Recall

As a set \( \text{Har} \subseteq \text{Mg} \)

\[
\text{Har} = \{ C \in \text{Mg} \mid \text{gen}(C) \leq \frac{1}{2^3} \}
\]

As a line bundle

\[
\text{Har} := \frac{1}{(2g-2)(g-1)!} \prod_{k=1}^g (E_{K_1} \otimes [2])
\]

\( p_c : E^k \rightarrow E, \quad \Pi_k : E^k \rightarrow \text{Mg} \)

\( K_1 := p_{c*}(\omega_{E^1}) \)

\( Z \) parameterizes \( (p_1, \ldots, p_k) \) s.t. \( h^0(C, \mathcal{O}_C) \geq 2 \)

\( [E_C] = \Pi_k E_{P_1} \ldots (P_k) \)

To construct it as a scheme

\[
\begin{align*}
0 & \rightarrow O_{E^1} \rightarrow O_{E^m} (\Sigma A_{i_1} \otimes i_{1m}) \rightarrow O \rightarrow 0 \\
0 & \rightarrow O_{E_{i_1}} \rightarrow O_{E_{i_{1m}}} \rightarrow \Sigma A_{i_1} \rightarrow 0
\end{align*}
\]
\[ p: E^k \rightarrow E_k \] is projection onto first \( k \) factors.

Applying \( p \)

\[ 0 \rightarrow P O \rightarrow \mathbb{R}^1 \rightarrow P O \rightarrow E^k \]

\[ \rightarrow \mathbb{R}^1 \rightarrow O \rightarrow (\leq \delta_{j+1} e^1) \rightarrow 0 \]

If \( (p_1, \ldots, p_k) \in E^k \)

\[ h(0) = k \Rightarrow P O \leq \delta_{j+1} \]

is local. free at \( k \)

\[ h'(c, 0) = \omega(c_0) = g \]

so \( \mathbb{R}^1 \rightarrow P O \rightarrow E^k \) is local. free at \( g \)

The fibre at \( \alpha \) at any point

\[ \varphi \leq \epsilon \psi \in E^k \]
\[ 0 \to H^0(\mathcal{O}_C) \to H^0(\Theta_{\mathcal{E}^p_i}) \xrightarrow{\alpha} H^0(\Theta_{\mathcal{E}^p_i} \otimes \mathcal{E}^p_i) \to H^1(\mathcal{O}_C) \]

Here \( \alpha \) is not injective.

\[ C \to H^0(\mathcal{O}_C(3\mathcal{E}^p_i)) \cong \mathbb{Z} \]

\( \mathbb{Z} \) is the locus of fixed points of \( g \).

\( z \) is not injective at \( g \).

To describe the class of \( Z \), we appeal to the following:

**Theorem (Porteous)**

\( X \) smooth / \( C \), \( \phi : E \to F \) morphism of vector bundles of ranks \( n \) and \( m \).

\[ X_{k} (\phi) := \{ p \in X \mid \text{rk} (\phi_p) \leq k \} \]

Assume \( X_{k} (\phi) \) has maximal codimension \( m - k \) \((n - k)\).
Then \[
\left[ \chi_k (\mathcal{O}) \right] = \Delta_{m-k+n-k} \mathcal{C}(F-E) /
\]
where if \( a(t) = \sum a_k t^k \) formal power series

\[
\delta_{p,q} := \text{det} \begin{bmatrix}
    a_p & a_{p+1} & \cdots & a_{p+q-1} \\
    a_{p-1} & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & \ddots \\
    a_{p-q+1} & \cdots & \cdots & a_p \\
\end{bmatrix}
\]

To apply this, we need to know \( \mathbb{Z} \mathcal{C}^{ab} \) has codimension \( g-(\frac{g}{2}+1) = k \).

This is easy to deduce from the following classical facts: Little Huraity space

\[
H_{g,d} := \sum \mathbb{C}^m \text{ of degree } d, \text{ genus } \frac{g}{2}, \text{ smooth,}
\]

is irreducible of \( \dim = 2g-5+2d \) (Clebsch)

(2) the map \( H_{g,d} \to \mathcal{M}_g \) is generically finite
This \( \Rightarrow \) \( \forall r \in \mathbb{R} \), \( a \in \mathbb{R}^{3g-4} \)

\( \mathbb{Z} \) \( \Rightarrow \) \( \mathbb{Z} \) is fixed, \( \text{dim} \, \mathbb{Z} = 3g-3 \)

so \( \mathbb{Z} \) is the cohomology \( H \).

Then (Harris–Mumford, Harris)

(i) \( \text{Euler} \) = \( c \alpha \) for some constant \( c \)

(ii) In fact \( c = \frac{(2h-4)!}{k! (h-2)!} \).

PP we will only do (ii). Harris–Mumford did (ii) by intersecting \( H \) with explicit test curves, Kempf did (ii) using Riemann–Roch.

Applying Riemann

\[
\{ z \} = \sum_{g \leq n \leq \frac{g}{2}} \varepsilon \left( C \cap (R_{\text{gen}}(E_{n}); \mathbb{C}^1) \right)
\]
\[ C_{g-2k+1} \left( \sum_{\Delta_1, \ldots, \Delta_k} e^{\Delta_1 + \cdots + \Delta_k} \right) \]

\[ p! \Theta(\varepsilon, \Delta_1, \ldots, \Delta_k) - 1 \]

Chern classes are expressible as polynomials in \( c_i \) \[ \text{Exercise} \]

so \( [E] \) is poly in \( c_i \) \( + \) \( p! \Theta(\varepsilon, \Delta_1, \ldots, \Delta_k) \)

and \( \text{GR} \rightarrow [E] \) is given by a poly. in class of the form \( \Phi_{\ast} \) \[ \text{poly. in } \left[ \sum_{j=1}^{r} \varepsilon_j \right] \left[ \sum_{j=1}^{r} \Phi_j \right] \] for varying \( j \)

terms of \( \text{ch}(\varepsilon, \Delta_1, \ldots, \Delta_k) \) \( \text{td}(w_1) \)

Let's simplify this further.

\[ \text{Part } [E] \left[ \varepsilon_1, \varepsilon_2 \right] = [E] \text{, } \text{"obvious"} \]

and the idea is to repeatedly use the
push-pull formula

$$P(\Delta^1_{k+1}, \phi^* C^3_2) = \gamma, \quad \gamma \in A^*(E_1, \mathcal{Q})$$

Try to express as many terms as possible in the form $\phi^* (g), \quad g \in A^*(E_1, \mathcal{Q})$

We have the following "obvious" identity:

$$[\Delta_i^{\gamma+1}] \cdot [\Delta_j^{\gamma+1}] = [\Delta_i^{\gamma+1}, \phi^* [\Delta_j^{\gamma+1}]]$$

If $(\gamma^i)$ s.t., $P_i = P_{k+1}$ and $P_j = P_{k+1}$

for $i \neq j$

$$[\Delta_i^{\gamma+1}] \cdot K_j = [\Delta_j^{\gamma+1}] \cdot P^* K_j$$

This lets us deal with any monomial in which no repeated $[\Delta_i^{\gamma+1}]$, $[K_j^{\gamma+1}]$ term appears.

E.g.: $P(\Delta_1^{k+1}, \Delta_2^{k+1}, \Delta_3^{k+1}, K^{k+1})$
\[
- \mathbb{P}(\Delta_{1,1}^{c_1+1} P^* \Delta_{1,2} P^* \Delta_{1,3} P^* K_1) \\
= \Delta_{1,2} \Delta_{1,3} K_1
\]

To deal with powers of \( [\Delta_{j,k+1}] \) use the self-intersection formula:

\[ e : Y \times X \to Y \text{ smooth} \]  
\[ Y \times Y = \bigotimes (C_0(N)) \]  
\[ N \text{ normal bundle} \]

[Ha, pg 431]

\[
[\Delta_{j,k+1}]^2 = c_1(N) \bigotimes_{j+1} \Delta_{j+1}
\]

\[
= - c_1(-J_2 q)
\]

\[
= - c_1(P_j^* S^i \epsilon_i/M)
\]

\[
[\Delta_{j,k+1}]^2 = -[\Delta_{j,k+1}] \cdot P^*(K_j)
\]

They \([Z_j]\) can be written as a polynomial on the cycle \([\Delta_{c_1+j}]\), \([K_j]\) and \(P^*(K_{j+1})\) for all \(c_1 \geq j\).
We rewrite the last term:

**Flat - Pullback of Cycles**

(Prop 1.7) says

\[
A \rightarrow B \quad \xi \downarrow \times \quad \xi \text{ proper} \quad C \rightarrow D \quad \bigwedge h
\]

flat

\[\xi^* f \alpha = h^* g_\ast \alpha \quad \text{for } \alpha \in A^k\]

Apply to

\[e^{k-1} P_{\text{flat}} e \quad \Rightarrow \quad P_{\ast e} (k^e_{\text{ker}}) \]

\[\pi \downarrow \times \downarrow \pi \quad e^k \rightarrow \text{Mg} \quad \pi_k \]

[clar] poly in \(\pi_k\), \(\pi_k^\ast \) (poly \(\Delta_{i,j}^k\), \(\pi_k^\ast \) \(\text{poly} \), \(\pi_k^\ast \) \(\text{ker}^e\))

Factor

\[\pi_k: e^k \rightarrow e^k \rightarrow \ldots \rightarrow \text{finally}\]
and using above identities
get C[t\nu] poly in T\nu (K^2)

(P_{\nu} C D c_{\nu}, \mu) = \mu^2 \delta_{\nu \mu}\  \text{in the final step there are no diagonals left)}

Only one with correct dim is

K = T\lambda (K^2)

\Rightarrow E[h\nu] = c_2 \ b\ \text{by Mukunda's formula}

B.