MINIMAL FREE RESOLUTIONS OF PARACANONICAL CURVES OF ODD GENUS

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ABSTRACT. We prove the Prym–Green conjecture on minimal free resolutions of paracanonical curves in the case where the genus is odd.

0. Introduction

Let $C$ be a curve of odd genus $g = 2n + 1$ for $n \geq 2$ and let $\tau \in \text{Pic}(C)$ be a non-trivial $k$-torsion, line bundle. Assume the pair $(C, \tau)$ is general. We prove in this paper the following two vanishing statements

$$K_{n-1,1}(C, K_C \otimes \tau) = 0, \quad \text{and} \quad K_{n-3,2}(C, K_C \otimes \tau) = 0.$$  

These two vanishings imply that the minimal free resolution of the paracanonnically embedded curve is natural. See [FaLu] and [CEFS] for background on this conjecture and its relationship to the birational geometry of moduli spaces of paracanonical curves as well as [FK1], [FK2] for previous results on the Prym–Green conjecture in odd genus. Also see [CFVV] for a counterexample to the analogous statement in even genus.

By semicontinuity, it is enough to establish the two vanishing statement for one particular example of a paracanonical curve. The example we use comes from [FT, §2]. Namely, let $E$ be a smooth elliptic curve, let $X = \mathbb{P}(O_E \oplus \eta)$, where $\eta \in \text{Pic}^0(C)$ is neither trivial nor torsion, and let $\phi : X \to E$ be the fibration. Let $a \in E$ be any fixed point and let $b \in E$ be such that $\eta = a - b$. Furthermore, choose $r \neq b \in E$ such that $\zeta := b - r$ is $k$-torsion.

The scroll $X \to E$ has two sections, $J_0$ respectively $J_1$ corresponding to $O_E \oplus \eta \to O_E$ resp. $O_E \oplus \eta \to \eta$. We have $J_1 \simeq J_0 - \phi^*\eta$, $J_0|_{J_0} \simeq \phi^*\eta|_{J_0}$, $J_1|_{J_1} \simeq -\phi^*\eta|_{J_1}$, where we freely mix notation for divisors and line bundles.

For any point $x \in E$ we denote by $f_x$ the fibre $\phi^{-1}(x)$. We let

$$C \in |gJ_0 + f_r|$$

be a general element: this is a smooth curve of genus $g \geq 5$. We further set

$$L := (g - 2)J_0 + f_a.$$  

Using that $K_X = -2J_0 + \phi^*\eta$, the adjunction formula shows that $(C, L|_C)$ is the paracanonical curve $(C, K_C + \tau)$, where $\tau$ is the restriction of $\phi^*\zeta$ to $C$. In this paper, we verify the Prym–Green conjecture for these particular paracanonical curves.

1. Vanishing of the linear syzygy group $K_{n-1,1}(C, K_C + \tau)$

In this section we establish the vanishing of the linear syzygy group $K_{n-1,1}(C, K_C + \tau)$. Before proceeding, we confirm that $\tau$ is actually nontrivial of order $k$, so that $(C, K_C + \tau)$ is a genuine paracanonical curve of level $k$.

Lemma 1.1. For any $1 \leq m \leq k - 1$, $m\tau \in \text{Pic}^0(C)$ is not effective.
First of all, note that the restriction map is injective, since Corollary 1.d.4, the vanishing of the Koszul group of graded modules, where the first map is defined by multiplication with the section defining $C$ and $M$ is defined by the exact sequence. By the long exact sequence of Koszul cohomology, [G] Corollary 1.d.4, the vanishing of the Koszul group $K_{p,1}(M, S)$ follows from $K_{p,1}(\tilde{X}, \tilde{L}) = 0$ and $K_{p-1,2}(\tilde{X}, -C; \tilde{L}) = 0$.

**Lemma 1.2.** Let $x_1 \in J_0$, $x_2 \in J_1$ be the two base points of $|C|$. Then $x_1, x_2 \notin \{p, q^{(g-2)}\}$.

**Proof.** As $J_1 \cap f_a \neq J_1 \cap f_s, p \neq x_2$. Next, $q^{(g-2)} = J_0 \cap f_s(q^{(g)})$ where $s^{(g-2)} = \eta^{g-2} + a \in \text{Pic}(E)$ and $x_1 = J_0 \cap f_s(q)$, where $t^{(g)} = \eta^g + r$. So we need to show $\eta^{g-2} + a \neq \eta^g + r$, which follows from the fact that $a - r = \eta + \zeta$, where $\zeta$ is torsion and $\eta$ is not. \hfill $\Box$

We will abuse notation by writing $C$ for $\pi^*(C)$. We set $S := \text{Sym} H^0(\tilde{X}, \tilde{L})$ and consider the short exact sequence

$$0 \to \bigoplus_{q \in \mathbb{Z}} H^0(\tilde{X}, q\tilde{L} - C) \to \bigoplus_{q \in \mathbb{Z}} H^0(\tilde{X}, q\tilde{L}) \to M \to 0,$$

of graded $S$ modules, where the first map is defined by multiplication with the section defining $C$ and $M$ is defined by the exact sequence. By the long exact sequence of Koszul cohomology, [G] Corollary 1.d.4, the vanishing of the Koszul group $K_{p,1}(M, S)$ follows from $K_{p,1}(\tilde{X}, \tilde{L}) = 0$ and $K_{p-1,2}(\tilde{X}, -C; \tilde{L}) = 0$.

**Lemma 1.3.** We have equality $K_{p,1}(M, S) = K_{p,1}(C, K_C + \tau)$ for any $p \geq 0$.

**Proof.** First of all, we claim that restriction induces an isomorphism $H^0(\tilde{X}, \tilde{L}) \simeq H^0(C, K_C + \tau)$. First of all, note that the restriction map is injective, since $\tilde{L} - C = -2J_0 + f_a - f_s - E_1 - E_2$ is not effective (as it has negative intersection with the nef class $f_s$). Next, $h^0(\tilde{X}, \tilde{L}) = h^0(X, L) = g - 1$ by a direct computation using the projection formula, see [FT], Lemma 2. As $h^0(C, K_C + \tau) = g - 1$, restriction induces the claimed isomorphism.

Let $M_d$ denote the $d$-th graded piece of $M$. We have an isomorphism $\mathbb{C} \simeq H^0(\tilde{X}, \mathcal{O}_C) \simeq M_0$. Next, we have already seen that $H^0(\tilde{X}, \tilde{L} - C) = 0$, so $H^0(\tilde{X}, \tilde{L}) \simeq M_1$. But since, $h^0(\tilde{X}, \tilde{L}) = g - 1 = h^0(C, K_C + \tau)$, we have equality $M_1 = H^0(C, \tilde{L}|_C) = H^0(C, K_C + \tau)$. So we have the commutative diagram

$$\begin{array}{cccccc}
\wedge^{p+1} H^0(\tilde{X}, \tilde{L}) \otimes M_0 & \to & \wedge^p H^0(\tilde{X}, \tilde{L}) \otimes M_1 & \to & \wedge^{p-1} H^0(\tilde{X}, \tilde{L}) \otimes M_2 \\
\downarrow & & & & \\
\wedge^{p+1} H^0(K_C + \tau) \otimes H^0(\mathcal{O}_C) & \to & \wedge^p H^0(K_C + \tau) \otimes H^0(\mathcal{O}_C) & \to & \wedge^{p-1} H^0(K_C + \tau) \otimes H^0(2K_C + 2\tau)
\end{array}$$

where the two leftmost vertical maps are isomorphisms and the rightmost vertical map is injective. Thus the middle cohomology of each row is isomorphic, so that we have the equality $K_{p,1}(M, S) = K_{p,1}(C, K_C + \tau)$ for any $p \geq 0$. \hfill $\Box$
Hence, in order to show $K_{n-1,1}(C, K_C + \tau) = 0$, it suffices to show

(1) \[ K_{n-1,1}(\tilde{X}, \tilde{L}) = 0, \] and
(2) \[ K_{n-2,2}(\tilde{X}, -C; \tilde{L}) = 0. \]

The first vanishing is fairly straightforward:

**Proposition 1.4.** We have $K_{n-1,1}(\tilde{X}, \tilde{L}) = 0$.

**Proof.** Let $D \in |\tilde{L}|$ be a general element. We have an isomorphism $K_{n-1,1}(\tilde{X}, \tilde{L}) \cong K_{n-1,1}(D, K_D)$, as $K_{\tilde{X}|D} \cong O_D$ and by [AN], Theorem 2.20 (note that one only needs that the restriction $H^0(\tilde{X}, L) \to H^0(D, K_D)$ is surjective, and not $H^1(\tilde{X}) = 0$, for this result). As $D$ is a smooth curve of odd genus $2n - 1$, it suffices to show that $D$ has maximum gonality $n + 1$ by [HR], [V1], [V2]. As $D$ is Brill–Noether general by [FT], Remark 2, it has maximal gonality. This completes the proof. \qed

We now turn our attention to the vanishing of the second group $K_{n-2,2}(\tilde{X}, -C; \tilde{L}) = 0$.

**Proposition 1.5.** Let $D \in |\tilde{L}|$ be general and let $p \geq 0$. Assume $K_{m,2}(D, O_D(-C); K_D) = 0$ for $m \in \{p, p + 1\}$. Then

\[ K_{p,2}(\tilde{X}, -C; \tilde{L}) = 0. \]

**Proof.** Let $S := \text{Sym}H^0(\tilde{X}, \tilde{L})$ and consider the short exact sequence

\[ 0 \to \bigoplus_{q \in \mathbb{Z}} H^0(\tilde{X}, (q-1)\tilde{L} - C) \to \bigoplus_{q \in \mathbb{Z}} H^0(\tilde{X}, q\tilde{L} - C) \to M \to 0, \]

of graded $S$ modules, where $M$ is defined by the above sequence and the first map is given by multiplication by a general section $s \in H^0(\tilde{X}, \tilde{L})$. We now argue as in [FK1], Lemma 2.2. Taking the long exact sequence of Koszul cohomology, and using that multiplication by a section $s \in H^0(\tilde{X}, \tilde{L})$ induces the zero map on Koszul cohomology, we get

\[ K_{p,q}(M, H^0(\tilde{X}, \tilde{L})) \cong K_{p,q}(\tilde{X}, -C; \tilde{L}) \oplus K_{p-1,q}(\tilde{X}, -C; \tilde{L}), \]

for all $p, q \in \mathbb{Z}$.

Let $D = Z(s)$ be the divisor defined by the section $s \in H^0(\tilde{X}, \tilde{L})$, and consider the graded $S$ module

\[ N := \bigoplus_{q \in \mathbb{Z}} H^0(D, qK_D - C_D). \]

We have the inclusion $M \subseteq N$ of graded $S$ modules. We claim $M_1 = N_1 = 0$. By intersecting with the nef class $f_2$, we see $H^0(\tilde{X}, \tilde{L} - C) = 0$ from which it follows that $M_1 = 0$. As $\text{deg}(K_D - C_D) = -4$, we have $N_1 = 0$. Upon taking Koszul cohomology, this immediately gives that

\[ K_{p,2}(M, H^0(\tilde{X}, \tilde{L})) \subseteq K_{p,2}(N, H^0(\tilde{X}, \tilde{L})). \]

In particular, $K_{p,2}(\tilde{X}, -C; \tilde{L}) \subseteq K_{p+1,2}(M, H^0(\tilde{X}, \tilde{L})) \subseteq K_{p+1,2}(N, H^0(\tilde{X}, \tilde{L}))$.

To finish the proof, it will suffice to show

\[ K_{p,2}(N, H^0(\tilde{X}, \tilde{L})) \cong K_{p,2}(D, O_D(-C); K_D) \oplus K_{p-1,2}(D, O_D(-C); K_D). \]

We have $h^0(\tilde{X}, \tilde{L}) = h^0(X, L) = (g - 2) + 1 = h^0(D, K_D) = 1$. As in the proof of [FK1], Lemma 2.2, the claim then follows by a direct computation using the Koszul sequence. \qed
It thus suffices to show
\begin{equation}
K_{n-2,2}(D, \mathcal{O}_D(-C); K_D) = 0 \quad \text{and} \quad K_{n-1,2}(D, \mathcal{O}_D(-C); K_D) = 0.
\end{equation}
Since \( H^0(D, K_D - C_D) = 0 \), these two statements are equivalent to
\begin{equation}
H^0(D, \bigwedge^{n-2} M_{K_D}(2K_D - C_D)) = 0 \quad \text{and} \quad H^0(D, \bigwedge^{n-1} M_{K_D}(2K_D - C_D)) = 0
\end{equation}
see [AF].

We now state a routine observation regarding difference varieties:

**Lemma 1.6.** Let \( C \) be a smooth curve and \( A \) a line bundle on \( C \). Let \( a \geq 2, b \geq 0, c > 0 \) be integers. Assume \( A - C_a \subseteq C_b - C_c \). Then \( A - C_{a-2} \subseteq C_{b+1} - C_{c-1} \).

**Proof.** Let \( B \) be an arbitrary effective divisor of degree \( a - 2 \), and let \( y \in C \) be any point. Since \( A - C_a \subseteq C_b - C_c \), we have a morphism

\[
f : C \to C_b - C_c \subseteq \text{Pic}^{b-c}(C)
\]
\[
x \mapsto A - (B + x + y)
\]
We further have the natural morphism

\[
g : C_b \times C_c \to \text{Pic}^{b-c}(C)
\]
\[
(D_1, D_2) \mapsto D_1 - D_2,
\]
as well as the projection \( p_2 : C_b \times C_c \to C_c \).

Suppose firstly that \( \dim p_2(g^{-1}(f(C))) \geq 1 \). As the divisor \( y + C_{c-1} \subseteq C_c \) is ample ([FuLa], Lemma 2.7), \( p_2(g^{-1}(f(C))) \) must meet \( y + C_{c-1} \). This means that there is some \( x \in C \) such that \( A - (B + x + y) = D_1 - D_2 \), \( D_1 \in C_b, D_2 \in C_c \) and such that \( D_2 - y \) is effective. But then

\[
A - B = (D_1 + x) - (D_2 - y) \in C_{b+1} - C_{c-1}.
\]

Lastly, assume \( p_2(g^{-1}(f(C))) \subseteq C_c \) is finite. Then there is some fixed \( D_2 \in C_c \) such that, for every \( x \in C \), there is an effective divisor \( D_x \) with \( A - (B + x + y) = D_x - D_2 \). Picking \( x \) in the support of \( D_2 \), we may find some \( x \) such that \( D_2 - x \) is effective. Then \( A - B \in (D_x + y) - (D_2 - y) \in C_{b+1} - C_{c-1} \).

We may now restate the vanishing conditions (3) in terms of difference varieties.

**Lemma 1.7.** Let \( D \in |(g - 2)J_0 + f_a - E_1 - E_2| \) be general as above, and assume \( n \geq 2 \) (so \( g \geq 5 \)). Suppose

\[ C_D - K_D - D_2 \notin D_n - D_{n-2}. \]

Then (3) holds.

**Proof.** By assumption, there is a degree two divisor \( T \in D_2 \) such that \( C_D - K_D - T \notin D_n - D_{n-2} \). By [FMP] §3, this is equivalent to \( H^1(D, \bigwedge^{n-2} Q_{K_D}((C_D - K_D - T))) = 0 \) for \( Q_{K_D} := M_{K_D} \).

This implies \( H^1(D, \bigwedge^{n-2} Q_{K_D}(C_D - K_D)) = 0 \) and so \( H^0(D, \bigwedge^{n-2} M_{K_D}(2K_D - C_D)) = 0 \) by Serre–Duality. This is equivalent to \( K_{n-2,2}(D, \mathcal{O}_D(-C); K_D) = 0 \).

Next, by Lemma 1.6, the assumption implies \( C_D - K_D - D_4 \notin D_{n-1} - D_{n-1} \). This implies \( H^1(D, \bigwedge^{n-1} Q_{K_D}(C_D - K_D - T)) = 0 \) for some \( T \in D_4 \). Thus \( H^1(D, \bigwedge^{n-1} Q_{K_D}(C_D - K_D)) = 0 \) which is equivalent to \( K_{n-1,2}(D, \mathcal{O}_D(-C); K_D) = 0 \) by Serre-Duality.

Any (smooth) divisor \( D \in |L| = |L| \) carries two natural points, namely the base points \( p \) and \( q^{(g-2)} \). We will prove that, if \( D \in |L| \) is general, then

\[ C_D - K_D - p - q^{(g-2)} \notin D_n - D_{n-2}. \]
Let us first introduce some notation. For an integer \( u \geq 1 \), define the line bundle \( M_u \in \text{Pic}(X) \) by \( M_u = uJ_0 + f_u \), thus \( L = M_{g-2} \). A general element \( D_u \in |M_u| \) is smooth of genus \( u \), with two base points \( p \in J_1 \) and \( q^{(u)} \in J_0 \). To help with the abundance of subscripts, for a smooth curve \( D_u \) of genus \( u \) write the divisorial difference varieties as

\[
\text{Diff}_j(D_u) := \text{Sym}_j(D_u) - \text{Sym}_{u-1-j}(D_u) \subseteq \text{Pic}(D_u)
\]

for \( 0 \leq j \leq u - 1 \). Furthermore, define \( \text{Diff}_j(D_u) = \emptyset \) for \( j < 0 \) or \( j > u - 1 \). Lastly write

\[
C^j_u := K_{D_u} - C_{D_u} \in \text{Pic}(D_u)
\]

for the adjoint bundle of \( C_{D_u} \). We aim to prove

\[
-C^j_{g-2} - p - q^{(g-2)} \notin \text{Diff}_n(D_{g-2})
\]

for general \( D_{g-2} \in M_{g-2} \). We will achieve this via induction. For the induction step, we will show that if

\[
-C^j_{g-2-j} - p - (2i + 1)q^{(g-2-j)} \in \text{Diff}_{-i}(D_{g-2-j}) \text{ for some } 0 \leq i \leq j
\]

holds for any fixed \( 0 \leq j \leq g - 3 \) and for the general \( D_{g-2-j} \in |M_{g-2-j}| \), then

\[
-C^{j'}_{g-3-j} - p - (2i' + 1)q^{(g-3-j)} \in \text{Diff}_{-i'}(D_{g-3-j}) \text{ for some } 0 \leq i' \leq j + 1
\]

holds for the general \( D_{g-3-j} \in |M_{g-3-j}| \). Notice that the assumption \( \text{Diff}_{-i}(D_{g-2-j}) \neq \emptyset \) gives

\[
0 \leq n - i \leq g - 3 - j,
\]

where \( g = 2n + 1 \).

Let \( D_{g-2-j} \in |M_{g-2-j}| \) be general. In order to prove the induction step, we will degenerate \( D_{g-2-j} \) to the reducible divisor

\[
D'_{g-2-j} := J_0 + D_{g-3-j}
\]

for a general \( D_{g-3-j} \in |M_{g-3-j}| \). Notice that \( J_0 \cap D_{g-3-j} = \{ q^{(g-3-j)} \} \). We will choose to degenerate \( K_{D_{g-2-j}} \) to the line bundle

\[
\tilde{K} \in \text{Pic}(D'_{g-2-j})
\]

defined by \( \tilde{K}|_{J_0} \simeq O_{J_0}, \tilde{K}|_{D_{g-3-j}} \simeq K_{D_{g-3-j}} + 2q^{(g-3-j)} \); i.e. the unique limit of canonical bundles which has bidegree \((0, 2(g - 2 - j) - 2)\).

**Lemma 1.8.** Notation as above. Assume the bounds (5). Then, for any \( 0 \leq i \leq j \leq g - 4 \) we have:

(i) \( h^0(D'_{g-2-j}, \tilde{K}) = h^0(D_{g-2-j}, K_{D_{g-2-j}}) \)

(ii) \( h^0(D'_{g-2-j}, C - J_1 - (2i + 1)J_0 - \tilde{K}) = h^0(D_{g-2-j}, -C^j_{g-2-j} - p - (2i + 1)q^{(g-2-j)}) = 0 \)

(iii) \( h^0(D'_{g-2-j}, C - J_1 - (2i + 1)J_0) = h^0(D_{g-2-j}, C - p - (2i + 1)q^{(g-2-j)}) \)

(iv) \( h^0(D'_{g-2-j}, C - J_1 - (2i + 1)J_0 + \tilde{K}) = h^0(D_{g-2-j}, C - p - (2i + 1)q^{(g-2-j)} + K_{D_{g-2-j}}) \).

In the above we are omitting subscripts when this is unlikely to cause confusion.

**Proof.**

(i) As \( \tilde{K} \) is a limit of canonical bundles on smooth curves, \( h^0(D'_{g-2-j}, \tilde{K}) \geq g - 2 - j = h^0(D_{g-2-j}, K_{D_{g-2-j}}) \). So it suffices to show \( h^0(D'_{g-2-j}, \tilde{K}) \leq h^0(D_{g-2-j}, K_{D_{g-2-j}}) \). We have the short exact sequence

(6) \[
0 \to O_{J_0}(-q^{(g-3-j)}) \to O_{D'_{g-2-j}} \to O_{D_{g-3-j}} \to 0.
\]
Twisting by $\widetilde{K}$, we get $0 \to O_{\mathcal{J}_0}(-q^{(g-3-j)}) \to \widetilde{K} \to K_{D_{g-3-j}} + 2q^{(g-3-j)} \to 0$, which gives $h^0(D'_{g-2-j}, \widetilde{K}) \leq h^0(D_{g-3-j}, K_{D_{g-3-j}} + 2q^{(g-3-j)}) = g - 2 - j$, as required.

(ii) Set $A := C - J_1 - (2i + 1)J_0 + l\mathcal{K} \in \text{Pic}(D'_{g-2-j})$. For (ii), it suffices by semicontinuity to show $h^0(D'_{g-2-j}, A - j) = 0$. We have $A_{-1}|_{J_0} \simeq (g - 2i - 1)\eta + r$, where we identify $J_0$ with $E$ via $\phi$. Further $\mathcal{O}_{J_0}(q^{(g-3-j)}) \simeq (g - 3 - j)\eta + a$, so $A_{-1}|_{J_0}(-q^{(g-3-j)}) \simeq (2 + j - 2i)\eta + r - a$. We have $h^0(E, (2 + j - 2i)\eta + r - a) = 0$ as

$$a - r = \eta + \zeta,$$

where $\zeta$ is $k$-torsion but $\eta$ is not torsion. From the short exact sequence (6) twisted by $A_{-1}$, it suffices to show

$$A_{-1}|_{D_{g-3-j}} \simeq ((g - 2i - 3)J_0 - J_1 + f_r - K_{D_{g-3-j}})|_{D_{g-3-j}} \simeq ((j + 1 - 2i)J_0 + f_r - f_a)|_{D_{g-3-j}}$$

is not effective. We will firstly show $H^0(X, (j + 1 - 2i)J_0 + f_r - f_a) = 0$. If $j + 1 - 2i < 0$, this is immediate since then $((j + 1 - 2i)J_0 + f_r - f_a) < 0$ and $f_r$ is nef. If $j + 1 - 2i \geq 0$, then

$$H^0(X, (j + 1 - 2i)J_0 + f_r - f_a) \simeq H^0(E, \mathcal{O}_E (r - a)) \otimes \text{Sym}^{j+1-2i}(\mathcal{O}_E \oplus \eta) = 0.$$

So it is enough to show $H^1(X, (j + 1 - 2i)J_0 + f_r - f_a - D_{g-3-j}) = 0$. By Serre duality, this is equivalent to

$$H^1(X, K_X + D_{g-3-j} + f_a - f_r - (j + 1 - 2i)J_0) = 0.$$

We have

$$K_X + D_{g-3-j} + f_a - f_r - (j + 1 - 2i)J_0 = (g - 6 + 2i - 2j)J_0 + \phi^*\eta + 2f_a - f_r,$$

where $g - 6 + 2i - 2j \geq -1$ by (5). If $g - 6 + 2i - 2j = -1$, then we have the short exact sequence

$$0 \to -J_0 + \phi^*\eta + 2f_a - f_r \to \phi^*\eta + 2f_a - f_r \to (\phi^*\eta + 2f_a - f_r)|_{J_0} \to 0.$$

Taking cohomology and using the Leray spectral sequence we get the long exact sequence

$$0 \to C \to C \to H^1(X, -J_0 + \phi^*\eta + 2f_a - f_r) \to 0,$$

so $H^1(X, -J_0 + \phi^*\eta + 2f_a - f_r) = 0$. If $g - 6 + 2i - 2j \geq 0$, then

$$H^1(X, (g - 6 + 2i - 2j)J_0 + \phi^*\eta + 2f_a - f_r) = H^1(E, \mathcal{O}_E(2a - r + \eta) \otimes \text{Sym}^{g-6+2i-2j}(\mathcal{O}_E \oplus \eta)),$$

which vanishes by Riemann–Roch. This completes the proof.

(iii) By Riemann–Roch and semicontinuity, it suffices to show $H^1(D'_{g-2-j}, A_0) = 0$. I.e. we need to prove

$$H^1(D'_{g-2-j}, (g - 2i - 1)J_0 - J_1 + f_r) = 0.$$

By (5), $i \leq n$. Consider first the case $i = n$. We have the short exact sequence

$$0 \to -J_1 + f_r \to f_r \to f_r|_{J_1} \to 0.$$

Taking the long exact sequence of cohomology and using the Leray spectral sequence gives

$$0 \to C \to C \to H^1(X, -J_1 + f_r) \to 0.$$
so $H^1(X, -J_1 + f_r) = 0$. Using the sequence
\[ 0 \to nJ_0 - J_1 + f_r \to (n + 1)J_0 - J_1 + f_r \to ((n + 1)J_0 - J_1 + f_r)|_{J_0} \to 0, \]
we get by induction that $H^1(X, (g - 2i - 1)J_0 - J_1 + f_r) = 0$.

It is now enough to show $h^2(X, (g - 2i - 1)J_0 - J_1 + f_r - D'_{g-2-j}) = 0$. We have
\[
\begin{align*}
h^2(X, (g - 2i - 1)J_0 - J_1 + f_r - D'_{g-2-j}) & = h^0(X, K_X + J_1 + D'_{g-2-j} - (g - 2i - 1)J_0 - f_r) \\
& = h^0(X, (2i - 4 - j)J_0 + f_a - f_r).
\end{align*}
\]
If $(2i - 4 - j) < 0$, then the class $(2i - 4 - j)J_0 + f_a - f_r$ is not effective on $X$ as it has negative intersection with $f_r$. If $2i - 4 - j \geq 0$ then this class is not effective by projecting to $E$.

(iv) It suffices to show $H^1(D'_{g-2-j}, A_1) = 0$. We use the exact sequence
\[
0 \to \mathcal{O}_{D_{g-2-j}}(-q(g-3-j)) \to \mathcal{O}_{D'_{g-2-j}} \to \mathcal{O}_{J_0} \to 0.
\]
As $\deg(A_1|_{J_0}) = 1$, it suffices to show $H^1(D_{g-3-j}, A_1|_{D_{g-3-j}}(-q(g-3-j))) = 0$. Now
\[
\deg A_1|_{D_{g-3-j}}(-q(g-3-j)) = \deg K_{D_{g-3-j}} + 2n - 2i + g - 3 - j.
\]
From (5), $n - i \geq 0$, whereas $j \leq g - 4$ by assumption, so $g - 3 - j > 0$ and $H^1(D_{g-3-j}, A_1|_{D_{g-3-j}}(-q(g-3-j))) = 0$ for degree reasons.

\[ \square \]

We now have all the pieces needed to prove the induction step:

**Proposition 1.9.** Fix $0 \leq j \leq g - 3$ and assume
\[-C^i_{g-2-j} - p - (2i + 1)q(g-2-j) \in \text{Diff}_{n-i}(D_{g-2-j}) \text{ for some } 0 \leq i \leq j\]
for the general $D_{g-2-j} \in |M_{g-2-j}|$. Then
\[-C^i_{g-3-j} - p - (2i' + 1)q(g-3-j) \in \text{Diff}_{n-i'}(D_{g-3-j}) \text{ for some } 0 \leq i' \leq j + 1\]
holds for the general $D_{g-3-j} \in |M_{g-3-j}|$.

**Proof.** By [FMP], §3, the assumption may be written as
\[ H^0(D_{g-2-j}, \bigwedge^{g-3-j-n+i} M^\vee_{K_{D_{g-2-j}}} \otimes (-C^i_{g-2-j} - p - (2i + 1)q(g-2-j))) \neq 0, \]
or, equivalently,
\[ H^0(D_{g-2-j}, \bigwedge^{n-i} M_{K_{D_{g-2-j}}} \otimes (C - J_1 - (2i + 1)J_0) \neq 0. \]
By Lemma 1.8, $H^0(D_{g-2-j}, C - J_1 - (2i + 1)J_0 - K_{D_{g-2-j}}) = 0$, so this is equivalent to
\[ K_{n-i,0}(D_{g-2-j}, C - J_1 - (2i + 1)J_0; K_{D_{g-2-j}}) \neq 0. \]
By semicontinuity for Koszul cohomology, [FK3] Lemma 1.1, together with Lemma 1.8, this implies
\[ K_{n-i,0}(D'_{g-2-j}, C - J_1 - (2i + 1)J_0; \widetilde{K}) \neq 0, \]
where $D'_{g-2-j} := J_0 + D_{g-3-j}$ for a general $D_{g-3-j} \in |M_{g-3-j}|$, as above. This is the same as saying that the map
\[ \bigwedge^{n-i} H^0(D'_{g-2-j}, \widetilde{K}) \otimes H^0(D'_{g-2-j}, A_0) \to \bigwedge^{n-i-1} H^0(D'_{g-2-j}, \widetilde{K}) \otimes H^0(D'_{g-2-j}, A_1) \]
is not injective, for \( A_l := C - J_1 - (2i + 1)J_0 + l\tilde{K} \). As seen in the proof of Lemma 1.8 (i), restriction induces an isomorphism

\[ H^0(D^l_{g-2-j}, \tilde{K}) \to H^0(D_{g-3-j}, K_{D_{g-3-j}} + 2q^{(g-3-j)}). \]

Moreover, as seen in the proof of Lemma 1.8 (ii), \( A_l|_{q} (-q^{(g-3-j)}) \simeq (2 + j - 2i)\eta + r - a \) is a nontrivial line bundle of degree 0 and so \( h^0(J_0, A_l(-q^{(g-3-j)})) = h^1(J_0, A_l(-q^{(g-3-j)})) = 0 \). Thus, restriction induces an isomorphism

\[ H^0(D^l_{g-2-j}, A_l) \simeq H^0(D_{g-3-j}, A_l). \]

Hence the map

\[ \bigwedge^{n-i} H^0(D_{g-3-j}, K_{D_{g-3-j}} + 2q^{(g-3-j)}) \otimes H^0(D_{g-3-j}, A_0) \]

\[ \to \bigwedge^{n-i} H^0(D_{g-3-j}, K_{D_{g-3-j}} + 2q^{(g-3-j)}) \otimes H^0(D_{g-3-j}, A_1), \]

is not injective. As \( H^0(D_{g-3-j}, A_1) \subsetneq H^0(D_{g-3-j}, A_1 + 2q^{(g-3-j)}), \) and since \( H^0(D_{g-3-j}, A_1 - 2q^{(g-3-j)}) \subsetneq H^0(D_{g-3-j}, A_1) = 0, \) we get

\[ K_{n-i,0}(D_{g-3-j}, A_0; K_{D_{g-3-j}} + 2q^{(g-3-j)}) \neq 0. \]

This can be rewritten as \( H^0(D_{g-3-j}, \bigwedge^{n-i} M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}}(A_0) \neq 0 \). We now compute

\[ (n - i)\mu(M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}}) + \deg(A_0) + 1 - (g - 3 - j) = 0, \]

where \( \mu(M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}}) = -2 \) is the slope of \( M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}} \). So by Riemann–Roch, \( H^1(D_{g-3-j}, \bigwedge^{n-i} M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}} \otimes A_0) \neq 0 \) and by Serre-Duality,

\[ H^0(D_{g-3-j}, \bigwedge^{n-i} M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}} \otimes K_{D_{g-3-j}} - A_0) \neq 0. \]

From [B], Prop. 2 we thus have

\[ A_0 - K_{D_{g-3-j}} \in \text{Diff}_{n-i}(D_{g-3-j}) \]

or

\[ A_0 - K_{D_{g-3-j}} - 2q^{(g-3-j)} \in \text{Diff}_{n-i-1}(D_{g-3-j}). \]

Whilst Beauville assumes that \( D_{g-3-j} \) is non-hyperelliptic, which in particular implies \( g - 3 - j \geq 3 \), note that the above statement is a triviality for \( g - 3 - j = 1 \) (in which case \( M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}} \simeq O_{D_{g-3-j}}(2q^{(g-3-j)}), \) whereas it follows directly from the argument in [B] Prop. 2 in the hyperelliptic case. Indeed, we have a short exact sequence

\[ 0 \to O_{D_{g-3-j}}(2q^{(g-3-j)}) \to M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}}^\vee \to M_{K_{D_{g-3-j}}}^\vee \to 0. \]

This gives the exact sequence

\[ 0 \to \bigwedge^{n-i-1} M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}}^\vee \to \bigwedge^{n-i} M_{K_{D_{g-3-j}} + 2q^{(g-3-j)}}^\vee \to \bigwedge^{n-i} M_{K_{D_{g-3-j}}}^\vee \to 0. \]

The claim now follows immediately from [FMP], §3. This completes the proof. \qed
By the above Proposition and induction, to finish the proof of (3), it suffices to show
\[-C_1^i - p - (2i + 1)q(1) \notin \text{Diff}_{n-1}(D_1)\]
for \(1 \leq i \leq g - 2\). Moreover, the bounds (5) force \(n = i\) in this case, so we just need to prove
\[-C_1^i - p - (2n + 1)q(1) \cong (f_r - J_1)|_{\mathcal{I}^1}\]
is not effective. As \(H^0(X, f_r - J_1) = 0\), it suffices to prove
\[H^1(X, f_r - J_1 - J_0 - f_a) = H^1(X, f_r - f_a + K_X) = 0.\]
By Serre duality, this is equivalent to \(H^1(X, f_a - f_r) = 0\). This follows immediately by the Leray spectral sequence, noting that \(H^0(E, a - \tau) = 0\), as we have seen.

2. VANISHING OF THE QUADRATIC SYZYGY GROUP \(K_{n-3,2}(C, K_C \otimes \tau) = 0\)

It remains to establish the vanishing of the group \(K_{n-3,2}(C, K_C \otimes \tau) = 0\). To begin, we recall the following theorem:

**Theorem 2.1** ([FK1], Thm. 1.7). Let \(C\) be a smooth curve of genus \(g\) and maximal Clifford index. In the case \(g = 2n\) is even, assume further that the Brill–Noether locus \(W_{n+1}^1(C)\) is zero-dimensional. Let \(L \in \text{Pic}^d(C)\) be a line bundle of degree
\[d = 2g + 1 - p - \text{Cliff}(C).\]
Assume that \(h^1(C, L) = 0\) and further \(h^0(C, L - K_C) = 0\). Assume further that
\[L - K_C + C_{d-g-2p-3} \notin \text{Diff}_{2g-d+p}(C).\]
Then \(K_{p,2}(C, L) = 0\).

Note that, whilst the version of the above theorem in [FK1] makes the stronger assumption that \(C\) is Brill–Noether–Petri general, from the proof the only hypotheses needed are those in the statement above (also note the condition \(h^0(C, L - K_C) = 0\), erroneously omitted in [FK1]). In our case, a general divisor
\[C \in |gJ_0 + f_r|\]
for \(g = 2n + 1\) has maximal Clifford index, and applying the above Theorem to \(L|_C = K_C + \tau\) it suffices to show
\[\tau + C_2 \notin \text{Diff}_{n-1}(C),\]
where \(\text{Diff}(C) := C_{g-1-l} - C_l\) as in the previous section. We have two natural points on \(C\), namely those cut out by intersection with \(J_0\) and \(J_1\), so it suffices to show
\[(\phi^* \zeta + J_0 + J_1)|_C \notin \text{Diff}_{n-1}(C).\]

In order to prove this, we let \(C_l \in |J_0 + f_r|\) be general and perform induction on the genus \(l\), as we did to establish the vanishing of the linear syzygies.

For \(0 \leq i \leq g - 1\), let \(C_{g-i} \in |(g - i)J_0 + f_r|\) be the union of \(J_0\) with a general \(C_{g-i-1} \in |(g - i - 1)J_0 + f_r|\). Let \(\tilde{K} \in \text{Pic}(C_{g-i})\) be the unique line bundle with \(\tilde{K}|_{J_0} \cong \mathcal{O}_{J_0}, \tilde{K}|_{C_{g-i-1}} \cong K_{g-i-1} + 2J_0 - J_1\), where \(2J_0 - J_1 := C_{g-i-1} \cap J_0\).

**Lemma 2.2.** Let \(C_{g-i} \in |(g - i)J_0 + f_r|\) for \(0 \leq i \leq g - 1\) be as above and assume \(j\) is an integer satisfying \(0 \leq j \leq n - 1\) and \(0 \leq i - j \leq n + 1\). Then we have:

(i) \(h^0(C_{g-i}, -\phi^* \zeta - (2j + 1 - i)J_0 - J_1) = 0\)

(ii) \(h^0(C_{g-i}, mK - \phi^* \zeta - (2j + 1 - i)J_0 - J_1) = h^0(C_{g-i}, mK_{C_{g-i}} - \phi^* \zeta - (2j + 1 - i)J_0 - J_1)\)
for \(C_{g-i} \in |(g - i)J_0 + f_r|\) general and \(m \in \{1, 2\}\)

(iii) \(h^0(C_{g-i}, \tilde{K}) = h^0(C_{g-i}, K_{C_{g-i}})\) for \(C_{g-i} \in |(g - i)J_0 + f_r|\) general.

**Proof.**
(i) Set $\ell = -(2j + 1 - i)$. If $\ell \leq 0$, then the statement is clear for degree reasons, so assume $\ell \geq 1$. Then
\[ H^0(X, -\phi^* \zeta + \ell J_0 - J_1) = H^0(E, (\eta - \zeta) \otimes \text{Sym}^{\ell - 1}(O_E \oplus \eta)) = 0, \]
so it suffices to show $H^1(X, -\phi^* \zeta + \ell J_0 - J_1 - C'_{g-i}) = 0$. By Riemann–Roch, this is equivalent to\[ H^1(X, (g - \ell - i - 1)J_0 + f_r + \phi^* \zeta) = 0. \]
Using the given bounds, $g - \ell - i - 1 \geq -1$, so it suffices to show $H^1(X, mJ_0 + f_r + \phi^* \zeta) = 0$ for $m \geq -1$. For $m = 0$ this follows from the Leray spectral sequence for $\phi$. For $m = -1$, the claim follows by taking the long exact sequence of cohomology associated to \[ 0 \to -J_0 + f_r + \phi^* \zeta \to f_r + \phi^* \zeta \to (f_r + \phi^* \zeta)|_{J_0} \to 0. \]
For $m \geq 0$, Leray gives
\[ H^1(X, mJ_0 + f_r + \phi^* \zeta) = H^1(E, (r + \zeta) \otimes \text{Sym}^m(O_E \oplus \eta)) = 0, \]
as required.

(ii) By Riemann–Roch and semicontinuity, it suffices to show $h^1(C'_{g-i}, m\tilde{K} - \phi^* \zeta - (2j + 1 - i)J_0 - J_1) = 0$ for $m \in \{1, 2\}$. Consider the exact sequence
\[ 0 \to O_{C_{g-i-1}}(-\beta(g-i-1)) \to O_{C'_{g-i}} \to O_{J_0} \to 0. \]
Then\[ (m\tilde{K} - \phi^* \zeta - (2j + 1 - i)J_0 - J_1)|_{J_0} \cong -\zeta - (2j + 1 - i)\eta \neq 0 \in \text{Pic}^0(E). \]
So it suffices to show $H^1(C_{g-i-1}, K_{C_{g-i-1}} - \phi^* \zeta - (2j - i)J_0 - J_1) = 0$ and $H^1(C_{g-i-1}, 2K_{C_{g-i-1}} - \phi^* \zeta - (2j - i - 2)J_0 - J_1) = 0$. The second of these vanishing is automatic for degree reasons (using the bounds on $i$ and $j$), so we just need to establish the first one. By Serre duality, this is equivalent to $H^0(C_{g-i-1}, \phi^* (\zeta - \eta) + \ell J_0) = 0$, for $\ell = 2j - i + 1$. This is obvious if $\ell < 0$, so assume $\ell \geq 0$. Then $H^0(X, \phi^* (\zeta - \eta) + \ell J_0) = 0$ by the Leray spectral sequence, so it suffices to prove\[ H^1(X, \phi^* (\zeta - \eta) + \ell J_0 - C_{g-i-1}) = 0. \]
By Serre duality, this is equivalent to\[ H^1(X, (g - 2j - 4)J_0 + \phi^* (\zeta + 2\eta) + f_r) = 0. \]
Using the given bounds, $g - 2j - 4 \geq -1$, so this goes through exactly as in the proof of (i).

(iii) This is exactly as in Lemma 1.8 (i). 

The following proposition provides the induction step:

**Proposition 2.3.** Let $0 \leq i \leq g - 2$ and suppose\[ (\phi^* \zeta + (2j - i + 1)J_0 + J_1)|_{C_{g-i}} \in \text{Diff}_{n-1-j}(C_{g-i}), \]
for some $0 \leq j \leq i$ and $C_{g-i} \in |(g - i)J_0 + f_r|$ general. Then\[ (\phi^* \zeta + (2j' - i)J_0 + J_1)|_{C_{g-i-1}} \in \text{Diff}_{n-1-j'}(C_{g-i-1}), \]
for some $0 \leq j' \leq i + 1$ and $C_{g-i-1} \in |(g - i - 1)J_0 + f_r|$ general.

\[ \square \]
Proof. By assumption $\text{Diff}_{n-1,j}(C_{g-i}) \neq \emptyset$ which gives $n-1-j \geq 0$ and $g-i-1-(n-1-j) \geq 0$, i.e. $j \leq n-1$ and $i-j \leq n+1$. By [FMP] §3, we have

$$H^1(C_{g-i}, \bigwedge^{n-1-j} M_{K_{C_{g-i}}}^\vee (\phi^* \zeta + (2j-i+1)J_0 + J_1)) \neq 0,$$

and hence

$$H^0(C_{g-i}, \bigwedge^{n-1-j} M_{K_{C_{g-i}}} (K_{C_{g-i}} - \phi^* \zeta - (2j-i+1)J_0 - J_1)) \neq 0,$$

by Serre duality. By Lemma 2.2 and semicontinuity, $h^0(C_{g-i}, -\phi^* \zeta - (2j+1-i)J_0 - J_1) = 0$, so the above is equivalent to

$$K_{n-1-j,1}(C_{g-i}, -\phi^* \zeta - (2j-i+1)J_0 - J_1; K_{C_{g-i}}) \neq 0.$$

By Lemma 2.2 and semicontinuity for Koszul cohomology [FK3] Lemma 1.1, we have

$$K_{n-1-j,1}(C'_{g-i}, -\phi^* \zeta - (2j-i+1)J_0 - J_1; \tilde{K}) \neq 0.$$

Consider the short exact sequence

$$0 \to \mathcal{O}_{C_{g-i-1}}(-\beta(g-i-1)) \to \mathcal{O}_{C'_{g-i}} \to \mathcal{O}_{J_0} \to 0.$$

We have $h^0(J_0, mJ_0 - \phi^* \zeta) = h^0(E, m\eta - \zeta) = 0$ for any $m \in \mathbb{Z}$, since $\eta$ is not torsion (and $\zeta$ is). Thus, for any integer $\ell$, inclusion gives an isomorphism

$$H^0(C_{g-i-1}, \ell K_{C_{g-i-1}} - \phi^* \zeta + (2\ell - 2j + i - 2)J_0 - J_1)$$

$$= H^0(C_{g-i-1}, \mathcal{O}(\ell \tilde{K} - \phi^* \zeta - (2j-i+1)J_0 - J_1)(-\beta(g-i-1)))$$

$$\simeq H^0(C'_{g-i}, \mathcal{O}(\ell \tilde{K} - \phi^* \zeta - (2j-i+1)J_0 - J_1)).$$

We have seen that restriction induces an isomorphism $H^0(C'_{g-i}, \tilde{K}) \simeq H^0(C_{g-i-1}, K_{C_{g-i-1}} + 2J_0)$. Let $A$ denote the graded $\text{Sym}(H^0(K_{C_{g-i-1}} + 2J_0))$ module

$$\bigoplus_{q \in \mathbb{Z}} H^0(C_{g-i-1}, q(K_{C_{g-i-1}} + 2J_0) - \phi^* \zeta - (2j-i+2)J_0 - J_1),$$

and let $B$ denote the graded $\text{Sym}(H^0(\tilde{K}))$ module

$$\bigoplus_{q \in \mathbb{Z}} H^0(C'_{g-i}, q(\tilde{K}) - \phi^* \zeta - (2j-i+1)J_0 - J_1).$$

We have a commutative diagram

$$\bigwedge^{n-1-j} H^0(K_{C_{g-i-1}} + 2J_0) \otimes A_1 \to \bigwedge^{n-2-j} H^0(K_{C_{g-i-1}} + 2J_0) \otimes A_2$$

$$\bigwedge^{n-1-j} H^0(\tilde{K}) \otimes B_1 \to \bigwedge^{n-2-j} H^0(\tilde{K}) \otimes B_2,$$

where the vertical arrows are isomorphisms. Thus

$$K_{n-1-j,1}(C_{g-i-1}, -\phi^* \zeta + (i-2j-2)J_0 - J_1; K_{C_{g-i-1}} + 2J_0) \neq 0,$$

or, equivalently,

$$H^0(C_{g-i-1}, \bigwedge^{n-1-j} M_{K_{C_{g-i-1}}}^{\vee} + 2\beta(g-i-1) \otimes A_1) \neq 0.$$

By a Riemann–Roch computation, $\chi(\bigwedge^{n-1-j} M_{K_{C_{g-i-1}}}^{\vee} + 2\beta(g-i-1) \otimes A_1) = 0$, so

$$H^0(\bigwedge^{n-1-j} M_{K_{C_{g-i-1}}}^{\vee} + 2\beta(g-i-1)(K_{C_{g-i-1}} - A_1)) = (H^1(\bigwedge^{n-1-j} M_{K_{C_{g-i-1}}}^{\vee} + 2\beta(g-i-1) \otimes A_1))^\vee \neq 0.$$
By [B], Proposition 2, we have either
\[(\phi^*\zeta + (2j - i)J_0 + J_1)|_{C_{g-1}} \in \text{Diff}_{n-1-j}(C_{g-1})\]
or
\[(\phi^*\zeta + (2(j + 1) - i)J_0 + J_1)|_{C_{g-1}} \in \text{Diff}_{n-2-j}(C_{g-1}),\]
which establishes the claim. \(\Box\)

To complete the proof of the vanishing of the quadratic syzygy group \(K_{n-3,2}(C, K_C \otimes \tau)\), it suffices to prove the base case of the induction, i.e. it suffices to prove
\[(\phi^*\zeta + (2j - g + 2)J_0 + J_1)|_{C_1} \notin \text{Diff}_{n-1-j}(C_1),\]
for any \(0 \leq j \leq g - 1\) and \(C_1 \in |J_0 + f_r|\) general. So suppose
\[(\phi^*\zeta + (2j - g + 2)J_0 + J_1)|_{C_1} \in \text{Diff}_{n-1-j}(C_1)\]
for some \(0 \leq j \leq g - 1\) and \(C_1 \in |J_0 + f_r|\) general. As \(\text{Diff}_{n-1-j}(C_1) \neq \emptyset\), this forces \(j = n - 1\) and
\[\phi^*(\zeta - \eta)|_{C_1} \simeq (\phi^*\zeta - J_0 + J_1)|_{C_1} \simeq \mathcal{O}_{C_1}.\]
Since \(H^0(X, \phi^*(\zeta - \eta)) = 0\), this implies \(H^1(X, \phi^*(\zeta - \eta) - J_0 - f_r) \neq 0\). Observe \(h^1(X, \phi^* (\zeta - \eta) - J_0 - f_r) = h^0(X, -J_0 + \phi^* (2\eta - \zeta) + f_r) = 0\) by Serre duality. But taking the long exact sequence of cohomology associated to
\[0 \to \phi^*(\zeta - \eta) - J_0 - f_r \to \phi^*(\zeta - \eta) - f_r \to (\phi^*(\zeta - \eta) - f_r)|_{J_0} \to 0\]
and using the Leray spectral sequence, we immediately get \(H^1(X, \phi^*(\zeta - \eta) - J_0 - f_r) = 0\). This contradiction finishes the proof.

References


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