Section 1: The Hilbert Scheme of Points

Let $X$ be a K3 surface.
Let $Z \subseteq X$ be a zero dimensional
scheme and let $n = \dim H^0(Z, \mathcal{O}_Z)$.
Fix a very ample line bundle $L$ on $X$
with $g^2 = 2g - 2$.

$\Phi : X \to P^g$.

May view $Z \subseteq X$ as a projective
subvariety of $P^g$.

Obviously $\text{X}(Z, \mathcal{O}_Z(nL)) = \text{X}(Z, \mathcal{O}_Z)$

$= n$ is constant,
so $Z$ has Hilbert polynomial equal to the
constant function $n$. 

(t) \text{Hilb}^n (k) := \{ x \in \mathcal{S} \} \times \text{CP}_9 \text{ s.t.} \]
\[ p_t (z) = v \quad \exists \]
\[ \exists \Omega \in \text{Hilb}^n (k) \text{ are regularly ord} \text{inary}
\[ \text{subspaces with } \text{Ho} (z, \emptyset ) = N.
\]

By the deformation theory of the Hilbert
\[ \text{scheme, if } \exists \Omega \in \text{Hilb}^n (k), \]
\[ \exists \Omega \in \text{Hilb}^n (k) = \text{Ho}(z, N, z(t)) \]
\[ \exists \Omega \text{ are } N \Omega \times = \text{Hom} (\Omega, z(t), \emptyset ) \]

the normal sheaf.

Let's try to understand what the tangent
\[ \text{space is in our situation.} \]

If \[ \exists \Omega \text{ has support } P_1, \ldots, P_r \text{ and} \]
\[ z \in \mathcal{S} \cap U_i, (U_i \text{ small open about } P_i, \]

Let $N_3 = \emptyset \in N_{z_1}$

So it suffices to analyze the case where $\mathbb{C}_{z_3}$ is supported at one point $p$.

In this case, let $A$ be the localization of an affine open of $X$ about $p$.

We wish to study:

$\text{Hom}_{A/I} \left( \frac{I}{I^2}, \frac{A}{I} \right)$

$(A_{-\text{max}})$

$\text{Hom}_A \left( I, \frac{A}{I} \right)$

$(f : \text{Hom}_A \left( I, \frac{A}{I} \right)$, $f(e_i) = i \cdot f(e_i)$)

The assumption $h^0(\mathcal{O}_X) = n$ becomes:

Precisely that $A[I]$ is an artinian
A module of length $n$

Recall:

length is the maximal $n$-1, 2 chain of submodules of $M$

$N_0 \subseteq N_1 \subseteq \ldots \subseteq N_k$

Step

Let $A$ be a regular local ring of dimension 2. Let $I \subseteq A$ be an ideal.

Assume $A/I$ has length $n$ (and dim $A/I = 0$)

Then, Hom$_A(I, A/I)$ has length $\leq 2n$.

Proof

Consider depth of the $A$-module $I$.

depth $= \min \{ \ell \in \mathbb{Z} \mid A$-reg sequence $(x_1, \ldots, x_\ell)$ of length $\ell \}$
Obviously $\text{clp}(I) \geq 1$ (totally vacous?)

And under -such- basic formula:

$$\text{clp}(I) + \text{proj. dim } I = 2 \quad (= \dim A)$$

As $I$ is not invertible ($I$ not a divider)

$I$ is not free as it is not a $A$-module

$\implies$ $I \subseteq A$ is not free

$\implies$ proj. dim $I > 0$

hence $\text{clp}(I) = 1 = \text{proj. dim } I$

By defn of proj. dim we have a free resolution

$$0 \to P_1 \to P_0 \to I \to 0$$

localising at the generic point $I$ becomes

free at $P_1$ (so $i^* : P_0 \to \Lambda_{k+1}$)

then $P_i \cong A_i$
\[ 0 \to A \to A^1 \to I \to 0 \]

Applying \( \text{Hom}_A(-, A/I) \):

\[ 0 \to \text{Hom}_A(I, A/I) \to \text{Hom}_A(A, A/I) \]

\[ \to \text{Hom}_A(A^1, A/I) \to \text{Ext}^1_A(I, A/I) \to 0 \]

Let \( e_A := A - \text{coker} f \)

\[ e_A(\text{Hom}_A(A^1, A/I)) = \text{id}_A(A/I) \]

As length is additive,

\[ e_A(\text{Hom}_A(I, A/I)) = \text{id}_A(A/I) + \text{id}_A(A/I) \]

\[ - e_A(\text{Ext}^1_A(I, A/I)) = 0 \]
i.e. 

$$
e_A (\text{Hom}_A (I, A/I)) = e_A (A/I)$$

$$+ e_A \left( \text{Ext}^1_A (I, A/I) \right)$$

$$= n + e_A \left( \text{Ext}^1_A (I, A/I) \right)$$

It remains to show

$$e_A \left( \text{Ext}^1_A (I, A/I) \right) \leq n.$$ 

Apply \( \text{Ext}^1 (-, A/I) \) to

$$0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0 \quad (\star)$$

$$0 \rightarrow \text{Ext}^1_A (I, A/I) \rightarrow \text{Ext}^2_A (A/I, A/I) \rightarrow 0$$

Then

$$\text{Ext}^1_A (A/I, A/I)$$

Need now

$$e_A \left( \text{Ext}^2_A (A/I, A/I) \right) \leq n.$$
Apply \( \text{Ext}^1(A/I, 1) \Rightarrow \text{Ext}^1 \)

\[
\text{Ext}^2(A/I, A) \Rightarrow \text{Ext}^2(A/I, A/I) \Rightarrow 0
\]

\[
\text{Ext}^3(A/I, A)
\]

since \( A \) is regular at \( 	ext{dim} \ 2 \).

So it suffice to show

\[
\text{Ext}^1(A/I, A) \leq n
\]

As \( A \) is regular at \( i \) in \( 	ext{Gorenstein} \), i.e.

\[
\text{Ext}^i(A/I, A) = 0 \quad \text{for} \quad 0 \leq i \leq 1
\]

\[
\text{Ext}^2(A/I, A) = A/I_A
\]

\( \text{G} \to \text{N} \Rightarrow A/I \to A/(nA) \Rightarrow \)

\( \text{N} \) is finite length \( \text{Ext}^1(A/I, A) = \text{Ext}^1(A/I, A) = 1 \)
\[
\text{Ext}^2(N, A) \to \text{Ext}^2(A, I, A) \to \text{Ext}^2(A, A)
\]

So we get the claim by induction.

Thus we get:

Then (Fogarty)

\[\text{Hilb}^n(X)\] is smooth (inside of dim 2.

PF.

- \text{Hilb} is connected by Hartshorne?) then

- \text{Sym}^n(X) \cong U \subseteq \text{Hilb} is open of

- dim 2n (so by connectedness

By the previous prop \text{T Hilly has dim } \leq 2n

at each pt. Thus it suffices to show
that there are no components of dim \( \leq 2n-1 \).

If there were, then by connectedness there would be cuts \( I \) of dim \( \leq 2n-1 \). For dim \( 2n \)

\[ \text{with } I \cap J \neq \emptyset \Rightarrow \dim T^I_\emptyset (I \cap J) \]

\[ \geq 2n + 1 \]

for \( p \in I \cap J \) (sing. pt)