LINEAR SYZYGIES OF CURVES WITH PRESCRIBED GONALITY

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Abstract. We prove two statements concerning the linear strand of the minimal free resolution of a $k$-gonal curve $C$ of genus $g$. Firstly, we show that a general curve $C$ of genus $g$ of non-maximal gonality $k \leq \frac{2g+1}{3}$ satisfies Schreyer’s Conjecture, that is, $b_{g-k,1}(C, \omega_C) = g - k$. This statement goes beyond Green’s Conjecture and predicts that all highest order linear syzygies in the canonical embedding of $C$ are determined by the syzygies of the $(k-1)$-dimensional scroll containing $C$. Secondly, we prove an optimal effective version of the Gonality Conjecture for general $k$-gonal curves, which makes more precise the (asymptotic) Gonality Conjecture proved by Ein-Lazarsfeld and improves results of Rathmann.

0. Introduction

1. The effective gonality conjecture. Let $C$ be a smooth complex algebraic curve and $L$ a very ample line bundle on $C$ inducing an embedding $\varphi_L : C \hookrightarrow \mathbb{P} H^0(C, L)$. In order to describe the equations of this embedding, after setting $r := r(L)$, we consider the finitely generated graded $S := \text{Sym} H^0(C, L) \cong \mathbb{C}[x_0, \ldots, x_r]$-module $\Gamma_C(L) := \bigoplus_n H^0(C, L^\otimes n)$. By the Hilbert Syzygy Theorem, one has a minimal free resolution

$$0 \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_0 \rightarrow \Gamma_C(L) \rightarrow 0,$$

where

$$F_p = \bigoplus_{q>0} K_{p,q}(C, L) \otimes S(-p-q),$$

with $K_{p,q}(C, L)$ being the Koszul cohomology group of $p$-th order syzygies of weight $q$. As usual, the graded Betti numbers of $(C, L)$ are defined by $b_{p,q} := \dim K_{p,q}(C, L)$. If $L$ is non-special, then $K_{p,q}(C, L) = 0$ for all $q \geq 3$. Accordingly, the graded Betti diagram of $(C, L)$ consists only of two non-trivial rows: the linear strand ($q = 1$) and the quadratic strand ($q = 2$).

The quadratic strand of the resolution is the subject of the Green-Lazarsfeld Secant Conjecture \cite{GL1} and has been studied extensively in \cite{FK}, \cite{K2}. The linear row is the subject of the Gonality Conjecture formulated in the same paper \cite{GL1}.

Assume $C$ is $k$-gonal and let $L$ be a line bundle on $C$ of degree $\deg(L) \geq 2g - 1 + k$. By the Green-Lazarsfeld Nonvanishing Theorem \cite[Appendix]{G}, one has $K_{h^0(L)-k-1,1}(C, L) \neq 0$. In a major breakthrough, generalizing results in \cite{AV} in the case of general $k$-gonal curves, Ein and Lazarsfeld \cite{EL} proved that for an arbitrary smooth curve $C$ of gonality $k$, if $\deg(L) \gg 0$, then

$$K_{h^0(L)-k,1}(C, L) = 0.$$  

This result has been significantly improved by Rathmann \cite{R}, who showed that the vanishing (1) holds for every smooth curve $C$ of genus $g$, when $\deg(L) \geq 4g - 3$. As already indicated in the original paper \cite{GL1} Conjecture 3.7, one can ask for an effective version of the Gonality Conjecture. We show the following:

**Theorem 0.1.** Let $C$ be a general $k$-gonal curve of genus $g \geq 4$. Then for each line bundle $L$ on $C$ of degree $\deg(L) \geq 2g - 1 + k$, one has

$$K_{h^0(L)-k,1}(C, L) = 0.$$
While the original Gonality Conjecture has been formulated as an asymptotic statement in \( \text{deg}(L) \), the bound appearing in Theorem 0.1 is already raised as a possibility in [GL1, page 86]. Clearly Theorem 0.1 implies \( K_{p,1}(C, L) = 0 \), for all \( p \geq h^0(C, L) - k \). The bound on \( \text{deg}(L) \) appearing in Theorem 0.1 is optimal. Indeed, if \( A \in W_k^1(C) \) is a pencil of minimal degree, then

\[
K_{g-1,1}(C, \omega_C \otimes A) \neq 0,
\]

by the Green–Lazarsfeld Nonvanishing Theorem, that is, on every curve there exist line bundles of degree \( 2g - 2 + k \) which do not verify (1).

In the interest of convenience, we say that a smooth curve \( C \) of genus \( g \) and gonality \( k \) satisfies the Effective Gonality Conjecture if for each line bundle \( L \in \text{Pic}^d(C) \), where \( d \geq 2g - 1 + k \), one has \( K_{h^0(L) - k,1}(C, L) = 0 \). Equivalently, if there exists a line bundle \( L \in \text{Pic}^{2g - 1 + k}(C) \) such that \( K_{g,1}(C, L) \neq 0 \), then \( \text{gon}(C) \leq k - 1 \). Theorem 0.1 can be reformulated as stating that a general \( k \)-gonal curve of genus \( g \geq 4 \) verifies the Effective Gonality Conjecture.

By Green’s \( K_{p,1} \)-theorem, see [G, Theorem 3.c.1], an arbitrary \( 3 \)-gonal curve of genus \( g \geq 4 \) satisfies the Effective Gonality Conjecture. The same conclusion holds for each \( 4 \)-gonal curve of genus \( g \geq 7 \), see [Te, Proposition 3.8] or [AS]. Note that Theorem 0.1 fails for \( g = 3 \). In this case, the general curve is trigonal and it is easy to see that \( K_{3,1}(C, \omega_C \otimes 2) \neq 0 \), using the fact that the canonical linear system embeds \( C \) in the plane.

For curves of maximal gonality of odd genus \( g \geq 5 \), our results are complete:

**Theorem 0.2.** Every smooth curve of odd genus \( g \geq 5 \) and maximal gonality satisfies the Effective Gonality Conjecture.

Theorem 0.2, which plays an essential role in the proof of Theorem 0.1 turns out to be intimately related to the divisorial case of the Green-Lazarsfeld Secant Conjecture proved in full generality [FK, Theorem 1.4]. We observe that using [FK], if \( C \) is a smooth curve of genus \( g = 2n + 1 \) and gonality \( n + 2 \), the following equivalence holds for a line bundle \( M \in \text{Pic}^{2g}(C) \):

\[
\text{(2)} \quad K_{n,1}(C, M) \neq 0 \iff M - K_C \in C_{n+1} - C_{n-1}.
\]

The right hand side denotes the divisorial difference variety \( C_{n+1} - C_{n-1} \subseteq \text{Pic}^2(C) \). An argument involving the geometry of secant varieties for line bundles on \( C \) then shows that (2) implies the vanishing \( K_{g,1}(C, L) = 0 \), for every line bundle \( L \in \text{Pic}^{2n+3}(C) \), thus establishing Theorem 0.2. In order to deduce Theorem 0.1, we fix a value for the gonality \( k \leq \frac{2g+3}{2} \) and perform induction on the genus \( g \); the initial step is Theorem 0.2. By induction, assume that the general smooth curve \( C \) of genus \( g \) and gonality \( k \) satisfies the Effective Gonality Conjecture. The stable curve \( X \) of genus \( g+1 \) obtained by adding an elliptic curve \( E \) at a point of ramification of a degree \( k \) pencil on \( C \) lies in the limit in \( \overline{M}_{g+1} \) of the locus of smooth \( k \)-gonal curves of genus \( g + 1 \). An analysis of syzygies of line bundles of bidegree \((2g + k, 1)\) on \( X \) allows us to deduce the Effective Gonality Conjecture for a smooth deformation of \( X \) having gonality \( k \).

**2. Schreyer’s Conjecture.** Consider a general \( k \)-gonal curve canonically embedded curve \( C \hookrightarrow \mathbb{P}^{g-1} \) of gonality \( k \). Green’s Conjecture, known in this case, see [V1], [V2], [Ap2], and asserting that

\[
K_{p,1}(C, \omega_C) = 0 \quad \text{if and only if} \quad p \geq g - k + 1,
\]

determines the length of the linear (as well as that of the quadratic) strand of the resolution of \( C \). Schreyer’s Conjecture [Sch3, §6] and [SSW] addresses the more refined question of what actually is the Betti diagram of \( C \), that is, determine the values \( b_{p,1}(C, \omega_C) \) for \( k - 2 \leq p \leq g - k \). Note that in the case when \( C \) has the same gonality as a general curve of genus \( g \), that is, \( \text{gon}(C) = \lceil \frac{2g+3}{2} \rceil \), and only in this case, Green’s Conjecture determines the entire resolution of \( C \). Indeed, in this case Green’s Conjecture is equivalent to the statement that the resolution of
$C \subseteq \mathbb{P}^{g-1}$ is natural, or equivalently
\[ b_{p,2}(C, \omega_C) \cdot b_{p+1,1}(C, \omega_C) = 0 \]
for all $p$. Since the differences $b_{p+1,1}(C, \omega_C) - b_{p,2}(C, \omega_C)$ are known and independent of $C$, knowing which Betti numbers vanish amounts to knowing the entire Betti diagram.

Assume now $\text{gon}(C) \leq \frac{g+1}{2}$, that is, $C$ has non-maximal gonality. In this case, Green’s Conjecture predicts the following resolution, where we observe that $b_{p,1}(C, \omega_C) \cdot b_{p,2}(C, \omega_C) \neq 0$ for $k - 2 \leq p \leq g - k$.

<table>
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<tr>
<th>1</th>
<th>2</th>
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<th>k - 3</th>
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<th>g - 2</th>
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<tr>
<td>$b_{1,1}$</td>
<td>$b_{2,1}$</td>
<td>\ldots</td>
<td>$b_{k-3,1}$</td>
<td>$b_{k-2,1}$</td>
<td>\ldots</td>
<td>$b_{g-k,1}$</td>
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| 0  | 0  | \ldots | 0   | $b_{k-2,2}$ | \ldots | $b_{g-k,2}$ | $b_{g-k+1,2}$ | \ldots | $b_{g-2,2}$ |

Table 1. The Betti table of a general canonical $k$-gonal curve of genus $g$.

It is known [AC] that such a curve $C$ carries a unique pencil $A \in W^1_k(C)$ of minimal degree, inducing a $(k - 1)$-dimensional scroll $X \subseteq \mathbb{P}^{g-1}$ swept out by the fibres of $|A|$. The Betti numbers of $(X, \mathcal{O}_X(1))$ are determined by the Eagon-Northcott complex, see [Sch1]. Since $C \subseteq X \subseteq \mathbb{P}^{g-1}$, one has the following inequality (see also Section 4)

\[ b_{p,1}(C, \omega_C) \geq b_{p,1}(X, \mathcal{O}_X(1)) = p \cdot \binom{g - k + 1}{p + 1}. \]

It was originally expected that the inequality (3) is always an equality for $p \geq \lceil \frac{g-1}{2} \rceil$. This, however, is now known to fail. Indeed, Bopp [B] showed that the for a general 5-gonal curve of sufficiently high genus, if $m := \lceil \frac{g-1}{2} \rceil$, then $b_{m,1}(C, \omega_C) > b_{m,1}(X, \mathcal{O}_X(1))$. Schreyer’s Conjecture [SSW] concerns the value of the highest non-zero Betti number in the linear strand and predicts that in this case, under suitable generality assumptions, inequality (3) is an equality.

**Conjecture 0.3** (Schreyer’s Conjecture). Let $C$ be a curve of genus $g$ and non-maximal gonality $3 \leq k \leq \frac{g+1}{2}$. Assume $W^1_k(C) = \{A\}$ is a reduced single point and $A$ is the unique line bundle of degree at most $g - 1$ achieving the Clifford index. Then

\[ b_{g-k,1}(C, \omega_C) = g - k \text{ and } b_{p,1}(C, \omega_C) = 0, \text{ for } p > g - k. \]

The converse statement is straightforward. Indeed, if $W^1_k(C)$ does not consist of a reduced single point, then $b_{g-k,1}(C, \omega_C) > g - k$, see [SSW, Proposition 4.10]. As already pointed out, Green’s conjecture is known for general curves in each gonality stratum. Thus Schreyer’s Conjecture 0.3 in the case of generic $k$-gonal curves, purely concerns the condition

\[ b_{g-k,1}(C, \omega_C) = g - k. \]

In fact, Schreyer further conjectures that unless $C$ is isomorphic to a smooth plane quintic, the condition $b_{g-k,1}(C, \omega_C) = g - k$ automatically implies the vanishing statements $b_{p,1}(C, \omega_C) = 0$, for $p > g - k$, see [SSW, Conjecture 4.3]. Conjecture 0.3 is known to hold for a general $k$-gonal curve provided $(k - 1)^2 < g$, see [Sch2]. An important piece of evidence for the conjecture is the case of general $k$-gonal curves of odd genus $2k - 1$. Such curves form a divisor $\mathfrak{Hur}$ in the moduli space $\mathcal{M}_{2k-1}$, much studied by Harris and Mumford in [HM]. Combining results in [HR] and those in [V2], it follows that Conjecture 0.3 holds in this case. Outside this divisorial range, little has been known. The main result of this paper is the following, showing that Schreyer’s conjecture holds for any curve satisfying an explicit growth condition on Brill–Noether loci.
**Theorem 0.4.** Assume $C$ is a $k$-gonal curve $C$ of genus $g \geq 2k - 1$ satisfying bpf-linear growth:

$$\dim G^1_{k+m}(C) \leq m, \text{ for } 0 \leq m \leq g - 2k + 1$$

and, further, $\dim G^{1,\text{bpf}}_{k+m}(C) < m$, for $0 < m < g - 2k + 1$.

Assume that there is a unique pencil in $A \in G^1_k(C)$, with simple ramification and with $h^0(A^{\otimes 2}) = 3$. Then Schreyer’s Conjecture holds for $C$, i.e. $b_{g-k,1}(C,\omega_C) = g - k$.

In particular, Schreyer’s Conjecture holds for a general $k$-gonal curve of genus $g \geq 2k - 1$.

In the theorem above, $G^{1,\text{bpf}}_d(C) \subseteq G^1_d(C)$ denotes the subvariety of base point free pencils of degree $d$ on $C$.

Part of Theorem 0.4 is that there is a canonical identification

$$K_{g-k,1}(C, \omega_C) \cong \bigwedge^{g-k+1} H^0(C, K_C \otimes A^\vee) \otimes \text{Sym}^{g-k-1} H^0(C, A) \otimes \bigwedge^2 H^0(C, A),$$

where $A$ is the unique degree $k$ pencil on $C$. All the $(g-k)$-th syzygies linear syzygies of the canonical curve $C \subseteq \mathbb{P}^{g-1}$ are of Eagon-Northcott type and can be written down explicitly. Precisely, if $(\tau_0, \ldots, \tau_{g-k})$ is a basis of $H^0(C, \omega_C \otimes A^\vee)$ and $\sigma \in H^0(C, A)$, then the syzygy corresponding to the power $\sigma^{g-k-1} \in \text{Sym}^{g-k-1} H^0(C, A)$ has the form

$$\sum_{j=0}^{g-k} (-1)^j (\sigma \tau_1) \wedge \cdots \wedge (\sigma \tau_j) \wedge \cdots (\sigma \tau_{g-k}) \wedge \{(\sigma \tau_0) \otimes (\sigma' \tau_j) - (\sigma' \tau_0) \otimes (\sigma \tau_j)\} \in \bigwedge^j H^0(\omega_C) \otimes H^0(\omega_C),$$

where $\sigma' \in H^0(C, A)$ is another section such that $(\sigma, \sigma')$ form a basis of $H^0(C, A)$.

It is tempting to interpolate between and link the two main results of this paper, namely Theorems 0.1 and 0.4, and conjecture that a statement analogous to Schreyer’s Conjecture holds not only for the canonical bundle, but for every sufficiently positive line bundle on $C$. We fix a general $k$-gonal curve $C$ of genus $g \geq 2k - 1$ and a line bundle $L$ on $C$ with $\deg(L) \geq 2g + k$.

**Conjecture 0.5.** If $r = r(L)$, one has $\dim K_{r-k,1}(C, L) = r - k$.

We expect that all syzygies in $K_{r-k,1}(C, L)$ are again of Eagon-Northcott type, being induced by the $k$-dimensional scroll induced by the unique pencil $A \in W^1_k(C)$ and which contains the embedded curve $\varphi_L : C \hookrightarrow \mathbb{P}^r$.

The proof of Theorem 0.4 begins in Section 3 with the already mentioned observation that via [HR] and [V2], a smooth curve $C$ of genus $2k - 1$ and gonality $k$ satisfies $b_{k-1,1,1}(C, \omega_C) = k - 1$, provided $W^1_k(C)$ is integral of dimension zero. Consider the Hurwitz space $\mathcal{H}_{2k-1,1}$ of smooth curves of genus $g$ which are $k$-fold covers of $\mathbb{P}^1$. We define the Eagon-Northcott divisor $\mathcal{E}N$ on $\mathcal{H}_{2k-1,1}$ parametrizing moduli points $[f : C \to \mathbb{P}^1] \in \mathcal{H}_{2k-1,1}$ with $b_{k-1,1,1}(C, \omega_C) > k - 1$. In other words, points of $\mathcal{E}N$ correspond to canonical curves $C \subseteq \mathbb{P}^{g-1}$ having a $(g - k)$-th order linear syzygy which is *not* of Eagon-Northcott type. We also consider the Brill-Noether type divisor $\mathfrak{B}N$ on $\mathcal{H}_{2k-1,1}$ consisting of points $[f : C \to \mathbb{P}^1]$, such that $C$ has an extra pencil of degree $k$. By the above discussion these two divisors coincide set-theoretically, that is,

$$\mathcal{E}N = \mathfrak{B}N.$$

Now suppose we are no longer in the divisorial case and choose $k \leq \frac{g+1}{2}$. We follow a strategy reminiscent of [Ap2]. Starting with a general $k$-gonal curve $C$ of genus $g$, we form the irreducible nodal curve $[D] \in \overline{\mathcal{M}}_{2g-2k+1}$ obtained by identifying $g - 2k + 1$ general pairs of points on $C$. Clearly $p_a(D) = 2g - 2k + 1$ and $\text{gon}(D) \leq g - k + 1$, that is, $[D]$ belongs to the closure $\overline{\mathcal{M}}_{gur} = \overline{\mathcal{M}}_{2g-2k+1, g-k+1}$ of the Hurwitz divisor, already considered in [HM], [HR] and [FK]. Let

$$\pi : \mathcal{H}_{2g-2k+1, g-k+1} \to \overline{\mathcal{M}}_{2g-2k+1}$$
denote the forgetful map from the space of admissible covers of degree \( g-k+1 \) compactifying the Hurwitz space \( \mathcal{H}_{2g-2k+1, g-k+1} \). Assuming the curve \( C \) we started with is sufficiently general, one checks directly that set-theoretically \( W_{g-k+1}^1(D) \) consists of one point (that is, \( \pi^{-1}([D]) \) consists of one admissible cover \([f]\)). This point corresponds to the torsion free sheaf on \( D \) given by pushing forward the unique degree \( k \) pencil on \( C \). By an argument inspired by limit linear series, we show that \([f] \notin \mathfrak{B}_{\text{EN}}\) therefore \([f] \notin \mathfrak{E}_{\text{EN}}\). To conclude \( b_{g-k+1}(C, \omega_C) = g-k \), we extend in Section 4 the determinantal structure of the Eagon-Northcott divisor \( \mathcal{E}_{\text{EN}} \) over a partial compactification of \( \mathcal{H}_{2g-2k+1, g-k+1} \) containing the moduli point of \([f]\). In the short Section 5, we then use \( K3 \) surfaces to show that this extended Eagon-Northcott divisor does not contain the unique boundary component of \( \mathcal{H}_{2g-2k+1, g-k+1} \) containing \([f]\). Since the injection \( K_{g-k, 1}(C, \omega_C) \hookrightarrow K_{g-k, 1}(D, \omega_D) \) always holds, this completes the proof of Theorem 0.4.\(^1\)

The organisation of the paper is as follows: We first review some background on syzygies of curves in Section 1. In Section 2, we prove Theorem 0.1. We prove Theorem 0.4 in Sections 3, 4 and 5.

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1. Background on Syzygies

We recall a few definitions and collect some basic results on syzygies that will be used throughout the paper. Let \( X \) be a (possibly singular) projective variety and let \( L, M \in \text{Pic}(X) \) be line bundles. Consider the graded \( S := \text{Sym} \text{ } H^0(X, L) \)-module

\[
\Gamma_X(M, L) := \bigoplus_{n \geq 0} H^0(X, L \otimes^n \otimes M).
\]

One defines the Koszul cohomology groups \( K_{p,q}(X, M; L) \) of \( p \)-th syzygies of weight \( q \) by resolving the module \( \Gamma_X(M, L) \) and computes them via the Koszul complex, see [G]. When \( M = \mathcal{O}_X \), we write \( K_{p,q}(X, L) := K_{p,q}(X, \mathcal{O}_X; L) \). The following fact is surely well-known:

**Lemma 1.1** (Semicontinuity). Let \( \pi : \mathcal{X} \rightarrow S \) be a flat, projective morphism of schemes over an integral base. Let \( \mathcal{L} \in \text{Pic}(\mathcal{X}) \) be a line bundle such that \( h^0(X_s, \mathcal{L}_s) = e \), for each \( s \in S \). Let \( \mathcal{M} \in \text{Pic}(\mathcal{X}) \) be a second line bundle, and assume

\[
h^0(X_s, \mathcal{L}_s \otimes^{(q-1)} \otimes \mathcal{M}_s) = r_1, \quad h^0(X_s, \mathcal{L}_s \otimes^q \otimes \mathcal{M}_s) = r_2, \quad h^0(X_s, \mathcal{L}_s \otimes^{(q+1)} \otimes \mathcal{M}_s) = r_3
\]

are also independent of \( s \in S \). Then the function

\[
\psi : s \mapsto \dim K_{p,q}(X_s, \mathcal{M}_s; \mathcal{L}_s)
\]

is upper semicontinuous on \( S \).

We collect some results on syzygies of curves which, taken together, reduce Theorem 0.1 to the extremal case of line bundles of degree \( d = 2g - 1 + \text{gon}(C) \). We quote from [AN], Theorem 4.27:

\(^1\)It might be tempting to carry out this argument at the level of \( \overline{\mathcal{M}}_{2g-2k+1} \) rather than pass to the Hurwitz space. However, the scheme structure of \( W_{g-k+1}^1(D) \) is difficult to analyse, in particular \([D]\) is a singular point of \( \text{Im}(\pi) = \overline{\mathcal{M}} \). Thus a degenerate version of results in [HR], does not actually lead to a proof of Conjecture 0.3.
Lemma 1.2. Let $C$ be a smooth curve of genus $g$ and $L$ a line bundle of degree $d \geq g$ with $h^1(C, L) = 0$. Assume $K_{p,1}(C, L) = 0$. Then $K_{p+1,1}(C, L(x)) = 0$, for any point $x \in C$ such that $L(x)$ is base point free.

It is standard, see e.g. [AN], Corollary 2.13, that if $L \not\cong O_C$ is a globally generated line bundle on a smooth curve $C$, if $K_{p,1}(C, L) = 0$, then $K_{p+1,1}(C, L) = 0$. Accordingly, there are several natural invariants which one can read directly off the Betti table of an embedded curve $C \hookrightarrow \mathbb{P}^r(L)$ and which measure the length of the linear and the quadratic strand respectively:

$$\ell_1(C, L) := \max \{ p \in \mathbb{N}_{>0} : b_{p,1}(C, L) \neq 0 \} \quad \text{and} \quad \ell_2(C, L) := \min \{ p \in \mathbb{N}_{>0} : b_{p,2}(C, L) \neq 0 \}.$$

Recalling that $K_{p,q}(C, L) = 0$ for $p \geq r(L)$, the invariants $\ell_1(C, L)$ are encoded in the more classical properties $(N_p)$ and $(M_q)$ defined in [GL1]. Precisely, $\ell_2(C, L)$ is the smallest integer such that $(C, L)$ fails property $(N_{\ell_2(C, L)})$, whereas $\ell_1(C, L)$ is the smallest integer such that $L$ fails property $(M_{r(L)} - \ell_1(C, L)).$

2. The Effective Gonality Conjecture for generic curves

We start by proving Theorem 0.2. It turns out that our proof of the generic Green-Lazarsfeld Secant Conjecture [FK] takes us a long distance towards finding a complete solution.

Proof of Theorem 0.2. Let $C$ be a curve of genus $2n+1$ and gonality $n+2$. Then using e.g. [HR, Remark 6.3], we observe that $\text{Cliff}(C) = n$, that is, $C$ has maximal Clifford index as well. We need to prove that for any line bundle $L \in \text{Pic}(C)$ of degree at least $5n+3$, we have $K_{1,1}(C, L) = 0$ for $i \geq h^0(C, L) - n - 2$. We may assume $n \geq 2$ and as explained in the previous section, it is enough to prove that for any line bundle $L \in \text{Pic}^{5n+3}(C)$, we have $K_{2n+1,1}(C, L) = 0$.

Theorem 1.4 of [FK] establishes the following equivalence for any line bundle $M \in \text{Pic}^{4n+2}(C)$:

$$K_{n-1,2}(C, M) \neq 0 \iff M - K_C \notin C_{n+1} - C_{n-1}.$$

For any line bundle $M \in \text{Pic}^{4n+2}(C)$ one has cf. [FK, formula (8)]

$$\dim K_{n,1}(C, M) = \dim K_{n-1,2}(C, M).$$

Thus, for any $M \in \text{Pic}^{4n+2}(C)$, the equivalence

$$K_{n,1}(C, M) \neq 0 \iff M - K_C \notin C_{n+1} - C_{n-1}$$

holds. Using Lemma 1.2 again, it thus suffices to show that for any line bundle $L$ of degree $5n+3$, there exists an effective divisor $D \in C_{n+1}$ such that

$$L - D - K_C \notin C_{n+1} - C_{n-1}.$$

Suppose this were not the case, that is,

$$L - K_C - C_{n+1} \subseteq C_{n+1} - C_{n-1}.$$

Then for every $D \in C_{n+1}$ there exists a divisor $E \in C_{n+1}$ such that $H^1(C, L(-D - E)) \neq 0$, that is, $D + E$ is an element of the (determinantal) secant variety $V_{2n+1}^{2n+1}(L)$ of effective divisors failing to impose independent conditions on $|L|$. In particular,

$$\dim V_{2n+2}^{2n+1}(L) \geq n + 1,$$

which is one higher than the expected dimension $n$. We observe that the morphism

$$\psi : V_{2n+2}^{2n+1}(L) \to C_{n-1},$$

$$A \mapsto K_C - L + A$$

is a bijection.
is well-defined, since \( h^0(C, K_C - L + A) = 1 \), for \( \text{gon}(C) > n - 1 \). Let \( I \) be any component of \( V^{2n+1}_{2n+2}(L) \) of dimension \( n + 1 \) and set \( r := n - 1 - \dim \psi(I) \). Then \( \psi|_I \) must have fibres of dimension at least \( 2 + r \). As all divisors in the inverse image \( \psi^{-1}(B) \) are clearly linearly equivalent, we have \( h^0(C, A) \geq 3 + r \) for all \( A \in V^{2n+1}_{2n+2}(L) \) such that \( \psi(A) = B \in \psi(I) \). By Riemann–Roch, this implies \( h^1(C, A) \geq 1 + r \), or \( h^0(C, K_C - A) = h^0(2K_C - L - B) \geq 1 + r \). The latter inequality holds for any effective divisor \( B \in \psi(I) \), so we must have
\[
\dim [2K_C - L] \geq r + \dim \psi(I) = n - 1.
\]
This implies \( h^1(C, 2K_C - L) \geq 3 \), or equivalently \( L - K_C \in W^2_{n+3}(C) \). But then \( \text{Cliff}(C) \leq n - 1 \) (if \( n = 2 \), then compute the Clifford index of \( 2K_C - L \) rather than \( L - K_C \)). Since we have \( \text{Cliff}(C) = n \), this is a contradiction. \( \square \)

The proof of Theorem 0.2 gives a characterisation of those line bundles \( L \in \text{Pic}^{2g-2+\text{gon}(C)}(C) \), such that \( K_{h^0(L)-\text{gon}(C),1}(C, L) \neq 0 \), in the case where \( C \) has odd genus and maximal gonality.

**Proposition 2.1.** Let \( C \) be a smooth curve of odd genus \( 2n + 1 \) and gonality \( n + 2 \). Let \( L \in \text{Pic}^{2n+2}(C) \) be such that \( K_{2n,1}(C, L) \neq 0 \). Then \( L - K_C \in W^1_{n+2}(C) \).

**Proof.** Following the proof of Theorem 0.2, we obtain \( \dim V^{2n}_{2n+1}(L) \geq n \). By studying the morphism
\[
\psi : V^{2n}_{2n+1}(L) \to C_{n-1}, \quad A \mapsto K_C - L + A.
\]
and arguing as in Theorem 0.2, we are again led to the statement \( h^0(C, 2K_C - L) \geq n \). The Riemann–Roch theorem gives \( h^0(C, L - K_C) \geq 2 \), as required. \( \square \)

We shall prove Theorem 0.1 by induction on the genus, fixing the gonality. To perform the induction step, let \( C \) be a smooth genus \( g \) curve of gonality \( k \) and denote by \( f : C \to \mathbf{P}^1 \) the induced degree \( k \) cover. We assume that \( C \) verifies the Effective Gonality Conjecture. Let \( p \in C \) be a branch point of \( f \), and consider the stable curve \( X = C \cup_p E \) obtained by glueing a smooth, genus 1 curve at \( p \). A standard argument with admissible covers or limit linear series shows that \( X \) is a limit of smooth, genus \( g + 1 \) curves of gonality \( k \), see [HM, §3.G].

**Proposition 2.2.** Let \( X = C \cup_p E \) be the genus \( g + 1 \) stable curve as above. Let \( L \) be a line bundle on \( X \) such that \( \deg(L_C) = 2g + k \) and \( \deg(L_E) = 1 \). Then, for a general point \( q \in E \setminus \{p\} \), we have
\[
K_{g,1}(X, L(-q)) = 0.
\]
Further, for such a point, \( h^1(X, L(-q)) = h^1(X, L^{\otimes 2}(-2q)) = 0 \).

**Proof.** We have the Mayer–Vietoris sequence on \( X \)
\[
0 \to L_C(-p) \to L(-q) \to L_E(-q) \to 0.
\]
For a general point \( q \in E \setminus \{p\} \), we have \( h^0(E, L_E^{\otimes j}(-jq)) = h^1(E, L_E^{\otimes j}(-jq)) = 0 \) for \( j = 1, 2 \), which implies \( h^1(X, L(-q)) = h^1(X, L^{\otimes 2}(-2q)) = 0 \). Further, we have a natural isomorphism \( H^0(C, L(-p)) \cong H^0(X, L(-q)) \), and we know, by the assumptions on \( C \), that
\[
K_{g,1}(C, L(-p)) = 0.
\]
We will use this to deduce \( K_{g,1}(X, L(-q)) = 0 \).

We have a natural commutative diagram
\[
\begin{array}{ccc}
\Lambda^{g+1} H^0(C, L(-p)) & \xrightarrow{d} & \Lambda^g H^0(C, L(-p)) \otimes H^0(C, L(-p)) \xrightarrow{d} \Lambda^{g-1} H^0(C, L(-p)) \otimes H^0(C, L^{\otimes 2}(-2p)) \\
\downarrow \alpha & & \downarrow \beta \\
\Lambda^{g+1} H^0(X, L(-q)) & \xrightarrow{d} & \Lambda^g H^0(X, L(-q)) \otimes H^0(X, L(-q)) \xrightarrow{d} \Lambda^{g-1} H^0(X, L(-q)) \otimes H^0(X, L^{\otimes 2}(-2q))
\end{array}
\]
where $\alpha, \beta$ are isomorphisms, and $\gamma$ is induced from the natural composition

$$H^0(C, L^{\otimes 2}(-2p)) \to H^0(C, L^{\otimes 2}(-p)) \cong H^0(X, L^{\otimes 2}(-2q)).$$

As $K_{g,1}(C, L(-p)) = 0$, the top row is exact and since $\beta$ is surjective and $\gamma$ is injective, the bottom row must also be exact, as required. \qed

From Proposition 2.2 we readily deduce Theorem 0.1, that is, the effective version of the Gonality Conjecture.

Proof. Fix $k \geq 4$. Assume that for the general $k$-gonal $C$ of genus $g$ one has $K_{g,1}(C, L) = 0$, for any line bundle $L \in \text{Pic}^{2g-1+k}(C)$. We claim there exists a smooth curve $C'$ of genus $g + 1$ and gonality $k$, such that $K_{g+1,1}(C', L') = 0$, for each line bundle $L' \in \text{Pic}^{2g+1+k}(C')$. By performing induction on $g$ and noting that the initial step is Theorem 0.2, this suffices to prove the theorem. By Lemma 1.2, it further suffices to prove that there exists a smooth curve $C'$ of genus $g + 1$ and gonality $k$ such that, for each line bundle $L' \in \text{Pic}^{2g+1+k}(C')$, there exists a point $q \in C'$ such that $K_{g,1}(C', L'(-q)) = 0$.

Let $X = C \cup_p E$ be the genus $g + 1$ stable curve introduced in Proposition 2.2. Consider a flat family $\pi : C \to S$ of stable curves over a smooth, pointed, one dimensional base $(S, 0)$, such that the central fibre is $X$ and $\pi^{-1}(s)$ is a smooth curve of gonality $k$ for all $0 \neq s \in S$. As $X$ is a curve of compact type, after shrinking $S$ and performing a finite base change if necessary, we have a relative Picard scheme

$$v : \mathcal{P}ic^{2g+1+k}(C/S) \to S,$$

with central fibre consisting of all line bundles of multidegree $(2g + k, 1)$ on $X = C \cup_p E$; this scheme is flat and proper over $S$, see [D, §4] and [EH], proof of Theorem 3.3.

Let $C_0 := C \setminus \{p\}$ be the open set of all points which are smooth in the fibres over $S$. By Proposition 2.2 together with semicontinuity for the dimension of Koszul groups, there is an open subset $U \subseteq \mathcal{P}ic^{2g+1+k}(C/S) \times_S C_0$ such that for each pair $(L', q') \in U$, one has $K_{g,1}(C', L'(-q)) = 0$, where $C' = \pi^{-1}(v(L'))$, and such that

$$0 \notin v\left(\mathcal{P}ic^{2g+1+k}(C/S) \setminus \text{pr}_1(U)\right),$$

where $\text{pr}_1 : \mathcal{P}ic^{2g+1+k}(C/S) \times_S C_0 \to \mathcal{P}ic^{2g+1+k}(C/S)$ is the projection. As flat morphisms are open, $\text{pr}_1(U)$ is open, and since $v$ is proper, the image

$$V := v\left(\mathcal{P}ic^{2g+1+k}(C/S) \setminus \text{pr}_1(U)\right)$$

is closed. Thus if $0 \neq t \in S \setminus V$ and $C_t := \pi^{-1}(t)$, then, for each $L \in \text{Pic}^{2g+1+k}(C_t)$ there exists $q \in C_t$ with $K_{g,1}(C_t, L(-q)) = 0$, as required. \qed

3. Schreyer's Conjecture for General Curves of Non-Maximal Gonality

In this section, we begin discussing Schreyer's Conjecture for general $k$-gonal curves of genus $g \geq 2k - 1$. We start by explaining the relevance of [HR] for Conjecture 0.3.

For $g = 2k - 1$, we consider two divisors on $\mathcal{M}_g$, which already played a role in [Ap2] or [FK]:

$$\mathfrak{H}_3 := \{[C] \in \mathcal{M}_g : K_{k-1,1}(C, \omega_C) \neq 0\}$$

$$\mathfrak{Hur} := \{[C] \in \mathcal{M}_g : W_k^1(C) \neq \emptyset\}.$$

Recall that $\mathfrak{H}_3$ has a structure of degeneracy locus whereas $\mathfrak{Hur}$ is the push-forward of the smooth Hurwitz space $\mathcal{H}_{2k-1,k}$ of degree $k$ covers of $\mathbb{P}^1$. We view both $\mathfrak{H}_3$ and $\mathfrak{Hur}$ as divisors
on the moduli stack of smooth curves of genus $2k - 1$, rather than on the associated coarse moduli space. It is proved in [HR], that one has the following relation at stack level:

$$[\mathfrak{S}_{\mathfrak{N}}] = (k - 1)[\mathfrak{N_{\mathfrak{ur}}}] \in CH^1(\mathcal{M}_{2k-1}).$$

**Theorem 3.1.** ([HR]) Let $C$ be a curve of genus $2k - 1$ and gonality $k$ such that the point $W^1_k(C)$ consists of a reduced single point. Then $b_{k-1,1}(C,\omega_C) = k - 1$.

**Proof.** For a smooth curve $C$, we denote by $\phi : X \to (S, 0)$ its versal deformation space, hence the associated moduli map $m(\phi) : S \to \mathcal{M}_g$ is an étale neighbourhood of the point $[C] \in \mathcal{M}_g$. For $s \in S$, set $C_s := \phi^{-1}(s)$, thus $C_0 = C$. From [HR], there exist two vector bundles $V$ and $W$ of the same rank over $S$ together with a morphism $\chi : V \to W$, such that, for any $s \in S$, we may identify $K_{k-1,1}(C_s, \omega_{C_s}) = \text{Ker}(\chi_s)$. Then the divisor $\mathfrak{S}_{\mathfrak{N}}(\phi) \subseteq S$ is defined by $\det(\chi)$. Suppose $b_{k-1,1}(C,\omega_C) \geq k$. Thus $\det(\chi)$ vanishes to order at least $k$, cf. [HR, Lemma 6.1]. By the equality of cycles $\mathfrak{S}_{\mathfrak{N}}(\phi) = (k - 1)[\mathfrak{N_{\mathfrak{ur}}}](\phi)$ on $S$, the function defining $\mathfrak{N_{\mathfrak{ur}}}(\phi)$ must vanish to order at least two. Thus $\mathfrak{N_{\mathfrak{ur}}}(\phi)$ is not smooth at the point $0 \in S$. On the other hand it is well-known, see [C], that $\mathfrak{N_{\mathfrak{ur}}}(\phi)$ is smooth at a point $0 \in S$ corresponding to a curve $C$ if and only if $W^1_k(C)$ consists of a single pencil $A$ and, moreover, $h^0(C, A^{\otimes 2}) = 3$. □

**Remark 3.2.** One can generalise Theorem 3.1 as follows. For an integral nodal curve $D$, we define $W^1_k(D) \subseteq \operatorname{Pic}^k(D)$ to be the closed subset of the compactified Jacobian of rank one, torsion free sheaves $A$ of degree $k$ on $C$ with $h^0(D, A) \geq 2$. Suppose $D$ is integral, nodal of genus $2k - 1$ and assume $W^1_k(D) = \{A\}$, where $A$ is locally-free and base-point free. Then the proof of Theorem 3.1 shows $b_{k-1,1}(D,\omega_D) = k - 1$.

We now turn our attention to curves of genus $g$ and non-maximal gonality $k \leq \frac{g+1}{2}$. Let $G^1_{d}(C) \subseteq G^1_d(C)$ be the subvariety of base point free pencils of degree $d$ on $C$ and further let $W^1_d(C)$ denote the Brill–Noether variety of line bundles of degree $d$ with at least two sections. Note that there is a morphism $G^1_d(C) \to W^1_d(C)$, with fibre over a point $[L] \in W^1_d(C)$ equal to the Grassmannian of pencils $V \subseteq H^0(C, L)$. The following observation is a slight modification of the linear growth condition of [Ap2, Theorem 2]:

**Lemma 3.3.** A general curve $C$ of genus $g$ and gonality $k \leq \frac{g+1}{2}$ satisfies bpf-linear growth:

$$\dim G^1_{k+m}(C) \leq m, \quad \text{for } 0 \leq m \leq g - 2k + 1$$

and, further, $$\dim G^1_{k+m}(C) < m, \quad \text{for } 0 \leq m \leq g - 2k + 1.$$

**Proof.** From [Ap2], we have $\dim W^1_{k+m}(C) = m$, for $0 \leq m \leq g - 2k + 1$. We observe that if $Z \subseteq W^1_d(C)$ is an irreducible component, then $Z \cap W^1_{d+1}(C)$ has codimension at least two in $Z$, provided $g - r + d \geq 0$. This follows from the fact that no component of $C^r_d$ is entirely contained in $C^r_{d+1}$, where $C^r_d$ is the variety parametrizing divisors $D$ of degree $d$ on $C$ with $\dim |D| \geq r$, see [ACGH, §IV.1].

We claim $\dim G^1_{d+m}(C) \leq m$, for $0 \leq m \leq g - 2k + 1$. Take an irreducible component $J \subseteq G^1_{d+m}(C)$ and consider the restriction to $J$ of the surjection $c : G^1_{k+m}(C) \to W^1_{k+m}(C)$. Assume $c(J) \subseteq W^1_{d+m}(C)$ and choose $j \geq 0$ maximal with this property. Then by the above, $\dim c(J) \leq m - 2j$. Since the general fibre of $c_j$ is isomorphic to the Grassmannian $G(2, 2 + j)$, it follows $\dim J \leq 2j + \dim c(J) \leq m$. By an identical argument and using [AC, Theorem 2.6], we similarly obtain that $\dim G^1_{k+m}(C) < m$, in the range $0 \leq m \leq g - 2k + 1$. □

The next Proposition is very similar to the proof of Theorem 2 in [Ap2] and we skip the details.
Proposition 3.4. Let $C$ be a smooth curve of genus $g$ and gonality $k \leq \frac{2g+1}{4}$. Assume $C$ satisfies bpf-linear growth and $W^1_k(C)$ consists of a single point $A$. If $(x_i, y_i)$ are general pairs of points on $C$, where $1 \leq i \leq g - 2k + 1$, let $D$ be the nodal curve obtained by gluing $x_i$ to $y_i$ for all $i$. Then $W^1_{g-k+1}(D) = \{ \nu(A) \}$, where $\nu: C \to D$ is the normalisation morphism. Furthermore, $\text{gon}(D) = g - k + 1$.

Consider the moduli space $\overline{\mathcal{H}}_{g,k}$ of degree $k$ admissible covers of genus $g$. Precisely,

$$\overline{\mathcal{H}}_{g,k} = \overline{\mathcal{M}}_{0,2g+2k-2}(\mathcal{B}\mathcal{S}_k) / \mathcal{S}_{2g+2k-2}$$

is the space of twisted stable maps from genus zero curves into the classifying stack $\mathcal{B}\mathcal{S}_k$ of the symmetric group $\mathcal{S}_k$ and which are simply branched over $2g + 2k - 2$ points which we do not order. We refer to [ACV] for the construction of this space. It is known that $\overline{\mathcal{H}}_{g,k}$ is the normalisation of the space of admissible covers constructed by Harris and Mumford in [HM].

There is a morphism $\pi: \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g$ given by stabilisation of the source curve of each admissible cover and then $\text{Im}(\pi) = \overline{\mathfrak{Hur}}$. The following result is the translation of Proposition 3.4 to the moduli space of admissible covers.

Proposition 3.5. Let $C$ be a smooth curve of genus $g$ and gonality $k \leq \frac{2g+1}{4}$. Assume $C$ satisfies bpf-linear growth and that $W^1_k(C)$ consists of a single point $A$, which we assume to have only simple ramification. For $1 \leq i \leq g - 2k + 1$, we choose general pairs of points $(x_i, y_i)$ on $C$ and let $[D] \in \overline{\mathcal{M}}_{2g-2k+1}$ be the nodal curve obtained by gluing $x_i$ to $y_i$. If

$$\pi: \overline{\mathcal{H}}_{2g-2k+1,g-k+1} \to \overline{\mathcal{M}}_{2g-2k+1}$$

is the forgetful map, then $\pi^{-1}([D])$ consists of a unique point $[f': B' \to T]$.

Proof. We show that the construction described in [HM, Theorem 5] is unique in our case. Let $[f': B' \to T] \in \overline{\mathcal{H}}_{2g-2k+1,g-k+1}$ be an admissible cover, where $p_a(T) = 0$ and $B'$ is a nodal curve whose stable model is isomorphic to $D$. There exists a unique component $C_0$ of $B'$ having positive genus. The restriction $f_0 := f|_{C_0}$ gives a morphism $f_0: C_0 \to P^1_T$ onto a smooth rational component $P^1_0$ of $T$. By admissibility, $C_0 \cong C$ and $\text{deg}(f_0) \geq k$.

Assume that $f_0(x_i) = f_0(y_i)$ if and only if $1 \leq i \leq j$. For $i = j + 1, \ldots, g - 2k + 1$, we denote by $R_{x_i}$ and $R_{y_i}$ respectively the irreducible components of $B'$ meeting $C$ at $x_i$ and $y_i$ respectively. As the stabilisation of $B'$ is $D$ and $f'(R_{x_i}) \cap f'(R_{y_i}) = \emptyset$, for each such $i$ there must be a component $\tilde{R}_i$ of the subcurve $B' - C_0$ of $B'$ such that $f'(\tilde{R}_i) = P^1_0$, or else $T$ contains a loop. As $\text{deg}(f') = g - k + 1$, this implies that $d := \text{deg}(f_0) \leq k + j$.

Since the pairs $(x_1, y_1), \ldots, (x_j, y_j)$ are general and $f_0$ gives rise to an element of $G^1_{bpf}(C)$, it follows $\dim G^1_{bpf}(C) \geq j$. If $d > k$, this contradicts the bpf-linear condition on $C$, which implies that $\text{deg}(f_0) = k$ and $f_0$ is the map induced by the pencil of minimal degree $A \in W^1_k(C)$. Each $\tilde{R}_i$ maps isomorphically onto $P^1_0$. Clearly $\text{deg}(f'|_{R_{x_i}}) \geq 2$ and $\text{deg}(f'|_{R_{y_i}}) \geq 2$, in particular $f'|_{R_{x_i}}$ and $f'|_{R_{y_i}}$ will both contain at least two ramification points of $f'$, for each $i = 1, \ldots, g - 2k + 1$ (note that being general points, $x_i, y_i$ are not among the ramification points of $f_0$). Counting the total number of ramification points of the cover $f'$, it follows that $\text{deg}(f'|_{R_{x_i}}) = \text{deg}(f'|_{R_{y_i}}) = 2$.

The morphism $f'$ is now uniquely determined, for $f'^{-1}(P^1_0) = C \cup \tilde{R}_1 \cup \ldots \cup \tilde{R}_{g-2k+1}$ and all the components of $f'^{-1}(f(R_{x_i}))$ and $f'^{-1}(f(R_{y_i}))$ other than $R_{x_i}$ and $R_{y_i}$ respectively are mapped isomorphically onto their images. 

Let $\mathfrak{HN} \subseteq \mathcal{H}_{2g-2k+1,g-k+1} \times_{\mathcal{M}_{2g-2k+1}} \mathcal{H}_{2g-2k+1,g-k+1}$ be the closure of the locus of pairs

$$([g_1: C \to P^1], [g_2: C \to P^1]),$$
where \( C \) is a smooth curve of genus \( 2g - 2k + 1 \) and \( g_1 \not\cong g_2 \). Applying [AC, Proposition 2.4], we know that \( \dim \mathfrak{B} \mathfrak{N} = \dim \mathcal{H}_{2g-2k+1,g-k+1} - 1 \). We introduce the Brill-Noether divisor on the Hurwitz space of curves possessing an extra pencil:

\[
\mathfrak{B} \mathfrak{N} := \text{pr}_1(\mathfrak{B} \mathfrak{N}') \subseteq \mathcal{H}_{2g-2k+1,g-k+1},
\]

where \( \text{pr}_1 \) is the projection on the first factor. Since \( \mathfrak{B} \mathfrak{N}' \) is birational to the Severi variety of nodal curves of type \((g-k+1, g-k+1)\) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) having geometric genus \( 2g - 2k + 1 \), using [Ty], we conclude that \( \mathfrak{B} \mathfrak{N} \) is an irreducible divisor. We also recall Coppens’ result [C] saying that if a curve \( C \in \mathcal{M}_{2g-2k+1} \) has a pencil \( A \in W^1_{g-k+1}(C) \) such that \( h^0(C, A^{(2)}) \geq 4 \), then \([C, A] \in \mathfrak{B} \mathfrak{N} \). The locus of such pairs \([C, A] \in \mathcal{H}_{2g-2k+1,g-k+1} \) is of pure codimension one in \( \mathfrak{B} \mathfrak{N} \).

Our next goal is to show that, in the notation of Proposition 3.5, the unique point of \( \pi^{-1}([D]) \) does not lie in the closure \( \overline{\mathfrak{B} \mathfrak{N}} \subseteq \overline{\mathcal{H}_{2g-2k+1,g-k+1}} \) provided the normalisation \( C \) is sufficiently general. To ease the notation, set \( a := g - k + 1 \) and assume \( a \geq 3 \). To carry out the argument, it is convenient to work with stable maps. Let

\[
\mathcal{G}_{2a-1,a} := \overline{\mathcal{M}_{2a-1}(\mathbb{P}^1, a)}
\]

denote the moduli space of finite stable maps \( f : C \to \mathbb{P}^1 \) of degree \( a \) such that \( C \) has genus \( 2a - 1 \), has only non-separating nodes and with \( h^0(C, f^*\mathcal{O}_{\mathbb{P}^1}(1)) = 2 \). Then \( \mathcal{G}_{2a-1,a} \) is an open subset of the projective moduli space \( \overline{\mathcal{M}_{2a-1}(\mathbb{P}^1, a)} \) of stable maps \( f : C \to \mathbb{P}^1 \) with \( f_*[C] = a[\mathbb{P}^1] \). Let

\[
\tilde{\pi} : \mathcal{G}_{2a-1,a} \to \overline{\mathcal{M}_{2a-1}}
\]

denote the natural projection.

Note that the Hurwitz space \( \mathcal{H}_{2a-1,a} \) can be realized as the quotient of an open set of \( \mathcal{G}_{2a-1,a} \) by \( PGL(2) \). We associate a stable map \([f : B \to \mathbb{P}^1] \in \mathcal{G}_{2a-1,a} \) to the unique point \([f' : B' \to T] \in \pi^{-1}([D])\) by letting \( B \) be the curve obtained from \( B' \) by contracting all components of \( B' \) whose image is different from \( f'(C) \), and then letting \([f : B \to \mathbb{P}^1]\) be the map which, on each component of \( B \), agrees with \( f' \) on the corresponding component of \( B' \) (this is only determined up to the \( PGL(2) \) action). Then

\[
B = C \cup R_1 \cup \ldots \cup R_{a-k}
\]

where \( R_i \simeq \mathbb{P}^1 \) meets \( C \) at two general points \((x_i, y_i)\) for each \( 1 \leq i \leq a - k \), and \( \text{deg} f_{R_i} = 1 \) for all \( i \).

The set \( \mathfrak{B}_a \subseteq \mathcal{G}_{2a-1,a} \times \mathcal{M}_{2a-1} \mathcal{G}_{2a-1,a} \) be the closure of the locus of pairs

\[
([g_1 : X \to \mathbb{P}^1], [g_2 : X \to \mathbb{P}^1]),
\]

where \( X \) is a smooth curve of genus \( 2a - 1 \) and there is no automorphism \( \sigma \in PGL(2) \) such that \([g_1] \cong \sigma \cdot [g_2] \).

In order to prove that the unique point of \( \pi^{-1}([D]) \) does not lie in the closure \( \overline{\mathfrak{B} \mathfrak{N}} \subseteq \overline{\mathcal{H}_{2a-1,a}} \) it is sufficient to prove that the point

\[
([f], [f']) \not\in \mathfrak{B}_a,
\]

since the stabilization procedure used to construct \([f]\) from the admissible cover \([f']\) can be performed in families.

For \( 0 \leq n \leq 2a - 5 \), let \( \overline{\mathcal{M}}_{2a-1-n}(\mathbb{P}^1, a - n; 2n) \) denote the moduli space of finite stable maps \( f : C \to \mathbb{P}^1 \) of degree \( a - n \) with \( 2n \) markings and such that \( C \) has genus \( 2a - 1 - n \).
non-separating nodes and $h^0(C, f^*\mathcal{O}_P(1)) = 2$. We have that $\tilde{\mathcal{M}}_{2g-1,n}^{\text{ns}}(\mathbb{P}^1, a-n; 2n)$ is smooth of dimension $\dim \tilde{G}_{2a-1,n}^{\text{ns}} - 2n$. Define a morphism

$$q_n : \tilde{\mathcal{M}}_{2g-1,n}^{\text{ns}}(\mathbb{P}^1, a-n; 2n) \to \tilde{G}_{2a-1,n}^{\text{ns}}$$

by sending a map $g : C \to \mathbb{P}^1$ with markings $x_1, \ldots, x_n, y_1, \ldots, y_n$ to the stable map

$$f(x_i, y_i) : B(x_i, y_i) \to \mathbb{P}^1,$$

where $B(x_i, y_i) = C \bigcup_{i=1}^n R_i$, with $R_i \simeq \mathbb{P}^1$, $R_i \cap C = \{x_i, y_i\}$, and with $(f(x_i, y_i))|_C = g$, $\deg(f(x_i, y_i))|_{R_i} = 1$, $1 \leq i \leq n$. We set

$$Z_n := q_n^{-1}(pr_1(\mathfrak{M}_a^{\text{ns}})).$$

Each component of $Z_n$ has dimension at least $\dim \tilde{G}_{2a-1,n}^{\text{ns}} - 2n - 1$. We further define $\tilde{B}(x_i, y_i)$ to be the stabilization of $B(x_i, y_i)$.

The goal in this section is to prove the following theorem, giving a classification of points in $Z_n$.

**Theorem 3.6.** Let $0 \leq n \leq 2a-5$, $a \geq 3$ and let $C$ be an integral, nodal curve of genus $2g - 1 - n$ and gonality $a - n$ with a unique pencil $f : C \to \mathbb{P}^1$ of degree $a - n$. Let $(x_i, y_i)$ be distinct pairs of points in the smooth locus of $C$, for, $1 \leq i \leq n$. Assume that for any $S \subseteq \{x_i, y_i \mid 1 \leq i \leq n\}$ of cardinality at most $n$,

$$h^0(C, f^*(\mathcal{O}_P(2))\left(\sum_{s \in S} s\right)) = 3.$$

Then if $\pi^{-1}(\tilde{B}(x_i, y_i))$ consists of a unique admissible cover, the marked stable map $[f, (x_i, y_i)]$ is not in $Z_n$.

Note that if $C$ be an integral, nodal curve of genus $2g - 1 - n$ and gonality $a - n$ with pencil $f$ of degree $a - n$ satisfying $h^0(f^*\mathcal{O}_P(2)) = 3$, and if $\{x_i, y_i \mid 1 \leq i \leq n\}$ is a general set of points on $C$, then

$$h^0(C, f^*(\mathcal{O}_P(2))\left(\sum_{s \in S} s\right)) = 3$$

for a general $S \subseteq \{x_i, y_i \mid 1 \leq i \leq n\}$ of cardinality at most $n + 1$. Indeed, $h^0(C, \omega_C \otimes f^*\mathcal{O}_P(-2)) = n+1$ by Riemann–Roch and since $(x_i, y_i)$ are general, $h^0(\omega_C \otimes f^*\mathcal{O}_P(-2)\left(\sum_{s \in S} s\right)) = n + 1 - |S|$. Thus $h^0(f^*\mathcal{O}_P(2)\left(\sum_{s \in S} s\right)) = 3$ as desired.

In particular, the above shows that the following statement is an immediate corollary of Theorem 3.6.

**Theorem 3.7.** Let $C$ be a curve of genus $g$ and gonality $k \leq \frac{g+1}{2}$ satisfying bpf-linear growth. Assume that there is a unique $A \in G^1_1(C)$ and that we have $h^0(A^\otimes 2) = 3$. Choose general pairs of points $(x_i, y_i)$ on $C$ for $1 \leq i \leq g-2k+1$ and let $D$ be the nodal curve obtained by identifying $x_i$ and $y_i$ for all $i$. Then $\pi^{-1}([D]) \cap \mathfrak{M}_a^{\text{ns}} = \emptyset$.

We will prove Theorem 3.6 by induction, the base case $n = 0$ being Remark 3.2.

### 3.1. Twists. A key idea in the proof of Theorem 3.6 is to construct twistings of line bundles on reducible curves. Suppose we have a family $\mathcal{B} \to \Delta$ of nodal curves over a smooth base $\Delta$, with general fibre smooth and with special fibre the “banana-like” curve $C \cup R$ for $R \simeq \mathbb{P}^1$ where $C$ is smooth and $R$ meets $C$ transversally in two points. If the total family $\mathcal{B}$ is smooth, then $\mathcal{O}_S(R)$ is a well-defined line bundle which is trivial outside of the special fibre. One can thus use this to define twists of natural line bundles on $\mathcal{B}$, such as $\omega_{\mathcal{B}/\Delta}$, which frequently prove useful, see for example [FP]. We will now explain how to generalise the definition of this twist
to the case where the general fibre is only integral and $\mathcal{B}$ is not necessarily smooth, so that $R$ does not define a Cartier divisor. This forces us to modify $\mathcal{B}$ via base changes and blow-ups.

We first introduce some convenient notation.

**Definition 3.8.** Let $X$ be a connected, nodal curve and $p \in X$. The “blow up” $c_p : \tilde{X} \to X$ of $X$ at $p$ is defined as such:

1. If $p \in X$ is a node, let $\nu : X' \to X$ denote the partial normalisation of $X$ at $p$ and let $\nu^{-1}(p) = \{a, b\}$. Then $\tilde{X}$ is defined to be $X' \cup E$ where $E \simeq \mathbb{P}^1$ and $E \cap X' = \{a, b\}$.

2. If $p$ is not a node, then define $\tilde{X} = X \cup E$ where $E \simeq \mathbb{P}^1$ with $X \cap E = p$. We further define the “strict transform” $X'$ of $X$ to be the closure $\tilde{X} \setminus E$.

In both cases, $c_p : \tilde{X} \to X$ is given by contracting the unstable component $X$ at a node $X$. Thus, after a finite base change, we have a birational morphism $\tau : \tilde{X} \to X$ of degree 2 such that there exists some nonzero $\nu \in H^0(\omega_{\tilde{X}})$ with $s(\alpha) = 0$. Such a pair $(C, \alpha)$ defines an element of $\mathcal{M}_{g,2g-2}$. Let $\mathcal{M}_g((1)^{2g-2}) \subseteq \mathcal{M}_{g,2g-2}$ denote the space of twisted abelian differentials of type $(1)^{2g-2}$, i.e. the closure of the space of abelian differentials of this type, see [FP].

The first twisting construction we will use is described below and is rather well-known. We attach a proof due to lack of a suitable reference.

**Proposition 3.9.** Let $(\Delta, 0)$ be an irreducible, pointed, variety and $\mathcal{B} \to \Delta$ be a flat family of nodal curves of genus $g$ such that $\mathcal{B}_t$ is irreducible for $t \in \Delta$ and general $\mathcal{B}_0 \simeq C \cup R$, $R \simeq \mathbb{P}^1$, $R \cap C = \{u, v\}$

for irreducible $C$. Then, after a base change, there is a birational morphism $\nu : \tilde{\mathcal{B}} \to \mathcal{B}$ between families of nodal curves of $\Delta$, together with a line bundle $\tilde{\tau} \in \text{Pic}(\tilde{\mathcal{B}})$ such that one of the following cases occur:

1. $\tilde{\mathcal{B}}_0 \simeq \mathcal{B}_0$ and further $\tau_{0|C} \simeq \mathcal{O}_C(u + v)$, $\deg \tau_{0|R} = -2$.

2. $\tilde{\mathcal{B}}_0$ is a blow-up of $\mathcal{B}_0$ at a node $p \in \{u, v\}$ with exceptional component $E$. Identifying $R, C$ with their strict transforms, $\tau_{0|C} \simeq \mathcal{O}_C(u + v)$, $\deg \tau_{0|R} = \deg \tau_{0|E} = -1$.

Morally speaking, case (1) corresponds to the case where there exists a twisting line bundle, whereas case (2) corresponds to twisting via a torsion-free sheaf.

**Proof.** Performing a base change if necessary, we have $g$ markings

\[ p_i : \Delta \to \mathcal{B}, \quad 1 \leq i \leq g \]

on all fibres, with $p_i(0) \in R \subseteq \mathcal{B}_0$ and $p_i(0) \in C$ general for $1 \leq i \leq g - 1$. For all $t \in \Delta$, there exists a unique $s_t \in H^0(\omega_{\mathcal{B}_t})$ which vanishes on $\{p_i(t) : 1 \leq i \leq g\}$. Notice that, by our choices, $s_0$ vanishes identically on $R$. Further, $s_0$ vanishes on an abelian differential on $C$ of type $(1)^{2g-4}$, by generality of $p_i(0) \in C$ for $1 \leq i \leq g - 1$. Hence, after shrinking $\Delta$, components of the vanishing set of $s_t$ provide, in particular, additional sections

\[ p_i : \Delta \to \mathcal{B}, \quad g + 1 \leq i \leq 2g - 3, \]

with $\sum_{i \neq g} p_i(0)$ defining an abelian differential on $C$ and, such that, for each $t$

\[ (\mathcal{B}_t, \{p_i(t) : 1 \leq i \leq 2g - 3\}) \]

lies in the image of the morphism $\mathcal{M}_g((1)^{2g-2}) \to \mathcal{M}_{g,2g-3}$, obtained by forgetting the last marking. Thus, after a finite base change, we have a birational morphism $\nu : \tilde{\mathcal{B}} \to \mathcal{B}$ between families of nodal curves, together with sections

\[ q_i : \Delta \to \tilde{\mathcal{B}}, \quad 1 \leq i \leq 2g - 2, \]
such that \((\tilde{B}_t, \{q_i(t)\}) \in \overline{M}_g((1)^{2g-2})\) and, further, these marked curves stabilize to \((B_t, \{p_i(t), 1 \leq i \leq 2g - 3\})\) after forgetting the last marking \(q_{2g-2}(t)\).

Notice that \(\nu_0(q_{2g-2}(0)) \in R\). We now have three cases. Suppose first of all that \(\nu_0(q_{2g-2}(0)) \notin \{u, v, p_g(0)\}\). Then \(\nu\) is an isomorphism, and we may take

\[
\tau := \omega_B/\Delta \left(\sum_{i=1}^{2g-2} \nu(q_i)(\Delta)\right).
\]

Secondly, suppose \(\nu_0(q_{2g-2}(0)) = p_g(0)\). Then \(\nu(q_{2g-2})\) defines a section in the smooth locus of each fibre \(B_t\), and we may define \(\tau\) by the same formula. Lastly, suppose \(\nu_0(q_{2g-2}(0)) \in R \cap C\).

We firstly observe that, for \(t\) general, \(\nu_t(q_{2g-2}(t))\) is not a node of \(B_t\). Indeed, otherwise \(\tilde{B}_t\) would be a blow-up of \(B_t\) at a node. Further, there must be a nonzero section of \(\omega_{\tilde{B}_t}\) vanishing at the exceptional component \(E\), as well as on \(2g - 3\) smooth points \(q_i(t), 1 \leq i \leq 2g - 3\) of \(\tilde{B}_t\), which is clearly not possible. Hence, in the last case, \(\tilde{B}\) is a blow-up of \(B\) at a node in the intersection \(C \cap R\), and we may now take

\[
\tau := \omega_{\tilde{B}}/\Delta \left(\sum_{i=1}^{2g-2} q_i(\Delta)\right).
\]

\(\square\)

The next twisting result we prove is more sophisticated. It requires the base \(\Delta\) to have dimension at least two and involves a condition on the locus where the family of curves has reducible fibre. It further requires that we have a family of stable maps \(G_1 : B_t \to \Delta\), in addition to the nodal curves \(B_t\). The benefit is that, under these additional assumptions, we can ensure that we can put ourselves in the more degenerate case (2) of Proposition 3.9.

**Proposition 3.10.** In the notation of Proposition 3.9, assume additionally that we have a family \(G : B \to P^1_\Delta\) of stable maps and that, further, \(h^0(G_1^*O_{P^1}(1)) = 2\),

\[\omega_C \otimes G_0^*O_{P^1}(-1)\]

is base-point free. Assume the locus of \(t \in \Delta\) with \(B_t\) reducible has codimension two about \(0 \in \Delta\). Then, after a base change, we have \(\nu : \tilde{B} \to B\) and \(\tau \in \text{Pic}(\tilde{B})\) as in case (2) of Proposition 3.9.

**Proof.** Let \(\overline{M}^\dagger(P^1,k) \subseteq \overline{M}_g(P^1,k;2g-2-k)\) denote the closed substack of the moduli space of degree \(k\), stable maps \(f : B \to P^1\) of genus \(g\) with markings \(p_1, \ldots, p_{2g-2-k}\) defined by the conditions:

\begin{itemize}
  \item \(h^0(B, f^*O_{P^1}(1)) = 2\).
  \item There exists a nonzero section \(\alpha \in H^0(B, \omega_B \otimes f^*O_{P^1}(-1))\) with \(\alpha\) vanishing on \(p_i, 1 \leq i \leq 2g - 2 - k\).
\end{itemize}

This moduli space \(\overline{M}^\dagger(P^1,k)\) may be constructed as an incidence variety in the obvious fashion (cf. [BCGGM, §2]). We have a forgetful morphism

\[\overline{M}^\dagger(P^1,k) \to \overline{M}_g(P^1,k)\]

to the moduli space of degree \(k\) stable maps. Let

\[r : \Delta^\dagger := \Delta \times \overline{M}_g(P^1,k) \overline{M}^\dagger(P^1,k) \to \Delta,\]

where \(\Delta \to \overline{M}_g(P^1,k)\) is induced by the family \(G : B \to P^1_\Delta\) of stable maps over \(\Delta\).
Let \( \widetilde{B}_0 \) denote the blow-up of \( B_0 \) at \( u \), let

\[ \widetilde{G}_0 : \widetilde{B}_0 \to \mathbb{P}^1 \]

be the morphism with \( \widetilde{G}_0 := G_0 \circ c_u \).

Let \( p_1, \ldots, p_{2g-3-k} \) be distinct points of \( C \), in the smooth locus of \( B_0 \), with

\[ \sum_{i=1}^{2g-3-k} p_i \in |\omega_C \otimes G_{0,c}^* \mathcal{O}_{\mathbb{P}^1}(-1)|, \]

and pick a general point \( p_{2g-2-k} \in E \). Then

\[ u = [\widetilde{G}_0, (p_i)] \in \Delta^1. \]

We claim for any component \( I \subseteq \Delta^1 \) containing the point \( u \), \( I \) dominates \( \Delta \) under the forgetful morphism \( r \). We are then done, by replacing \( \Delta \) with \( I \), setting \( \widetilde{G} : \widetilde{B} \to \mathbb{P}^1 \) to be the universal family and setting

\[ \tau := \widetilde{G}^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \omega_{\widetilde{B}/I}(-\sum_{i=1}^{2g-2-k} p_i), \]

where \( p_i : I \to \widetilde{B} \) are the markings.

The construction of \( \Delta^1 \) as an incidence variety shows

\[ \dim I \geq \dim \Delta + \dim |\omega_{B_0} \otimes \widetilde{G}_0^* \mathcal{O}_{\mathbb{P}^1}(-1)|. \]

Since \( \dim |\omega_C \otimes G_{0,c}^* \mathcal{O}_{\mathbb{P}^1}(-1)| = \dim |\omega_{B_0} \otimes \widetilde{G}_0^* \mathcal{O}_{\mathbb{P}^1}(-1)| + 1 \) near \( u \). Hence, \( r(I) \) has dimension \( \dim |\omega_{B_0} \otimes \widetilde{G}_0^* \mathcal{O}_{\mathbb{P}^1}(-1)| + 1 \). Thus \( I \) dominates \( \Delta \).

\[ \square \]

3.2. Induction Step. Our task is now to prove the induction step of Theorem 3.6. For the remainder of this section, we fix some integer \( m \geq 0 \) and assume Theorem 3.6 holds for \( n = m \).

We first prove a weakening of the induction step, using the more basic twisting result Proposition 3.9.

**Proposition 3.11.** Assume Theorem 3.6 holds for \( n = m \). Let \( C \) be a general curve of genus \( 2a - 2 - m \) and gonality \( a - m - 1 \) with pencil \( f : C \to \mathbb{P}^1 \) of degree \( a - m - 1 \). Let \( (x_i, y_i) \) be general pairs of points in \( C \) for \( 1 \leq i \leq m + 1 \). Then \( [f, (x_i, y_i)] \) is not in \( Z_{m+1} \).

In particular, \( Z_{m+1} \) has codimension one in \( \widetilde{M}^{ns}_{2a-1-(m+1)}(\mathbb{P}^1, a - (m + 1); 2(m + 1)) \).

The conclusion that \( Z_{m+1} \) has codimension one in \( \widetilde{M}^{ns}_{2a-1-(m+1)}(\mathbb{P}^1, a - (m + 1); 2(m + 1)) \) will allow us to apply the more sophisticated twisting result of Proposition 3.10, which will be crucial to finish the proof of Theorem 3.6.

**Proof.** Suppose \( [f, (x_i, y_i)] \in Z_{m+1} \). Let \( B_{m+1} := C \cup R_{m+1} \) with \( R_{m+1} \simeq \mathbb{P}^1, R_{m+1} \cap C = \{x_{m+1}, y_{m+1}\} \) and let

\[ f_{m+1} : B_{m+1} \to \mathbb{P}^1 \]

be the map with \( f_{m+1}|_C = f \), \( \deg f_{m+1}|_{R_{m+1}} = 1 \). Since \( [f, (x_i, y_i)] \in Z_{m+1} \),

\[ p = [f_{m+1}, (x_i, y_i)_{i \leq m}] \in Z_m. \]
Let $J \subseteq Z_m$ be any component containing $p$. As $\dim J \geq \dim \overline{\mathcal{G}}_{2a-1,a}^{\text{ns}} - 2m - 1$ but $\dim Z_{m+1} \leq \dim \overline{\mathcal{M}}_{2a-2-m}^{\text{ns}}(P^1, a - m - 1; 2(m+1)) = \dim \overline{\mathcal{G}}_{2a-1,a}^{\text{ns}} - 2m - 2$, the general point

$$[g : T \to P^1, (x'_i, y'_i)]$$

of $J$ is a marked stable map with irreducible base $T$. Let $\overline{B}_{m+1}$ be the curve obtained from $B_{m+1}$ by glueing $x_i$ to $y_i$ for $1 \leq i \leq m$ and contracting the unstable component $R_{m+1}$. Then $\overline{B}_{m+1}$ has gonality $a$ by Proposition 3.4. It follows that $\text{gon}(\overline{T}(x'_i, y'_i)) \geq a$, where $\overline{T}(x'_i, y'_i)$ is the curve obtained from $T$ by glueing $x'_i$ to $y'_i$ for $1 \leq i \leq m$. In particular, $\text{gon}(T) = a - m$.

Let $S \subseteq \{x'_i, y'_i \mid 1 \leq i \leq m\}$ be any set of cardinality at most $m$. In the following two lemmas, we prove:

1. $h^0(T, g^*\mathcal{O}_{P^1}(2)(\sum_{s \in S} s)) = 3$.
2. $\pi^{-1}(\overline{T}(x'_i, y'_i))$ consists of a unique admissible cover.

Putting all these facts together gives a contradiction by the assumption that Theorem 3.6 holds for $n = m$.

Here is the first lemma, with notation as above.

**Lemma 3.12.** We have $h^0(T, g^*\mathcal{O}_{P^1}(2)(\sum_{s \in S} s)) = 3$.

**Proof.** Note first of all that, by the Riemann–Roch calculation preceding the proof:

$$h^0(C, f^*\mathcal{O}_{P^1}(2)(\sum_{s \in S'} s)) = 3$$

for any $S' \subseteq \{x_i, y_i \mid 1 \leq i \leq m + 1\}$ of cardinality $|S'| \leq m + 2$.

We have

$$\mathcal{G}_1 : B_1 \to P^1_\Delta, \quad x_i, y_i : \Delta \to P^1$$

a family of marked stable maps over a smooth, pointed curve $(0, \Delta)$ such that, for $t \neq 0 \in \Delta$ the fibre $(\mathcal{G}_{1,t} : B_{1,t} \to P^1, x_i(t))$ is a marked stable map in the component $J \subseteq Z_m$ with $B_{1,t}$ integral, whereas the fibre over $0$ is the marked stable map $f_{m+1} : B_{m+1} \to P^1$ above. Let $S \subseteq \{x_i, y_i \mid 1 \leq i \leq m\}$ be a set of cardinality at most $m$ and set

$$\mathcal{M} = \mathcal{G}_1^*\mathcal{O}_{P^1}(2)(\sum_{s \in S} s).$$

We need to prove $h^0(\mathcal{B}_{1,t}, \mathcal{M}_t) = 3$ for general $t \in \Delta$.

By Proposition 3.9, we have $\nu : \overline{B}_1 \to B_1$ together with a line bundle $\tau$ on $\overline{B}_1$, which is trivial on the generic fibre. Set

$$\mathcal{L} := \nu^*\mathcal{M} \otimes \tau, \quad \overline{B}_1 := \mathcal{B}_1, \mathcal{L}_0 := \mathcal{B}_1.$$

By semicontinuity, it suffices to show $h^0(\overline{B}_1, \mathcal{L}_0) = 3$.

Assume $\overline{B} = B_{m+1}$ as in case (1) of Proposition 3.9. Then $C \subseteq \overline{B}$ is a component and

$$\mathcal{L}_{0|C} \simeq f^*\mathcal{O}_{P^1}(2)(\sum_{s \in S} s + x_{m+1} + y_{m+1})$$

where $S = \{s(0) \mid s \in S\}$. We further have $\mathcal{L}_{0|R_{m+1}} \simeq \mathcal{O}_{R_{m+1}}$ and the claim follows by twisting the exact sequence

$$0 \to \mathcal{O}_{R_{m+1}}(-2) \to \mathcal{O}_{B_{m+1}} \to \mathcal{O}_C \to 0$$

by $\mathcal{L}_0$ and equation (4) above.

Lastly, assume $\overline{B}$ is as in case (2) of Proposition 3.11. Then

$$(\deg \mathcal{L}_{0|R_{m+1}}, \deg \mathcal{L}_{0|E}) = (1, -1), \quad \mathcal{L}_{0|C} \simeq f^*\mathcal{O}_{P^1}(2)(\sum_{s \in S} s + x_{m+1} + y_{m+1}).$$
Set $B' = C \cup R_{m+1} \subseteq B$. From
\[ 0 \to \mathcal{O}_{B'}(-z - w) \to \mathcal{O}_B \to \mathcal{O}_E \to 0, \]
where $w = E \cap R_{m+1}$ and $z = C \cap E$, it is enough to have $h^0(B', \mathcal{L}_0(z - w)) = 3$. Then from
\[ 0 \to \mathcal{O}_{R_{m+1}}(-1) \to \mathcal{O}_{B'} \to \mathcal{O}_C \to 0 \]
this is equivalent to having
\[ h^0(C, f^*\mathcal{O}_{\mathbb{P}^1}(2)(\sum_{s \in S} s + z')) = 3 \]
where $z' = C \cap R_{m+1} \in \{x_{m+1}, y_{m+1}\}$, which follows by equation (4).

To finish the proof of Theorem 3.11, it suffices to show the following lemma.

**Lemma 3.13.** Let $\mathcal{G}_1 : B_1 \to \mathbb{P}^1_{\Delta}$ be as in the proof of Lemma 3.12. For $t \in \Delta$ is general, let $\tilde{B}_{\{x_i(t), y_i(t)\}}$ be the curve obtained from the fibre $B_{1,t}$ by identifying $x_i(t)$ and $y_i(t)$ for $1 \leq i \leq m$. Then $\pi^{-1}(\tilde{B}_{\{x_i(t), y_i(t)\}})$ consists of a unique admissible cover.

**Proof.** Suppose otherwise. As $C, (x_i, y_i)$ are general for $1 \leq i \leq m + 1$, there is a unique admissible cover in $\pi^{-1}(\tilde{B}_{m+1})$ (Prop. 3.5), which further must correspond to the cover constructed from $\mu_* f^* \mathcal{O}_{\mathbb{P}^1}(1)$ in [HM, Thm. 5], where $\mu : C \to \tilde{B}_{m+1}$ is the normalization morphism. Thus, for general $t$, there is an admissible cover $[f'_t : B'_t \to T_t]$ over $\tilde{B}_{\{x_i(t), y_i(t)\}}$, which:

1. Specializes to the unique element of $\pi^{-1}(\tilde{B}_{m+1})$ as $t \to 0$.
2. Is distinct from the admissible cover constructed from $\mu_* (\mathcal{G}_{1,t}^* \mathcal{O}_{\mathbb{P}^1}(1))$ as in [HM, Thm. 5], where $\mu_t : B_{1,t} \to \tilde{B}_{\{x_i(t), y_i(t)\}}$ is the partial normalization morphism.

For general $t$, there exists a subcurve $B^n_{1,t} \subseteq B'_t$, such that $B^n_{1,t}$ is isomorphic to the normalization of $B_{1,t}$. After restricting $f'_t$ to $B^n_{1,t}$, we have a family of stable maps $\mathcal{G}_{2,t} : B^n_{1,t} \to \mathbb{P}^1$.

To simplify notation let $C_2 \subseteq B_{m+1}$ denote $B_{m+1}$ (respectively $C$) depending on whether $B_{1,t}$ is smooth (respectively singular). By (1) above, the stable maps $\mathcal{G}_{2,t}$ tend to a finite stable map of the form

$\mathcal{G}_{2,0} : B'_{m+1} \to \mathbb{P}^1,$

where $B'_{m+1}$ is the blow-up of $C_2$ at some set $S \subseteq \{(x_i, y_i) \mid 1 \leq i \leq m\}$ of cardinality at most $m$.

If $E_1, \ldots, E_\ell, \ell \leq m$ are the exceptional components of $B'_{m+1}$, then
\[ \text{deg}(\mathcal{G}_{2,0})|_{E_i} = 1, \ 1 \leq i \leq \ell, \ \text{deg}(\mathcal{G}_{2,0})|_{C_2} = f_{m+1}|_{C_2}. \]

If $\ell = 0$, the second item above implies $\mathcal{G}_{2,t}$ is not isomorphic to the composition
\[ \mathcal{G}_{1,t}^0 := \mathcal{G}_{1,t} \circ \mu_t, \]

where $\mu_t$ is the normalization of $B_{1,t}$, cf. the proof of Proposition 3.5.

Thus, after a finite base change, we have a family of stable maps
\[ \mathcal{G}_{2} : B' \to \mathbb{P}^1_{\Delta} \]
with fibre $\mathcal{G}_{2,0} : B'_{m+1} \to \mathbb{P}^1$ for $t = 0$ and fibre $\mathcal{G}_{2,t} : B^n_{1,t} \to \mathbb{P}^1$ for $t \neq 0$. Let $\mu' : B' \to B_2$ be the morphism which contracts the unstable components $E_i$ for $1 \leq i \leq \ell$ and is otherwise an isomorphism (this exists after a further base change). By Lemma 3.14 below applied to $\mathcal{G}_2^* \mathcal{O}_{\mathbb{P}^1_{\Delta}}(1)$, we have a line bundle
\[ \mathcal{M}_2 \in \text{Pic}(B_2), \]
with $\mathcal{M}_{2,t} \simeq \mathcal{G}_{2,t}^* \mathcal{O}_\mathbb{P}^1(1)$ for $t \neq 0$ and 
\[
\mathcal{M}_{2,0} \simeq f_{m+1|c_2}^* \mathcal{O}_\mathbb{P}^1(1)(\sum_{s \in S} s).
\]

Let $B_1^n$ denote the fibre surface obtained by first normalizing $B_1$ and then contracting any exceptional components. We will now show that we may assume $B_1^n \simeq B_2$. Firstly, after a finite base change $B_1^{n*} \simeq B_2^* := B_2 \times_\Delta \Delta^*$, for $\Delta^* = \Delta \setminus \{0\}$.

The claim is clear when $C_2 = C$, as then $B_2^n$ and $B_2$ are both families of stable curves. So assume $B_1 \simeq B_2^n$ is normal, which has $C_2 = B_{m+1}$. For $\alpha \in \{0, 1, \infty\}$ may assume we have sections $s_\alpha : \Delta \to B_1$ which specialize to general points of $R_{m+1}$. Further, by varying the natural PGL$(2)$ action, we may assume $\mathcal{G}_1 \simeq B_2 \times_\Delta \Delta^*$, for $\Delta^* = \Delta \setminus \{0\}$.

Let $\mathcal{M}_{1,0} \simeq f^* \mathcal{P}_1^*(1)(z)$, $z \in \{x_{m+1}, y_{m+1}\}$. By the base-point free pencil trick $h^0(\mathcal{M}_{1,0} \mathcal{M}_{2,t}) \geq 4$ for general $t$. But our hypotheses give $h^0(\mathcal{M}_{1,0} \mathcal{M}_{2,0}) = 3$ which is a contradiction.

Now consider the case $C_2 = B_{m+1}$, in which case $B_2^n \simeq B_1$. In the notation of the proof of Proposition 3.9 consider 
\[
\mathcal{L}_2 := \nu^* \mathcal{G}_1^* \mathcal{O}_{\Delta^*}^1(1) \mathcal{M}_2 \tau
\]
on a blow-up $\nu : B_1 \to B_1$. By precisely the same argument as for the line bundle $\mathcal{L}$ of Lemma 3.12, we have $h^0(\mathcal{L}_{2,0}) \leq 3$, which again gives a contradiction. \(\square\)

The lemma below was needed in the proof of Proposition 3.11.

**Lemma 3.14.** Let $(0, \Delta)$ be a smooth pointed curve and $\mathcal{C} \to \Delta$ a flat, proper family of nodal curves with $\mathcal{C}_t$ smooth for $t \neq 0$ and let $\mathcal{L}$ be a line bundle on $\mathcal{C}$. Assume the central fibre $\mathcal{C}_0$ has the form 
\[
\mathcal{C}_0 = T \cup E,
\]
where $T$ is nodal and connected and $E \simeq \mathbb{P}^1$, with $z = E \cap T$ a single point. Assume $\deg(\mathcal{L}_0)|_E = 1$. Let $\mu_E : \mathcal{C} \to \mathcal{C}_2$ be a contraction morphism of $E$, [L, §8.3.3]. Then there exists a line bundle $\mathcal{L}'$ on $\mathcal{C}_2$ with $\mathcal{L}' \simeq \mathcal{L}$ for $t \neq 0$ and $\mathcal{L}' \simeq (\mathcal{L}_0)_{\gamma^*}(z)$.

Note that after a finite base change a contraction morphism as in the lemma above always exists.

**Proof.** We omit the proof of this well-known fact. \(\square\)

**Remark 3.15.** Note, in the context of the proof of Theorem 3.11, if one can ensure that $\mathcal{G}_1 : B_1 \to \mathbb{P}_\Delta^*$ is as the case (2) of Proposition 3.9, we only need equation (4) of Lemma 3.12 to hold for sets $S'$ of cardinality at most $m+1$.

We now bootstrap on Proposition 3.11 in order to prove the induction step for Theorem 3.6. We will utilise Remark 3.15, together with Proposition 3.10.

**Proof of Theorem 3.6.** It remains to prove the induction step. Let $C$ be as in the hypotheses of the theorem for $n = m+1$. Set $A := f^* \mathcal{O}_\mathbb{P}^1(1)$. The assumptions imply that $\dim W_{a-m-1}^1(C) = 0$ and hence $h^0(A) = 2$. 
We claim that $\omega_C \otimes A^*$ is base point free. Indeed, otherwise there is some point $p \in C$ with $H^0(ev_p)$ base point free. Let $N := \text{Ker}(ev_p : \omega_C \otimes A^* \to O_p)$. There is some partial normalization $\mu : \tilde{C} \to C$ at $\delta \geq 0$ nodes with $\mu_* \tilde{N} = N$, for some line bundle $\tilde{N}$. Then $\omega_{\tilde{C}} \otimes \tilde{N}^* \in W_{a-m-\delta}(C)$ which implies $\dim W_{a-m-1-\delta}(C) \geq 1$ and thus $\dim W_{a-m-1}(C) \geq 1$, which is a contradiction.

We now argue exactly as in Proposition 3.11, with the variation that we allow $\Delta$ to be integral, nodal and that we replace Proposition 3.10 in place of Proposition 3.9, with $\Delta$ taken to be the component $J$ of $Z_m$, to construct $\nu : \tilde{G}_1 : \tilde{B}_1 \to \tilde{P}_1$ and $\tau$. Note that the conclusion of Proposition 3.11 implies that the locus of reducible curves in $J$ has codimension two about $p = [f_{m+1}, (x_i, y_i)]$, so that all the hypotheses of Proposition 3.10 are satisfied. Using Remark 3.15, this completes the proof of the induction step.

We end this section with a definition.

**Definition 3.16.** The Eagon-Northcott divisor $\mathcal{EN} \subseteq H_{2g-2k+1,g-k+1}$ is defined as the locus of covers $[f : C \to P^1]$ such that $\dim K_{g-k,1}(C, \omega_C) > g - k$.

In the next section, we shall extend $\mathcal{EN}$ as a determinantal locus over a partial compactification of $H_{2g-2k+1,g-k+1}$. From Theorem 3.1 and [SSW, Proposition 4.10], observe that we have the equality of subsets of $H_{2g-2k+1,g-k+1}$:

$$\mathcal{BN} = \mathcal{EN}.$$

4. **Extending the Eagon-Northcott Divisor**

In this section we construct an extension of the Eagon-Northcott divisor $\mathcal{EN}$ on the moduli space $\tilde{G}_{2a-1,a}^{\text{ns}} := \tilde{M}_{2a-1}(P^1, a)$ of stable maps from the previous section.

We shall construct the extended Eagon-Northcott divisor

$$\mathcal{EN} \subseteq \tilde{G}_{2a-1,a}^{\text{ns}}$$

by studying the minimal free resolutions of the scrolls attached to a cover $[f : C \to P^1] \in \tilde{G}_{2a-1,a}^{\text{ns}}$. Set $A := f^*(O_{P^1}(1)) \subseteq W_1^a(C)$. Since $f$ is finite and flat, $f_* O_C$ is locally free and we write $f_* O_C \cong O_{P^1} \oplus E_f^\tau$, where $E_f$ is the so-called Tschirnhausen bundle of $f$, admiting a splitting

$$E_f = O_{P^1}(e_1) \oplus \cdots \oplus O_{P^1}(e_{a-1}),$$

where $e_1 \leq \cdots \leq e_{a-1}$ are the scrollary invariants of $f$ and satisfy $e_1 + \cdots + e_{a-1} = 3a - 2$.

Dualising the morphism $f_* O_C$ leads to an exact sequence

$$0 \to E_f \to f_* \omega_f \to O_{P^1} \to 0.$$ 

We tensor the morphism $f^* (E_f) \to \omega_f$ by $f^* \omega_{P^1}$ and produce a morphism $f^* (E_f(-2)) \to \omega_C$, inducing a closed immersion, see [Sch1], or [CE]

$$j : C \to P(E_f(-2)).$$

Note that $E_f(-2)$ is a globally generated vector bundle on $P^1$ with $\deg(E_f(-2)) = a$. Denoting by $q : X := P(E_f(-2)) \to P^1$ the associated $(a - 1)$-dimensional scroll, we have a morphism

$$\iota : X \to P(H^0(P^1, E_f(-2))) \cong P^{2a-2},$$

such that $\iota \circ j : C \to P^{2a-2}$ is the canonical morphism of $C$, cf. [Sch1]. Observe that since $C$ has no disconnected nodes, $\omega_C$ is globally generated. Also observe that if $h^0(C, A^{\otimes 2}) = 3$, then $e_1 \geq 3$ and $\iota$ is a closed immersion.

The Picard group of the scroll $X$ is generated by the class of a ruling $R := \varphi^*(O_{P^1}(1))$ together with $H := O_X(1)$. Note that $H^0(X, H) \cong H^0(C, \omega_C)$, whereas $H^0(X, R) \cong H^0(C, A)$...
and $H^0(X, \mathcal{O}_X(H - R)) \cong H^0(C, \omega_C \otimes A^V)$. As already mentioned in the Introduction, the Eagon-Northcott complex, explicitly describes the minimal free resolution of

$$
\Gamma_X(H) := \bigoplus_{q \in \mathbb{Z}} H^0(X, H^{\otimes q}),
$$

as a $\text{Sym} H^0(X, H)$-module, see [Sch1]. This gives that

$$
K_{p,0}(X, H) = 0 \quad \text{for} \; p > 0, \; \text{whereas} \; K_{p,q}(X, H) = 0, \; \text{for} \; q \geq 2 \; \text{and any} \; p,
$$
as well as the canonical identifications

$$
K_{p,1}(X, H) \cong \bigwedge^{p+1} H^0(X, H - R) \otimes \text{Sym}^{p-1} H^0(X, R) \otimes \bigwedge^2 H^0(X, R)
$$

$$
\cong \bigwedge^{p+1} H^0(C, \omega_C \otimes A^V) \otimes \text{Sym}^{p-1} H^0(C, A) \otimes \bigwedge^2 H^0(C, A).
$$

In particular, $b_{p,1}(X, H) = p \binom{a}{p+1}$.

We record the following lemma, while skipping the proof:

**Lemma 4.1.** We have the vanishing $H^i(X, H^{\otimes q}) = 0$, for $i \geq 1$ and $q \geq 0$. Furthermore, $H^i(X, \mathcal{O}_X(-H)) = 0$, for $i \geq 2$.

Define the kernel bundles $M_H$ and $M_{\omega_C}$ on $X$ and $C$ respectively by the exact sequences

$$
0 \rightarrow M_H \rightarrow H^0(X, H) \otimes \mathcal{O}_X \rightarrow H \rightarrow 0
$$

$$
0 \rightarrow M_{\omega_C} \rightarrow H^0(C, \omega_C) \otimes \mathcal{O}_C \rightarrow \omega_C \rightarrow 0.
$$

As $C \subseteq X$ is linearly normal, $j^* M_H \cong M_{\omega_C}$. Note that $H^0(X, \wedge^p M_H) = H^0(C, \wedge^p M_{\omega_C}) = 0$, for $p \geq 1$. Further, we record the following short exact sequences:

$$
0 \rightarrow \bigwedge^{p+1} M_H \otimes \mathcal{O}_X ((q - 1)H) \rightarrow \bigwedge^{p+1} H^0(X, H) \otimes \mathcal{O}_X ((q - 1)) \rightarrow \bigwedge^p M_H \otimes \mathcal{O}_X (qH) \rightarrow 0,
$$

$$
0 \rightarrow \bigwedge^{p+1} M_{\omega_C} \otimes \omega_C^{\otimes (q-1)} \rightarrow \bigwedge^{p+1} H^0(C, \omega_C) \otimes \omega_C^{\otimes (q-1)} \rightarrow \bigwedge^p M_{\omega_C} \otimes \omega_C^{\otimes q} \rightarrow 0.
$$

We shall make use of the following vanishing statement.

**Lemma 4.2.** We have $H^i(X, \wedge^p M_H \otimes H^{\otimes q}) = 0$ for $i \geq 2$ and arbitrary $p, q \geq 0$.

**Proof.** By the sequence (5) and Lemma 4.1, it suffices to show $H^{i-1} \left( \wedge^{p-1} M_H \otimes H^{\otimes (q+1)} \right) = 0$. Continuing in this fashion, it suffices to show $H^1 \left( X, \wedge^{p-i+1} M_H \otimes H^{\otimes (q+i-1)} \right) = 0$. Since $H^1(X, H^{\otimes (q+i-1)}) = 0$, this amounts to $K_{p,q+i}(X, H) = 0$, which holds as $q + i \geq 2$. □

**Lemma 4.3.** There is an injective restriction map of linear syzygies

$$
\alpha_f : K_{a-1,1}(X, H) \rightarrow K_{a-1,1}(C, \omega_C).
$$

The map $\alpha_f$ is surjective if and only if the restriction map

$$
\beta_f : H^0 \left( X, \wedge^{a-2} M_H \otimes H^{\otimes 2} \right) \rightarrow H^0 \left( C, \wedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2} \right)
$$
is injective.
Proof. The map $\alpha_f$ fits into a commutative diagram with exact rows:
\[
\begin{array}{cccc}
0 & \to & \bigwedge^a H^0(X, H) & \to & H^0(X, \bigwedge^{a-1} M_H \otimes H) & \to & K_{a-1,1}(X, H) & \to & 0
\end{array}
\]
\[
\begin{array}{cccc}
\approx & & \text{res}_C & & \alpha_f
\end{array}
\]
\[
\begin{array}{cccc}
0 & \to & \bigwedge^a H^0(C, \omega_C) & \to & H^0(C, \bigwedge^{a-1} M_{\omega_C} \otimes \omega_C) & \to & K_{a-1,1}(C, \omega_C) & \to & 0
\end{array}
\]
Since $C \subseteq X$ is linearly normal, it follows that $\text{res}_C$ is injective, therefore $\alpha_f$ is injective as well.
On the other hand, by the snake lemma the surjectivity of $\alpha_f$ is equivalent to the surjectivity of $\text{res}_C$. From the kernel bundle description of Koszul cohomology, we write
\[
K_{a-2,2}(X, H) = \ker \left\{ H^1(X, \bigwedge^{a-1} M_H \otimes H) \to \bigwedge^a H^0(X, H) \otimes H^1(X, H) \right\}.
\]
Since $H^1(X, H) = 0$ and $K_{a-2,2}(X, H) = 0$, it follows $H^1(X, \bigwedge^{a-1} M_H \otimes H) = 0$. We write the following diagram with exact rows:
\[
\begin{array}{cccc}
0 & \to & H^0(X, \bigwedge^{a-1} M_H \otimes H) & \to & \bigwedge^{a-1} H^0(X, H) \otimes H^0(X, H) & \to & H^0(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}) & \to & 0
\end{array}
\]
\[
\begin{array}{cccc}
\text{res}_C & & \approx & & \beta_f
\end{array}
\]
\[
\begin{array}{cccc}
0 & \to & H^0(C, \bigwedge^{a-1} M_{\omega_C} \otimes \omega_C) & \to & \bigwedge^{a-1} H^0(C, \omega_C) \otimes H^0(C, \omega_C) & \to & H^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2})
\end{array}
\]
By the snake lemma, the surjectivity of $\text{res}_C$ is equivalent to the injectivity of $\beta_f$. \(\square\)

Koszul duality gives an isomorphism $K_{a-2,2}(C, \omega_C) \cong K_{a-1,1}(C, \omega_C)^\vee$, therefore we have a surjection
\[
H^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}) \to H^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}) / \bigwedge^{a-1} H^0(C, \omega_C) \otimes H^0(C, \omega_C) \cong K_{a-1,1}(C, \omega_C)^\vee.
\]
The composition of this map with $\alpha_f$ gives rise to a surjection
\[
\psi_f : H^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}) \to K_{a-1,1}(X, H)^\vee.
\]
Because $K_{a-2,2}(X, H) = 0$, from the second diagram in the proof of Lemma 4.3, it follows $\psi_f \circ \beta_f = 0$.

Lemma 4.4. We have a natural isomorphism $\ker(\psi_f) \cong H^2(X, \bigwedge^a M_H \otimes I_{C/X})^\vee$.

Proof. Since $H^1(X, \mathcal{O}_X) = 0$, the description of Koszul cohomology via kernel bundles yields the identification $K_{a-1,1}(X, H)^\vee \cong H^1(X, \bigwedge^a M_H)^\vee$. Using that $\bigwedge^{a-2} M_{\omega_C} \otimes \omega_C \cong \bigwedge^a M_{\omega_C}^\vee$, Serre-Duality gives the isomorphism
\[
H^0(C, \bigwedge^2 M_{\omega_C} \otimes \omega_C^{\otimes 2})^\vee \cong H^1(C, \bigwedge^a M_{\omega_C}),
\]
which enables us to identify the dual map $\psi_f^\vee$ with the restriction
\[
H^1(X, \bigwedge^a M_H) \to H^1(C, \bigwedge^a M_{\omega_C}).
\]
Then $\ker(\psi_f) \cong \text{coker}(\psi_f^\vee)^\vee \cong H^2(X, \bigwedge^a M_H \otimes I_{C/X})^\vee$, using also Lemma 4.2. \(\square\)

Putting the above pieces together, we have constructed a natural map
\[
\beta_f : H^0(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}) \to H^2(X, \bigwedge^a M_H \otimes I_{C/X})^\vee
\]
such that $b_{a-1,1}(C, \omega_C) > a - 1$ if and only if $\beta_f$ fails to be injective. We shall see that both sides of this map have the same dimension. This allows us to construct $\mathcal{EN}$ as the degeneracy locus of a morphism between vector bundles of the same rank on the space of stable maps.

Lemma 4.5. We have:

$$h^0(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}) = h^2(X, \bigwedge^a M_H \otimes I_{C/X}) = (2a - 2) \binom{2a - 1}{a} - a + 1.$$ 

Proof. As already pointed out $H^1(X, \bigwedge^{a-1} M_H \otimes H) = 0$. Therefore

$$h^0(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}) = (2a - 1) \binom{2a - 1}{a} - h^0(X, \bigwedge^{a-1} M_H \otimes H),$$

by the short exact sequence (5). We further have a short exact sequence

$$0 \rightarrow \bigwedge^a H^0(X, H) \rightarrow H^0(X, \bigwedge^a M_H \otimes H) \rightarrow K_{a-1,1}(X, H) \rightarrow 0,$$

thus using that $b_{a-1,1}(X, H) = a - 1$, we find $h^0(X, \bigwedge^{a-1} M_H \otimes H) = a - 1 + \binom{2a - 1}{a}$, which leads to the claimed formula for $h^0(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2})$.

Using Lemma 4.4, we compute:

$$h^2(X, \bigwedge^a M_H \otimes I_{C/X}) = \dim(\ker \psi_f) = h^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}) - b_{a-1,1}(X, H).$$

Recall that $b_{a-1,1}(X, H) = a - 1$. The Riemann-Roch theorem (still valid for a nodal curve $C$ with no disconnecting nodes) gives

$$h^0(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}) = \chi(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}) = (4a - 2) \binom{2a - 2}{a},$$

which finishes the proof. □

We now explain how the above considerations can be carried out in a relative setting. Let

$$\xymatrix{ \mathcal{C} \ar[d]_{\nu} \ar[r]^{f} & \mathcal{P} \ar[d]^\mu \\
\tilde{\mathcal{G}}^{\text{ns}}_{2a-1,a} & }$$

be the universal degree $a$ cover, where $\mathcal{P} = \tilde{\mathcal{G}}^{\text{ns}}_{2a-1,a} \times \mathbb{P}^1$. The universal Tschirnhausen bundle $\mathcal{E}_f$ on $\mathcal{P}$ fits into an exact sequence:

$$0 \rightarrow \mathcal{E}_f \rightarrow f_* \omega_f \rightarrow \mathcal{O}_{\mathcal{P}} \rightarrow 0.$$ 

We further have the projective bundle $\varphi : \mathcal{X} := \mathbb{P}(\mathcal{E}_f \otimes \omega_\mu) \to \mathcal{P}$ and a closed immersion $j : \mathcal{C} \hookrightarrow \mathcal{X}$. Set $h := \mu \circ \varphi : \mathbb{P}(\mathcal{E}_f \otimes \omega_\mu) \to \tilde{\mathcal{G}}^{\text{ns}}_{2a-1,a}$. By Grauert’s Theorem, $h_*(\mathcal{O}_\mathcal{X}(1))$ is a vector bundle of rank $2a - 1$. Define the determinant $\xi := \det h_* (\mathcal{O}_\mathcal{X}(1))$. The evaluation map $h^* h_*(\mathcal{O}_\mathcal{X}(1)) \to \mathcal{O}_\mathcal{X}(1)$ is furthermore surjective, thus we can define the kernel bundle $\mathcal{M}$ by

$$0 \rightarrow \mathcal{M} \rightarrow h^* h_*(\mathcal{O}_\mathcal{X}(1)) \to \mathcal{O}_\mathcal{X}(1) \rightarrow 0.$$
Then $\mathcal{M}$ restricts to the kernel bundle $M_H$ for each scroll induced by an element $[C \to \mathbb{P}^1]$. Note that $j$ is defined by the surjection $f^*(E \otimes \omega_B) \to \omega_f \otimes f^*\omega_B \cong \omega_f$, hence $\mathcal{O}_C(1) \cong \omega_f$. Set
\[
\mathcal{F}_1 := h^a\left(\bigwedge^2 \mathcal{M} \otimes \mathcal{O}_C(2)\right) \otimes \xi^\vee,
\]
which is a vector bundle of rank $(2a - 2)\binom{2a-1}{a} - a + 1$, by Lemma 4.5. Set
\[
\mathcal{F}_2 := h^a\left(\bigwedge^{a-2} \mathcal{M} \otimes \mathcal{O}_C(2)\right) \otimes \xi^\vee,
\]
which is a vector bundle of rank $(2a - 2)\binom{2a-1}{a}$. Restriction to $C$ induces a morphism
\[
\beta : \mathcal{F}_1 \to \mathcal{F}_2.
\]
Relative duality gives the isomorphism
\[
R^1\nu_*\left(\bigwedge^a \mathcal{M}|_C\right) \cong \left(\nu_*\left(\bigwedge^a \mathcal{M}|_C \otimes \omega_B\right)\right)^\vee \cong \mathcal{F}_2^\vee,
\]
using $\det(\mathcal{M}) \cong h^*\xi \otimes \mathcal{O}_C(-1)$. Define the rank $a - 1$ vector bundle by $\mathcal{F}_3 := R^1h_*\left(\bigwedge^a \mathcal{M}\right)^\vee$.

The dual of the restriction morphism $\psi^\vee : R^1h_*\left(\bigwedge^a \mathcal{M}\right) \to R^1\nu_*\left(\bigwedge^a \mathcal{M}|_C\right)$ gives a morphism
\[
\psi : \mathcal{F}_2 \to \mathcal{F}_3
\]
with fibre over a moduli point $[f : C \to \mathbb{P}^1]$ equal to $\psi_f$. As already explained, $\psi \circ \beta = 0$.

We get a short exact sequence of vector bundles over $\mathcal{G}_{2a-1,a}^{\text{ns}}$:
\[
0 \to R^1h_*\left(\bigwedge^a \mathcal{M}\right) \to R^1\nu_*\left(\bigwedge^a \mathcal{M} \otimes \mathcal{O}_C\right) \to R^2h_*\left(\bigwedge^a \mathcal{M} \otimes \mathcal{I}_{C/X}\right) \to 0,
\]
where $\mathcal{F}_4 := R^2h_*\left(\bigwedge^a \mathcal{M} \otimes \mathcal{I}_{C/X}\right)$ is a vector bundle of rank $(2a - 2)\binom{2a-1}{a} - a + 1$ by Lemma 4.5. Thus we may canonically identify
\[
\text{Ker}(\psi) \cong \mathcal{F}_4^\vee
\]
and we have an induced morphism between vector bundles $\beta : \mathcal{F}_1 \to \mathcal{F}_4^\vee$ globalizing the morphisms $\beta_f$ as the moduli point $[f] \in \mathcal{G}_{2a-1,a}^{\text{ns}}$ varies. Since $\text{rk}(\mathcal{F}_1) = \text{rk}(\mathcal{F}_4)$, we define the extended Eagon-Northcott divisor
\[
\mathcal{E}\mathcal{N} \subseteq \mathcal{G}_{2a-1,a}^{\text{ns}}
\]
as the degeneracy locus of $\beta$. By the results of the previous chapter, this is a genuine divisor.

Define $\mathcal{E}\mathcal{N}^{\text{sm}}$ as the union of all components of $\mathcal{E}\mathcal{N}$ containing an element $[f : C \to \mathbb{P}^1]$, with $C$ being a smooth curve and all ramification simple. The following lemma is a direct consequence of Theorem 3.7.

**Lemma 4.6.** Let $C$ be a curve of genus $g$ and gonality $k \leq \frac{g+1}{2}$ satisfying bpf-linear growth. For $1 \leq i \leq g - 2k + 1$, choose $(x_i, y_i)$ to be general pairs of points on $C$ and let $B$ be the semistable curve given as the union of $C$ with $g - 2k + 1$ smooth rational curves $R_i$ such that each $R_i$ meets the rest of $B$ precisely at $x_i, y_i$ for $1 \leq i \leq g - 2k + 1$. Let
\[
[f : B \to \mathbb{P}^1] \in \mathcal{G}_{2g-2k+1,g-k+1}^{\text{ns}}
\]
be a morphism with $\deg(f|_C) = k$ and $f|_{R_i}$ an isomorphism. Assume that $f|_C$ is the unique minimal pencil on $C$ and, further, $h^0(C, f^*\mathcal{O}_{\mathbb{P}^1}(2)) = 3$. Then $[f] \notin \mathcal{E}\mathcal{N}^{\text{sm}}$. 

Proof. Consider the closure $\overline{\mathcal{E}N}^\text{sm} \subseteq \mathcal{M}_{2g-2k+1}(\mathbb{P}^1, g-k+1)$ in the moduli space of stable maps. We have the projections $\pi' : \mathcal{M}_{2g-2k+1}(\mathbb{P}^1, g-k+1) \to \mathcal{M}_{2g-2k+1}$, as well as the projection $\pi$ from the space of admissible covers. There is an equality of closed sets $\pi'(\overline{\mathcal{E}N}^\text{sm}) = \pi(\mathcal{E}N)$, since $\mathcal{M}_{2g-2k+1}(\mathbb{P}^1, g-k+1)$ is a $PGL(2)$-cover of $\overline{\mathcal{H}}_{2g-2k+1, g-k+1}$ over the open set of morphisms with smooth source and simple ramification. By Theorem 3.7, the point $[D] \in \mathcal{M}_{2g-2k+1}$ defined by the stabilization of $B$ does not lie in $\pi'(\overline{\mathcal{E}N}^\text{sm})$, therefore, $[f] \notin \overline{\mathcal{E}N}^\text{sm}$.

To complete the proof of Theorem 0.4 we need to show that, in the situation of Lemma 4.6, the point $[f]$ does not lie in the extended Eagon-Northcott divisor $\overline{\mathcal{E}N}$. Note that $[f]$ lies in precisely one boundary divisor of $\mathcal{G}^\text{ns}_{2g-1,a}$, namely the divisor $\Delta$ whose general point corresponds to maps $h : C \to \mathbb{P}^1$, where $C$ is a union of two curves $C_1$ and $C_2$ of genera $g-1$ and 0 respectively, meeting at two points, and such that deg($h_{|C_1}$) = $a-1$ and deg($h_{|C_2}$) = 1. Since $\overline{\mathcal{E}N}$ is pure of codimension one, we need to show that $\overline{\mathcal{E}N}$ does not contain $\Delta$. We carry this out in the next section, using $K3$ surfaces.

5. K3 SURFACES AND SCHREYER’S CONJECTURE

We start by considering a $K3$ surface $X = X_d$ with Picard group generated by two classes $L$ and $E$ with self intersections given by $(L)^2 = 4d - 4$, $(E)^2 = 0$ and $(L \cdot E) = d$, for $d \geq 3$. By performing Picard-Lefschetz transformations and a reflection if necessary, we may assume that $L$ is big and nef.

Lemma 5.1. For $X$ as above, the class $L$ is base point free and $E$ is the class of a smooth elliptic curve.

Proof. We firstly show that $L$ is base point free. As $L$ is big and nef, it suffices to show that there is no smooth elliptic curve $F$ with $(L \cdot F) = 1$, see [M, Proposition 8]. Assume such $F$ exists, and write $F = aL + bE$ for $a, b \in \mathbb{Z}$. As $F$ is smooth and elliptic, $(F)^2 = 0$, implying $0 = (aL + bE) \cdot F = a + b(E \cdot F) = a(1 + db)$. If $a = 0$, then $(L \cdot F) = bd \neq 1$, since $d \geq 2$, so $db = -1$, which is again impossible. Thus $L$ is base point free.

We next show that $E$ is the class of a smooth elliptic curve. As $(E)^2 = 0$ and $E$ is primitive, it suffices to show that $E$ is nef. Since $(E : L) > 0$, and $L$ is big and nef, $E$ is effective. Suppose $E$ is not nef. Then there exists a smooth, rational curve $R$ with $(R \cdot E) < 0$. Write $R = aL + bE$ for $a, b \in \mathbb{Z}$. Then $(R \cdot E) < 0$ implies $a < 0$. As $(R)^2 = -2$ and $R$ is effective, we must have $b > 0$. We have $-2 = (R)^2 = R \cdot (aL + bE) = a(R \cdot L) + b(R \cdot E) = a((R \cdot L) + bd)$, which is impossible for $d \geq 3$.

We now discuss the Brill-Noether theory of a smooth curve $C \in |L|$. To that end, we follow [K1, §2] which works in the situation of a higher rank Picard lattice containing the lattice Pic($X_d$).

Lemma 5.2. Let $D \in \text{Pic}(X_d)$ be effective with $(D)^2 \geq 0$. Assume in addition $L - D$ is effective and $(L - D)^2 > 0$. Then $D = cE$, for some integer $c$.

Proof. This is a slight modification of [K1, Lemma 2.5]. Write $D = aL + bE$. As $L - D$ is effective and $E$ nef, $(L - D) \cdot E = (1 - a)(L \cdot E) \geq 0$, so $a \leq 1$. From $(D \cdot E) \geq 0$, we obtain $a \geq 0$. If $a = 1$, then $(L - D)^2 = b^2(E)^2 = 0$, so we must have $a = 0$ as required.

The next lemma describes the Brill-Noether behaviour of curves in the linear system $|L|$.

Lemma 5.3. Let $C \in |L|$ be a smooth curve. Then Cliff($C$) = $d - 2$ and $W^1_d(C)$ is reduced and consists of the single point $\mathcal{O}_C(E)$.
Proof. The proof that Cliff$(C) = d - 2$ is essentially the same as [K1, Lemma 2.6]. Arguing as in [K1, Lemmas 2.7, 2.8], we see that $W^1_d(C)$ is set-theoretically a single point, namely $\mathcal{O}_C(E)$.

It remains to establish that $W^1_2(C)$ is reduced, which amounts to showing that $h^0(\mathcal{O}_C(2E)) = 3$. From the exact sequence

$$0 \to \mathcal{O}_X(E) \to \mathcal{O}_X(2E) \to \mathcal{O}_E(2E) \cong \mathcal{O}_E \to 0,$$

we deduce $h^1(X, 2E) = 1$ and then $h^0(X, 2E) = 3$ by Riemann–Roch. By the exact sequence

$$0 \to \mathcal{O}_X(2E - C) \to \mathcal{O}_X(2E) \to \mathcal{O}_C(2E) \to 0,$$

it suffices to show that neither $2E - C$ nor $C - 2E$ is effective. As $(E \cdot 2E - C) < 0$ and $E$ is nef, $2E - C$ is not effective. Now suppose $C - 2E$ is effective with integral components $R_1, \ldots, R_\ell$, for $\ell \geq 1$. We write $R_i = a_i L + b_i E$, for integers $a_i, b_i$, with $\sum_{i=1}^\ell a_i = 1$ and $\sum_{i=1}^\ell b_i = -2$. As $(E \cdot R_i) \geq 0$, we find $a_i \geq 0$ for all $i$. Without loss of generality, we may assume $a_1 = 1$ and $a_i = 0$ for $2 \leq i \leq \ell$. As $R_i$ is integral, we must then have $b_1 = 1$ for $i > 1$. Thus $R_1 = L - (\ell + 1)E$, which implies $(R_1)^2 = 4d - 4 - 2d(\ell + 1) \leq -4$, contradicting that $R_1$ is integral. $\square$

We can now prove Theorem 0.4, that is, establish the Schreyer Conjecture.

Proof of Theorem 0.4. Let $[f : B \to \mathbb{P}^1]$ be as in the statement of Lemma 4.6. By an argument along the lines of [V1, Corollary 1], we have an injection $K_{g-k,1}(C, \omega_C) \hookrightarrow K_{g-k,1}(B, \omega_B)$. For the sake of completeness we recall the proof.

The Mayer-Vietoris sequence induces an injection $H^0(C, \omega_C) \hookrightarrow H^0(B, \omega_B)$, as well as the composition of injections $H^0(C, \omega_C^2) \hookrightarrow H^0(C, \omega_C^2(\sum_{i=1}^{g-2k+1}(x_i + y_i))) \hookrightarrow H^0(B, \omega_B^2)$. We then get a commutative diagram

$$
\begin{array}{ccc}
\wedge^{g-k+1}H^0(\omega_C) & \overset{\delta_0}{\longrightarrow} & \wedge^{g-k}H^0(\omega_C) \otimes H^0(\omega_C) \\
\downarrow & & \downarrow \\
\wedge^{g-k+1}H^0(\omega_B) & \overset{\delta_0'}{\longrightarrow} & \wedge^{g-k}H^0(\omega_B) \otimes H^0(\omega_B)
\end{array}
$$

The conclusion now follows from the existence of maps, see also [AN, Lemma 7.1]

$\wedge : \wedge^{g-k}H^0(\omega_C) \otimes H^0(\omega_C) \to \wedge^{g-k+1}H^0(\omega_C)$ and $\wedge' : \wedge^{g-k}H^0(\omega_B) \otimes H^0(\omega_B) \to \wedge^{g-k+1}H^0(\omega_B)$, with $\wedge \circ \delta_0 = \pm(g-k)\text{Id}$ and $\wedge' \circ \delta_0' = \pm(g-k)\text{Id}$.

We secondly claim that $[f]$ does not lie in the extended Koszul divisor $\mathcal{E}\mathcal{N}$. In light of the injective map above, this will complete the proof. As $[f]$ lies in exactly one boundary divisor, namely $\Delta$, all that remains is to show that the divisor $\mathcal{E}\mathcal{N}$ does not contain $\Delta$. By Lemma 5.3, we know that any smooth curve $C \in |L|$ on the K3 surface $X = X_{g-k+1}$ satisfies $b_{g-k,1}(C, \omega_C) = g - k$. By the Lefschetz Theorem for Koszul cohomology [G], the same holds for any integral nodal curve $C_0 \in |L|$. As any integral, nodal curve $C_0$ (with at least one node) defines a point in $\Delta$, it suffices to show that such curves exist for the general $X_{g-k+1}$.

In order to do this, it suffices to take $2g - 2k + 1 \geq 8$, as the conclusion of the Theorem is well-known for $g \leq 8$ by [Sch1]. Indeed, if $g - k \leq 3$, then, since we are assuming $g \geq 2k - 1$, we must have $k \leq 4$ and $g \leq 7$. The class $L - E$ is very ample for a general K3 surface $X_{g-k+1}$ general with the given Picard lattice, by degenerating to the K3 surface $Y_{2g-2k+1}$ from [K1, Lemma 2.3]. Choose a curve $C_1 \in |L - E|$ meeting a smooth elliptic curve $E_0 \in |E|$ transversally, and consider the nodal curve $C_1 \cup E_0$. Pick any node $p_1 \in C_1 \cup E_0$. Then, by [Ta, Theorem 3.8], the moduli space $V_{1}(X_2)$ parametrising deformation of $C_1 \cup E_0$ preserving the assigned node $p_1$ is smooth near $(C_1 \cup E_0, p_1)$ of dimension $2g - 2k$. As $\dim |L - E| + \dim |E| = g - k + 1 < g - 1$
for \( k \geq 3 \), there exist integral, nodal curves \( C_0 \in |L| \) with exactly one node, completing the proof.

\[ \square \]

**References**


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