Voisin's Description of Syzygies

Let $X$ be a smooth K3 surface. It was shown last time that the Hilbert scheme $X^N$ of $n$-points is smooth of dimension $2n$ (this is true for any surface $X$).

Let $Z \subset X \times X$ be the universal subscheme.

Have two projections

\[ Z \to X \]

\[ \downarrow P \]

\[ X \]

Define $L^{[n]} = q_* P^* L$ the "hyperbolic sheaf". If $y \in X^{[n]}$, then $L^{[n]}$ has
Let $\mathcal{O}_3$ be the structure sheaf of $\mathbb{C}^n$.

Claim: $\mathcal{O}_3$ is locally free of rank $n$.

Prop (Ellingsrud–Göttsche–Lehn):

\[ H^0(\mathbb{C}^n, \det \mathcal{O}_3) \cong \Lambda^n H^0(\mathbb{C}^n) \]

Proof (Sketch, following Voisin):

Consider the evaluation map

\[ H^0(X, L) \otimes \mathcal{O}_3 \rightarrow \mathcal{L} \]

over $\mathbb{C}^n$. Let $H^0(\mathcal{L}) \rightarrow H^0(\mathbb{C}^n)$.

Taking the $n$th wedge of the global sections, we get a map
$\mu^1 H_0(\mathbb{A}^n \times \mathbb{P}^1) \rightarrow H^0 \mathbb{C}(\mathbb{P}^3, \text{det } \mathcal{O}_{\mathbb{P}^3})$

we will prove this is an isomorphism by constructing an inverse.

Let $X^n$ be the $n$-fold symmetric product of $X$. Have the Hilbert-Chow morphism $p: X^n \rightarrow X$.

Let $U \subseteq X^n$ be the open subset of zero cycles of degree $n$ with support consisting at least $n-1$ points.

$\pi: X^n \rightarrow X$ the natural quotient

$U = \pi^{-1}(U)$

Further, let $\overline{U}$ be the blow-up of $U$. 
along \( \mathcal{B} = \bigcup \mathcal{B}_i \), i.e., \( \mathcal{B}_i = \{ \mathcal{B}_i \} \).

Fact: the quotient of \( \mathcal{B} \) under the (finitely) saturated \( \mathcal{S}_0 \) action is an open subset of \( \mathcal{G} \otimes \mathcal{K}_2 \).

Hence \( \mathcal{B} \rightarrow \mathcal{B} \subset \mathcal{K}_2 \).

Consider \( \pi^*: \mathcal{K}_2 \rightarrow \mathcal{P} (\mathcal{K}_2^* \mathcal{L}) \)

\[ \pi^*: \mathcal{X}_n \rightarrow \mathcal{X} \]

If \( z \in \mathcal{V} \), the fibre of the above map
is $H^0(C \otimes \mathcal{O}_E(\mathfrak{p})) \rightarrow \bigotimes_i H^0(x_i, \mathcal{O}_{x_i})$

where $q(z) = (x_1, \ldots, x_n)$

Thus $f$ is an iso over the open set corresponding to $n$ distinct points.

$\Rightarrow f$ is injective.

Consider $f$ supported on exceptional divisor $E$ of $U$

$\Rightarrow \operatorname{det} (q^* (\mathcal{O}_E)) \sim q^* \mathcal{L}^{-E}$

Fact: If $USX$ has codim 2,

Let $Pic X$

then $H^0(X, \mathcal{O}) \cong H^0(C, \mathcal{O})$

Using this $H^0(X^{\otimes 2}, \operatorname{det} \mathcal{L}^{\otimes 2}) = H^0(\omega \otimes \mathcal{L}^{\otimes 2})$
Since \( w = V/s_n \)

\[
H^0(W, \det(C^{n,2})) \cong H^0(V, \varphi \cdot \det(C)^{\ast})^s_n
\]

\[
\cong H^0(V, \varphi \cdot \mathbb{L}^{(-E)})^s_n
\]

\( S_n \) acts by permuting factors of \( X^n \) gotten by factoring determinant

Get

\[
H^0(X^n, \det(C^{n,2}))
\]

\[
\cong H^0(V, \varphi \cdot \mathbb{L}^{(-E)})^s_n
\]

\[
\cong H^0(V, \varphi \cdot \mathbb{L}^{(-E)})^s_n
\]

\[n\]

Explicitly if \( f_1, \ldots, f_n \in \mathbb{H}^0(V)\)
\( \sigma \in S_n \)
\( \sigma = (p_1 \pm \epsilon \cdot \ldots \cdot \pm p_n \pm \epsilon) \)
\( = \text{sgn}(\sigma) \cdot p_1 \pm \epsilon \cdot \ldots \cdot \pm p_n \pm \epsilon \cdot \text{sgn}(\sigma) \)

So
\( H^0(X, \mathcal{O}_X) \cong \mathbb{R}^n \cong H^0(X, \mathcal{L}) \)

Thus we have
\( H^0(X, \mathcal{O}_X) \cong \mathbb{R}^n \cong H^0(X, \mathcal{L}) \)

which can be checked to be \( \mathbb{Z} \)

Carnilinar Subschemes

Let \( E \in \mathbb{P}^n \) (value \( \mathbb{Z} \cap X \) is a 0-dim subscheme, we call \( E \) carnilinear of \( f \), locally about each \( p \in \text{Supp} \mathcal{E} \))
$O_{Z, p} \cong C([\mathcal{T}]) / (t^e)$ for some $e$.

There is an open $U \subseteq X^{-3}$ of codimension $\geq 2$ parameterizing curvilinear subvarieties.

If $C^{-3} x \subseteq X^{-3}$ and $p \in \text{Supp}(Z)$ then the residual subvariety $\Gamma_x(Z) \subseteq X$ in $Z$ is defined to be the subvariety with $(\Gamma_x(Z))_y = \begin{cases} O_{Z, y} & \text{for } y \neq x \\ C[-e] / (t^{e-1}) & \text{if } y = x \end{cases}$

and $O_{Z, x} \cong C[-e] / (t^e)$

Thus, if $X^{-3} \subseteq X^{-3}$ is the open set of curvilinear subvarieties.
and $\exists C X_{\text{can}} \xrightarrow{\text{can}}$ the universal

subscheme, have residual map

\[ y : Z \rightarrow \text{can} \]

\[ (y, x) \mapsto t_x (y) \]

Using this map and the previous prop /

Voevodsky proves

\[ Z \xrightarrow{\phi} X \]

Prop (Voevodsky)

let I be a \[ \Leftrightarrow \]

line bundle on X

there is an isomorphism

\[ K_{n+1} \xrightarrow{\text{can}} \text{can} \]

\[ q^* \circ \text{can} \]
This description is the key step to deriving familiar proof of Green's conj.
for general k3 surfaces.