Roadmap to the proof of Schreyer's Conjecture

We will explain here the proof of the following:

Theorem (Farkas - Kemeny)

Let $C$ be a general curve of genus $g$ and gonality $k$ with $g \geq 2k-1$. Then $b_{g-k+1}^g(C, w) = g-k$.

This was originally conjectured by Schreyer. The idea is to capitalize on:

Theorem (Hirschowitz - Ramanan)

Assume equality $g = 2k-1$. If $C_{g, g}$ is general of gonality $k$, then $b_{k-1, 1}^{g-1}(C, w) = k-1$.

More precisely, if $C$ is any smooth point of the Hurwitz divisor
The basic strategy is to reduce to HR's result using a trick of Aprodu & Voisin. Suppose $C$ has genus $g$ and gonality $k$. Let $n = g+n-2k$ and pick $n$ general pairs $(x_i, y_i) \in C \times C$.

Consider:

$$
\begin{array}{ccc}
    u : & C & \rightarrow & D \\
    \begin{cases}
    x_i \\
    x_i \\
    \vdots \\
    x_i \\
    y_i \\
    y_i \\
    \vdots \\
    y_i \\
\end{cases}
    & \rightarrow & \\
    \begin{cases}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
\end{cases}
\end{array}
$$

Then $\text{gon}(D) = \text{gon}(C) + n$, $g(D) = g + n$ and now $g(D) = 2 \text{gon}(D) - 1$.

If $\text{dim} \mathcal{H}ur$ were a smooth point...
then \( HR \Rightarrow b_{g-k,1}(D, \omega_D) = g-k \).

\[ \Rightarrow b_{g-k,1}(C, \omega_C) = g-k \] by a result of Voisin.

**Major Problem** \( HR \) has been described by Harris–Mumford using the admissible cover space

\[
\overline{H}_{2(k+n)-1, k+n}^{\text{ad}} = \{ B \supset \mathbb{P}^1 \}
\]

\( \xrightarrow{\pi} \overline{M}_{2(k+n)-1} \)

\( \exists! \) admissible cover over \( \mathbb{D}_g \) corresponding to the stable map

\[
g \quad \xrightarrow{\text{degree 1}} \quad \mathbb{P}^1
\]

\[
\sum_{i=1}^{b_{g-k,1}} \mathbb{P}^1 \quad \xrightarrow{f_n} \quad \mathbb{D}_{g-k-1}
\]

\[
\sum_{i=1}^{b_{g-k,1}} \mathbb{P}^1 \quad \xrightarrow{\text{degree 1}} \quad \mathbb{P}^1
\]
\( \deg (f_{n_i} \mid \mathcal{P}_i) = 1, \quad 1 \leq i \leq n \), \( f_{n_i} \) is the unique minimal pencil \( f: \mathbb{C} \rightarrow \mathbb{P}^1 \).

**Problem** It is very hard to analyse the scheme structure of \( \pi^{-1}(\mathcal{E}D) \). In particular it seems very difficult to decide when \( f \) is smooth at \( (\mathcal{E}, D) \).

However, we may still expect
\[ b_{g-k, 1} (\mathcal{O}_D, \omega_0) = g-k, \]
as the following example shows.

**Example** (Case \( n=1 \))

Let \( g=2k \). Consider a K3 surface \( X \) with \( \text{Pic } X = \mathbb{Z} C \oplus 2\mathbb{Z} E \oplus 2\mathbb{Z} \) and \( C^2 = 2(2g-2), \ E^2 = 0, \ P^2 = -2 \)
\[ C(\mathcal{E}) = k, \ \mbox{C}(\mathcal{E}P) = 2, \ \mbox{C}(\mathcal{E}P) = 1. \]

This is a K3 with elliptic fibration \( X \xrightarrow{\varphi_E} \mathbb{P}^1 = \mathbb{E} \) with section \( P \).
The general element $D \subset |C + \mathbb{P}|$ is smooth of genus $g+1$ and gonality $k+1$. We have $b_{g-k-1,1}(D_+ \cup \mathcal{O}_D) = g - k$ using HR. Since Betti numbers are constant in the linear system $|C + \mathbb{P}|$ (Green) we get

$$b_{g-k-1,1}(D_0 \cup \mathcal{O}_D) = g - k$$

for $D_0 = \mathcal{O}_C$.

This deals with the case $n=1$.

Idea: use induction

- $D_{n-1}^{g-1}$ degenerate $\Rightarrow$ $D_{n-1}^{g}$

- $D_{n-1}^{g}$ genus $g$$\Rightarrow$ $D_{n-1}$ genus $g+1$ Have statement by induction hypothesis.
Hope to deduce claim on \( \text{Der} \) by an analogue of constancy of Betti \#s on K3s. We cannot use K3s for the case of arbitrarily many tails (e.g. \( \text{Pic}_X \) has bounded rank for \( X \) K3). As a replacement for constancy of Betti \#s on K3s, we do the following. On a suitable partial compactification

\[
\mathcal{H} \to 2(k+n) - 1, k+n
\]
of the space of covers \( B \to \mathbb{P}^1 \), \( g(B) = 2(k+n) - 1 \) we construct a divisor

\[
\mathcal{E}_N \subseteq \mathcal{H}
\]

as

\[
2(k+n) - 1, k+n
\]

"Eagon-Northcott divisor" parametrizing stable maps \( B \to \mathbb{P}^1 \) such that \( b_{k+n-1,1} (B, \omega_B) > k+n-1 \). (This follows ideas due to Schreyer).
We need to show

$$\exists \beta \in \mathbb{N}$$

We may now state our main theorem more precisely.

**Theorem (Farkas–Kemery)**

Let $C$ be an integral, nodal curve of genus $g = 2a - 1 - n$ with a unique pencil $f: C \to \mathbb{P}^1$ of degree $a - n$.

Let $(x_i, y_i) \in C \times C$, $1 \leq i \leq n$ be distinct points on $C$. Assume, for

$$\begin{cases} x_i & \to y_i \\ x_i & \to D \\ x_i & \to x_i, y_i \end{cases}$$

that there is a unique admissible cover of degree $a$ over $ED$. Assume further that $\forall S \subseteq \exists x_i, y_i \mid 1 \leq i \leq n$ of
Then for
\[
\begin{align*}
\text{cardinality} & \leq n \\
h^0(C, f^* \mathcal{O}(2) \otimes \mathcal{S}) & = 3 \\
\end{align*}
\]

This theorem implies Schreyer's conjecture for general curves.

Proof of Theorem

We argue by induction on \( n \). The base case \( n = 0 \) follows from Hirschowitz–Ramanan's result.

Suppose, for a contradiction

\[
\exists n \geq 1 \quad [fn] \notin \mathbb{E}N
\]
Since $EN$ is a divisor, a dimension count shows that we may degenerate $[f_n]$ to a map:

$$
\begin{array}{ccc}
\{ x' \} & \to & \mathbb{P}^n \\
g' : & & \\
\{ x' \} & \to & \mathbb{P}^n \\
\{ x_n' \} & \to & c' \\
\{ y_n' \} & \to & c' \\
\end{array}
$$

with $c'$ integral of genus $2a-n$.

By induction, it suffices to show:

(A) $f'$, admissible cover of degree $a$ over $D'$, where:

$$
\begin{array}{ccc}
\{ x' \} & \to & \mathbb{P}^n \\
g' : & & \\
\{ x_n' \} & \to & c' \\
\{ y_n' \} & \to & c' \\
\end{array}
$$

(B) $A \leq s \leq 3x' \cup y' \cup 3$ of cardinality $\leq n-1$,

$$
h^0(\mathcal{O}(\mathbb{P}^n) \otimes (2s) \leq s) = 3
$$

where $f' = g' | c'$, $s \leq 5$. 
In order to verify (A), (B) we will need to abstractify one other important feature from the K3 example above. The K3 surface has a bundle \( O_x(C) \) such that \( (O_x(C))|_{D+E} = 0 \) for general \( D+E \in C+P1 \), whereas \( (O_x(C))|_{x} \) is nontrivial and has degree \(-2\) on \( P1 \). Do this non-trivial limit of trivial bundles is very useful.

In the general case we have:

**Proposition ("Existence of Twistings")**

Suppose \( X \xrightarrow{g} \mathbb{P}^1 \) is a family of stable maps with \( X_0 = \bigvee \mathbb{P}^1 \), \( \deg G_0 = 1 \), and \( X_+ \) irreducible for \( t \in \Delta \) general.
Then, after a finite base change $\Delta \to \Delta$, there is a birational morphism $v : \tilde{X} \to X$ and $T \in \text{Pic} \tilde{X}$ such that
\[ v_* \mathcal{O}_T = \mathcal{O}_{\tilde{X}} \] for $t \in \Delta$ general and either:

**CASE 1** \( \tilde{X} = X \) and $T_1 x_0$ has degree $-2$ on $R$, whereas $T_1 B = \mathcal{O}_B(u + v)$.

**CASE 2** $X_0$ is a blow-up
\[
\begin{array}{c}
\text{p} \\
\text{R} \\
\text{B}
\end{array}
\]
of $X_0$ at $p \in \Sigma u, \Sigma j$, and we have
\[ \deg T_1 r = \deg T_1 \epsilon = -1, \quad T_1 B = \mathcal{O}_B(u + v). \]

Furthermore, if we assume additionally $h^0(G^* \mathcal{O}_P(1)) = 2$, $\omega_B \otimes G^* \mathcal{O}_P(-1)$ is base-point free and:

The locus $t \in \Delta$ with $X_t$ reducible has codim $\geq 2$.
Then we may arrange things so as to be in \textit{CASE 2}.

We may now complete the proof of the Theorem. We need to first verify statement (\textit{2}): there exists a unique cover \(\overline{\mathcal{D}}\).

Suppose we have a second admissible cover \(\overline{\mathcal{D}}\). For simplicity, assume it corresponds to a stable map

\[
\begin{array}{ccc}
\mathbb{P}^1 & \to & \mathbb{P}^1 \\
\mathcal{D} & \to & \mathbb{P}^1 \\
\mathcal{R}_{n-1} & \to & \mathbb{P}^1 \\
\mathcal{C}' & \to & \mathbb{P}^1
\end{array}
\]

of the same "shape" as \(g'\) (there are other possibilities, e.g.\)

\[
\begin{array}{ccc}
\mathbb{P}^1 & \to & \mathbb{P}^1 \\
\mathcal{D} & \to & \mathbb{P}^1 \\
\mathcal{P} & \to & \mathbb{P}^1 \\
\mathcal{C}' & \to & \mathbb{P}^1
\end{array}
\]

which also must be dealt with). Then we have a second stable map

\(f'' := g'_{|\mathcal{C}'}\) of degree \(a+1-n\) on \(\mathcal{C}'\).
By the base-point free pencil tric
\[ h^0(C', f'^*O_{\mathbb{P}^n}(1)) = f^*O_{\mathbb{P}^n}(1) > 3 \]
Both \( f' \) and \( f'' \) must specialize to the same map
\[
\begin{array}{ccc}
\mathbb{P}^n & \overset{f}{\longrightarrow} & \mathbb{P}^1 \\
c & \downarrow & \\
\end{array}
\]
By "Existence of Twistings" we have
\[ X \to \Delta \] with general fibre \( C' \), special fibre \[ E \], and \( Ye \in \text{Pic}X \) such that the degrees of \( Y_{\frac{X}{X_0}} \) are as indicated:
\[
\begin{array}{ccc}
2 & -1 & \in \\
\end{array}
\]
(\text{Modulo verifying the dimension hypothesis of the Twisting Thm. This is itself done by induction})
we have a stable map \( \tilde{f} \) on \( \tilde{X}_0 \), obtaining by "blowing-up" \( f \) (so \( \deg \tilde{f} = 0 \)).
By semi-continuity, we reach a contradiction.
if the following line bundle on $\Sigma_0$ has at most 3 sections (labelling each component with the restriction of the bundle)

$$
\begin{array}{c}
\text{The bundle}
\end{array}
$$

$$f^*\mathcal{O}(2) \otimes \mathcal{L}_0$$

By the Mayer-Vietoris sequence, it suffices to show $h^0(f^*\mathcal{O}_0(2)(v)) = 3$, which holds by hypothesis.

The arguments for the other "shapes" are similar (and use the hypothesis for $\#S > 1$).

The argument for (B) is very similar (and easier).