, irreducible,

(ii) \( A \in \mathcal{W}_{\mathfrak{r}}(C) \) is a pure bundle

s.t. \( A \) and \( \mathcal{W}_{\mathfrak{r}}(C) \) are stable

b.p. f.

(iii) \( \tau \) is a torsion

\[ \nu \odot \theta = \iota_{\mathfrak{r}} A \]
\[ \text{st. } \hat{\mathcal{H}}^0(U) : U \to \hat{\mathcal{H}}^0(C(A)) \]

Next, we identify two triple \( C(A_1, \lambda_1), C(A_1, \lambda_2) \) where
\[ \lambda = C(A_3), \quad C \subseteq C(\mathbb{R}^3). \]

Hence, \( F \) a quasi-projective

scheme \( P^r \) parameterizing triple \( C(A, A_1, \lambda) \) as above.

If we construct \( P^r \) as an

open sublocale of \( \text{Hilb} (\kappa \times P(U)) \)

Namely, \( C(A, A_1, \lambda) \) determines an
\[ \Delta_c : \nu \circ \epsilon_c \to \Delta \]

By Hartshorne's ch. on projective bundles, \( \Delta_c \) determines an embedding:

\[ C \subset \mathbb{P}(\nu \circ \epsilon_c) = \times \times \mathbb{P}(\nu) \]

Conversely, \( C \subset \mathbb{P}(\nu \circ \epsilon_c) \) defines an injection \( \Delta_c \) as above.

\( \Phi^r \) is a quasi-projective open subset of \( \mathbb{H}^n \times \mathbb{P}(\nu) \).

There is an isomorphism:

\[ \Pi : \Phi^r \cong \mathbb{H} \]

\[ C \cap A \ni \alpha \mapsto [c_j] \in \mathbb{H} \]
we wish to study the differential of \( \Pi \).

If \( C(A, X) \in \mathbb{P}^n \), get

\[
C \subset \times X \times \mathbb{P}(V)
\]

\[
g : C \to \mathbb{P}(V) \text{ the composition with } \quad \text{pr}_2 : X \times \mathbb{P}(V) \to \mathbb{P}(V)
\]

Have: stand and S.E.S.

\[
0 \to g^* T \mathbb{P}(V) \to N \mathbb{C}X (X \mathbb{P}(V)) \to 0
\]

By the Prop from last time on the Hilbert scheme, \( H^0 (X \mathbb{C}X (X \mathbb{P}(V))) \) may be identified with the tangent
Space of flat (\(\mathfrak{g}(u)\)) at 
\[ C \subset \mathfrak{g}(u) \], i.e. with 
\[ T_{(c \subset A, \mathfrak{g})} \mathbb{P}_d. \]

OTOH (by the same result)

\[ f^0(\mathcal{N}\mathcal{C}_x) = T_{(c \subset A)} \mathbb{H}_1 \]

is the same thing as the 
\[ \text{fibered Scheme of Curves in } X \text{ over } [c \subset A] \]

we may identify

\[ d_{\mathbb{P}_d}: T_{(c \subset A, \mathfrak{g})} \mathbb{P}_d \to T_{c \subset A, \mathfrak{g}} \]

with 
[\( f^0(x) \)]
Prop Assume \( \text{Pic } X = \mathbb{Z} \) as usual, then \( \text{cl} \) is surjective, i.e.

\[ \text{the multiplication map} \]

\[ \mu: H^0(A) \otimes H^0(\mathcal{O}_X(-A)) \to H^0(\mathcal{O}_X) \]

is injective.

Proof To begin, we claim that

\[ H^1(x): H^1(Nc/\mathcal{O}_X) \to H^1(Nc/\mathcal{O}_X) \]

is bijective.

\[ 0 \to F \to \mathcal{O}_X \to c_A \to 0 \]

(1)
Let's try and describe the embedding \( C \cong X \times \mathcal{P}(U) \) better.

Think of \( V \) as \( f^* \mathcal{P}(U) \otimes \mathcal{O}(1) \)

\( \phi \) induces a morphism

\[
\text{pr}_1^* (\mathcal{F}) \otimes \text{pr}_2^* (\mathcal{O}(\mathcal{P}(U))) \rightarrow \mathcal{O}
\]

\( \otimes + \quad \rightarrow + (\mathcal{E}(5)) \)

\( \mathcal{C} \in \text{Hom} (\mathcal{O}(\mathcal{P}(U)), \mathcal{O}) \)

This produces a section

\[
\mathcal{O} \otimes \mathcal{P}(U) \rightarrow \text{pr}_1^* (\mathcal{F}^*) \otimes \text{pr}_2^* (\mathcal{O}(\mathcal{P}(U)))
\]
Using \( C \) and unwrapping how this section is defined, you can check that the resulting form of \( Z \) is precisely

\[ C \leq \times \times PC(\mathfrak{u}) \]

so

\[ N_{C/\times \times PC(\mathfrak{h})} \]

\[ = F^\circ \circ \odot C_1 \]

\[ \approx F^\circ \circ \circ A \]

Let's compute \( h^\circ (N_{C/\times \times PC(\mathfrak{h})}) \)

\[ = G F^\circ \circ \circ A \]

Have

\[ 0 \rightarrow F^\circ A \rightarrow \rightarrow U \circ F^\circ \rightarrow A \circ F^\circ \rightarrow 0 \]
Take cohomology:

$F_A$ is simple, so $h^0(F_A \otimes F_A^*) = h^0(F_A \otimes F_A^*) = 1$

$h^1(U \otimes F^*) = 0 \quad \Rightarrow \quad h^1(F) = h^1(U \otimes F^*) = 0$

From:

$0 \rightarrow F \rightarrow U \otimes F^* \rightarrow i^\ast A \rightarrow 0$

$h^2(U \otimes F^*) = 0 = h^0(F)$

$\Rightarrow h^1(F \otimes i^\ast A) = h^2(U \otimes F^*) = 1$

Further: $h^1(N_c(x)) = h^1(C(x)) = h^1(\omega) = 1$

Recalling ses (1): $g: C \rightarrow \mathbb{A}^n$

$0 \rightarrow g \rightarrow N \rightarrow \mathbb{A}^n \rightarrow 0$

$\Rightarrow h^2(C \otimes \mathbb{A}^n) = 0$

we get

$h^*(F): h^1(N \otimes \mathbb{A}^n(\nu)) \Rightarrow h^1(\nu)$
Finishing the proof. From claim
\[ \text{co} \text{ur of } d \Pi \quad (c/\Pi, \gamma) \Rightarrow H^1(C \otimes T_{\Phi(J)}) \]

\[ \Phi \text{ is the map claim} \]
\[ c \Rightarrow \Phi(c) \]

Prop (Petri)
\[ H^1(C \otimes T_{\Phi(J)}) \]
\[ \Rightarrow \ker (H^0(C \otimes A) \otimes H^0(C \otimes A)) \]
\[ \Rightarrow H^0(C \otimes A) \]

This finishes the claim.