Abstract. We construct new Eagon–Northcott cycles of arbitrary codimension on Hurwitz space and compare their classes to Kleiman’s multiple point loci. Applying this construction towards the classification of Betti tables of canonical curves, we find that the value of the extremal Betti number records the number of minimal pencils. The result holds under natural transversality hypotheses equivalent to these virtual cycles having a geometric interpretation. We further analyse the classically studied case of two minimal pencils, showing that, in this case, the transversality hypotheses hold generically.

0. Introduction

The canonical ring $\Gamma_C(\omega_C) := \bigoplus_{m \in \mathbb{N}} H^0(C, \omega_C^{\otimes m})$ of a smooth curve $C$ has long been of central interest in algebraic geometry. In order to describe its structure, define

$$b_{i,j}(C, \omega_C) := \dim \text{Tor}^i_S(\Gamma_C(\omega_C), C)^{i+j},$$

where $S$ denotes the polynomial algebra $\text{Sym}(H^0(C, \omega_C))$. The invariants $b_{i,j}(C, \omega_C)$ determine the terms appearing in the minimal free resolution of $\Gamma_C(\omega_C)$ as an $S$ module and encode deep information about the algebraic structure of the canonical ring. For example, the classical theorem of Noether–Babbage–Petri states that a non-hyperelliptic, non-trigonal canonical curve which is not a plane quintic is an intersection of quadric hypersurfaces. This can be made more concise by stating that, for any curve $C$ of gonality at least 4 which is not a plane quintic, we have $b_{1,q}(C, \omega_C) = b_{0,q}(C, \omega_C) = 0$ for $q \geq 2$. Furthermore, the number of quadrics required to cut out $C$ is given by the quantity $b_{1,1}(C, \omega_C)$.

One sees from Noether–Babbage–Petri’s result that the first nontrivial Betti number $b_{1,1}(C, \omega_C)$ carries interesting geometric information. It is natural to ask what can be said about the remaining Betti numbers. As we explain below, the last Betti number on the linear strand $b_{i,1}(C, \omega_C)$ is the most interesting of all these invariants. Moreover, the loci of curves $C$ where the extremal Betti number is kept constant are themselves very interesting from the point of view of moduli theory. These loci turn out to have a surprisingly close relationship to multiple point loci, a fertile area of research within intersection theory, see [Kl]. This provides a connection between syzygies and the study of the enumerative geometry of Hurwitz cycles in $\mathcal{M}_g$, see for instance [FaP].

Thanks to the combined work of Green, Teixidor i Bigas, Voisin and Aprodu, one knows precisely which invariants $b_{i,j}(C, \omega_C)$ are nonzero for a general curve $C$ of gonality $k$, see [G1], [Te], [V2], [V3], [Ap2]. Let $C$ be a curve of genus $g$ and gonality $k$, satisfying the following linear growth condition on the dimension of the moduli space of linear series of dimension two:

$$\dim G^1_{k+m}(C) \leq m, \quad \text{for } 0 \leq m \leq g - 2k + 1.$$ 

Then the shape of the Betti tables of the canonical curve, i.e. the table with $(i,j)^{th}$ entry $b_{j,i}(C, \omega_C)$, is illustrated in Figure 1 below.

By a well-known result of Hilbert

$$\sum_i (-1)^i b_{i,j-i}(C, \omega_C) = \sum_i (-1)^i \binom{g-1}{i} h^0(\omega_C^{\otimes j-i})$$
and so we know the alternating sum of entries along the diagonal of the Betti table. Hence, in order to classify Betti tables of a canonical curve of fixed gonality, it suffices to consider the first row of entries $b_{i,1}$. Thus the only interesting Betti numbers are the portion
\[ b_{k-1,1}, \ldots, b_{g-1,1}, \]
of the 2-linear strand $b_{i,1}$, where there exist two non-zero entries on the diagonal (the numbers $b_{g-2,3} = b_{0,0} = 1$ being uninteresting).

Very little has been known about the values of the non-zero entries $b_{p,q}$ of the table and how they reflect the geometry of the curve $C$. In genus 9 or less, a classification of the possible Betti tables of canonical curves is available due to the work of Schreyer and Sagraloff [Sch1], [S1], whereas conjectural tables for genus 10 and 11 have been produced via computer experiment [Sch2]. The “extremal” Betti number $b_{g-1,1}$ stands out from these tables as the most interesting of all the $b_{i,j}$, as it seems to be responsible for much of the variance in the Betti tables. Furthermore, the extremal Betti number seems to have a remarkably close relationship to the geometry of the curve $C$. Indeed, Schreyer’s experiments seem to suggest the intriguing formula
\[ b_{g-1,1}(C, \omega_C) = m(g - k) \]
where $m$ counts the number of minimal pencils of $C$, i.e. degree $k$ maps $C \rightarrow \mathbb{P}^1$, with multiplicity.

The objective of this paper is to provide an explanation for Schreyer’s observation by utilising the geometry of Hurwitz space and Kleiman’s study of multiple-point loci. Before stating our result, note that certain exceptions to the above formula are immediately apparent. For example, if $C$ is a smooth plane sextic, then $g = 10$ and the extremal Betti number is $b_{6,1}(C, \omega_C) = 27$, which is not even divisible by 6 (and, further $k = 5$, so the linear strand has the wrong length). For another example, if $C$ is a genus 11 curve admitting a degree three cover of an elliptic curve, then one again expects $b_{6,1}(C, \omega_C) = 27$.

To avoid such exceptions, and guided by the results on Green’s Conjecture stated above, it is very natural to first impose a regularity assumption on the dimension of Brill–Noether loci before trying to classify Betti tables. Following [FK3], a curve $C$ of genus $g$ and gonality $k \leq \frac{g+3}{2}$ satisfies bpf-linear growth provided we have the dimension estimates
\begin{align*}
\dim G^1_{k+m}(C) &\leq m, \quad \text{for } 0 \leq m \leq g - 2k + 1 \\
\dim G^1_{k+m, \text{bpf}}(C) &< m, \quad \text{for } 0 < m \leq g - 2k + 1.
\end{align*}
Bpf-linear growth holds for a general element of $M_g(2,k)$, assuming $8 \geq g > (k-1)^2$, see [FK3].

The condition above appears implicitly in the well-known works of Martens–Mumford and Keem on the dimensions of Brill–Noether loci, see [ACGH, Ch. IV], in particular §5 and Exercise G.

For low values of $k$, curves violating bpf-linear growth tend to be either plane curves or low degree covers of curves of low genus. Furthermore, work of Aprodu–Farkas easily gives that if $C$ is a curve of non-maximal gonality which can be abstractly embedded on a K3 surface and
if, furthermore, the only line bundles computing the Clifford index are minimal pencils, then $C$ satisfies bpf-linear growth. See the Appendix for more on how the bpf-linear growth condition appears in the literature.

Let $C$ be a canonical curve satisfying bpf-linear growth. A variational approach of Hirschowitz–Ramanan converts the problem of determining special Betti numbers of $C$ into a study of the geometry of moduli spaces and their effective cone, [HR]. Bounds on the number of syzygies then come from computations of the order of vanishing of particular divisors at the point.

Whilst Hirschowitz–Ramanan compare the classes of divisors on the moduli space of curves, we work here with cycles of higher codimension, using as input Kleiman’s proof of Herbert’s multiple point formula, [Kl]. Let

$$f : X \to Y$$

be an unramified, proper morphism of smooth varieties. For any fixed integer $m$, Herbert’s multiple point formula computes the class of the loci of those $y \in Y$ with

$$\#f^{-1}(y) \geq m,$$

under the assumption that $f$ is self-transverse meaning that $T_{x_1}(X), \ldots, T_{x_m}(X)$ are in general position in $T_y(Y)$, where $\{x_1, \ldots, x_m\} = f^{-1}(y)$.

To apply this to our situation, consider the moduli space $\overline{M}_{g,k}(\mathbb{P}^1, \{0, 1, \infty\})$ of stable maps of genus $g$ and irreducible component $k$ to $\mathbb{P}^1$, with fixed base points over $0, 1, \infty$. Denote by $M_{g,k}(\mathbb{P}^1, \{0, 1, \infty\})$ the unique irreducible components of $\overline{M}_{g,k}(\mathbb{P}^1, \{0, 1, \infty\})$ such that the general point is a morphism with smooth base $C$. This space comes with a projection

$$\pi : M_{g,k}(\mathbb{P}^1, \{0, 1, \infty\}) \to \overline{M}_{g,3},$$

defined by sending a marked stable map to its base.

We first focus on the divisorial case $g = 2k - 1$. Consider the space $M^0_{2k-1,3}$ of irreducible, automorphism-free curves and let $\mathcal{H}(0) \subseteq M^0_{2k-1,3}$ be the locus such that $\pi$ is self-transverse over $\mathcal{H}(0)$. Set $\mathcal{H}(1) = \pi^{-1}(\mathcal{H}(0))$. Our first task is the construction of a virtual cycle $\mathcal{E}N$ in the chow group $A^m(\mathcal{H}(1))$. Under a transversality assumption as explained below, $\mathcal{E}N$ represents the locus

$$\{C \to \mathbb{P}^1 \in \mathcal{H}(1) \mid b_{g,k,1}(C, \omega_C) > m(g-k)\}.$$

The construction of $\mathcal{E}N$ requires a considerable amount of work, but it has the advantage that it is constructed in an iterative way out of determinantal loci, whose classes one may always, in theory, compute [HT]. Furthermore, following Kleiman we have the multiple-point cycle $B\mathcal{N}_{m+1} \in A^m(\mathcal{H}(1))$ corresponding to curves with $m + 1$ minimal pencils. Our first result is then:

**Theorem 0.1.** We have the following equality of virtual cycles in $A^m(\mathcal{H}(1))$

$$\mathcal{E}N_m = (k - 1)B\mathcal{N}_{m+1}.$$

**Theorem 0.1** can be seen as an intersection–theoretic explanation for the experimental observation $b_{g,k,1}(C, \omega_C) = m(g-k)$ with $m = \#W^1_k(C)$, in the case $g = 2k - 1$.

We next upgrade the above virtual computation into a geometric statement. In order to do this, we need to demand that the pencils are in general position as we now explain. Let $C$ be a smooth curve of gonality $k$ and choose three general points $p, q, r \in C$. We say that the minimal pencils are infinitesimally in general position if $\pi : M_{g,k}(\mathbb{P}^1, \{0, 1, \infty\}) \to \overline{M}_{g,3}$ is self-transverse over $(C, (p, q, r))$. This condition essentially means that the deformation theory of the collection of pencils is unobstructed, i.e. each subset of pencils can be independently deformed and is necessary for the Brill–Noether cycles $B\mathcal{N}_{m+1}$ to carry geometric meaning. See Section 3.3 for an in-depth discussion on this condition.
In order for the Eagon–Northcott cycles to carry their natural geometric meaning, we need to impose a second condition which is more global in nature. Let \( L_1, \ldots, L_m \) be minimal, pencils of type I\(^1\) and fix a general divisor \( T \) of degree \( g - 1 - k \) general points on \( C \). For each minimal pencil \( L_i \) of type I one may naturally associate a rank 4 quadric \( Q_i \subseteq \mathbb{P}^{g-1} \) following an idea of Green [G2], see Section 3.1. We say that \( L_1, \ldots, L_m \) are in geometrically general position if there are no linear relations amongst the associated quadrics \( \{Q_1, \ldots, Q_m\} \subseteq |\mathcal{O}_{\mathbb{P}^{g-1}}(2)| \). If the set \( \{L_1, \ldots, L_m\} \) of minimal pencils are both infinitesimally and geometrically in general position, then we say that the minimal pencils are in \textit{general position}.

We may now state our second result. Recall that for any pencil \( L \) on a projective variety, one can associate the scroll \( X_L \) swept out by the span of the divisors \( D \in |L| \), [Sch1].

**Theorem 0.2.** Let \( C \) be a smooth curve of genus \( g \) and non-maximal gonality \( k \leq \lceil \frac{g+1}{2} \rceil \), satisfying bpf-linear growth. Assume the minimal pencils are in general position and have ordinary ramification. Then

\[
b_{g-k,1}(C, K_C) = m(g - k),
\]

where \( m = #W^1_k(C) \). Furthermore, under these circumstances all extremal linear syzygies arise from scrolls, i.e. there is an isomorphism of syzygy spaces

\[
\bigoplus_{i=1}^{m} K_{g-k,1}(X_{f_i}, \mathcal{O}_{X_{f_i}}(1)) \simeq K_{g-k,1}(C, \omega_C)
\]

where \( X_{f_1}, \ldots, X_{f_m} \) are the scrolls associated to the minimal pencils.

Since the syzygies of the scrolls \( X_{f_i} \) can be explicitly described via the Eagon–Northcott complex, the above theorem provides an explicit description of syzygies in the group \( K_{g-k,1}(C, \omega_C) \).

The assumptions of Theorem 0.2 are necessary. Some version of infinitesimal general position is certainly required, as, for instance, the statement fails if \( m = 1 \) but \( W^1_k(C) \) is not reduced, [SSW, Prop. 10]. Geometrically general position is required to ensure that the syzygies of the \( m \) scrolls contribute independently to the syzygies of the canonical curve in the final position of the linear strand. This fails for a general curve of even genus and maximal gonality.

It is a very interesting problem to try and understand under which conditions there exist genus \( g \) curves admitting precisely \( m \) minimal pencils as above. If \( m = 1 \), such curves always exist for \( k \leq \lceil \frac{g+1}{2} \rceil \) by a result of Arbarello and Cornalba, [AC2]. In the case \( m = 2 \), Jongmans proved that if there are genus \( g \) curves with two mutually independent pencils, then \( g > (k - 1)^2 \), [J]. Assuming \( g \geq 8 \), Coppens has proven that this condition is sufficient, [C]. We show:

**Theorem 0.3.** Let \( g \geq 8 \), \( k \geq 6 \) and \( g > (k - 1)^2 \). Then there exist smooth curves of genus \( g \) with precisely two minimal pencils satisfying the assumptions of Theorem 0.2. In particular, \( b_{g-k,1}(C, K_C) = 2(g - k) \) for such curves.

The space of curves with two independent minimal pencils is irreducible, [Ty].

For \( m \geq 3 \), there are few decisive results on the existence of curves with \( m \) pencils. To give an example of what can occur, based on computer experiments one expects the existence of curves of genus 11 and gonality 6 with \( m \) pencils in general position for \( 1 \leq m \leq 10 \), [Sch2] and [BS]. On the other hand, is seems that there do not exist curves of genus 11 and gonality 6 with \( 11 \) pencils in general position. More precisely, as soon as a curve with \( g = 11 \), \( k = 6 \) has 11 pencils it appears to additionally have a \( 12^{th} \) pencil and, likewise, as soon as \( b_{g-k,1} > 50 \), one in fact has \( b_{g-k,1} \geq 60 \).

\(^1\)A line bundle \( L \) with \( h^0(L) = 2 \) is said to be of type I if \( W^1_k(C) \) is smooth and zero dimensional at \( [L] \). This is equivalent to having \( h^0(L^2) = 3 \).
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Glossary of Moduli Spaces

$\overline{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\})$: The moduli space of genus $g$ stable maps to $(\mathbb{P}^1)^m$ in the class $k[\Delta]$ and with three base points over $(\alpha, \ldots, \alpha)$ for $\alpha \in \{0, 1, \infty\}$.

$\mathcal{M}^{ns}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\})$: The open substack of $\overline{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\})$ parametrising morphisms $f : C \to (\mathbb{P}^1)^m$ such that $C$ has non-separating nodes and further $f_i$ is finite with $h^0(f_i^*\mathcal{O}_{\mathbb{P}^1}(1)) = 2$, for each factor $f_i$ of $f$. Further, we demand that $f_i$ is etale near the base points $(p, q, r) \in X$ for all $1 \leq i \leq m$.

$\mathcal{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\})$: The closure of $\mathcal{M}^{ns}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\})$ in $\overline{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\})$.

$\pi_k : \mathcal{M}_{g,k}(\mathbb{P}^1, \{0, 1, \infty\}) \to \overline{M}_{g,3}$: The natural forgetful morphism.

$\mathcal{H}(m)$: This is defined to be $\mathcal{M}^{ns}_{2k-1,k}((\mathbb{P}^1)^m, \{0, 1, \infty\})$.

$\mathcal{H}(1)$: The largest open substack of $\mathcal{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\})$ such that $\pi_k$ is unramified and self-transverse on $\mathcal{H}(1)$ and, further, for any $x \in \mathcal{H}(1)$, $y \in \pi_k^{-1}(\pi_k(x))$, the base of $y$ is irreducible and automorphism-free.

1. Notation and Set-Up

Convention: In this paper, all Chow groups are taken with $\mathbb{Q}$ coefficients. All schemes and stacks are defined over $\mathbb{C}$.

Let $\mathcal{X}, \mathcal{Y}$ be smooth varieties over $\mathbb{C}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be proper and unramified. We inductively define schemes $\mathcal{X}(m)$ and proper, unramified morphisms

$$f(m) : \mathcal{X}(m) \to \mathcal{X}(m-1).$$

Set $\mathcal{X}(1) := \mathcal{X}$, $\mathcal{X}(0) := \mathcal{X}$, and $f(1) := f$. Assuming we have defined $f(m-1)$, the diagonal morphism $\Delta_{f(m-2)}$ is an open immersion. Define

$$\mathcal{X}(m) := \mathcal{X}(m-1) \times_{\mathcal{X}(m-2), \mathcal{X}(m-1)} \mathcal{X}(m-1) \setminus \text{Im}(\Delta_{f(m-2)}),$$

and let $f(m) : \mathcal{X}(m) \to \mathcal{X}(m-1)$ be projection to the first factor.

Definition 1.1. We say $f : \mathcal{X} \to \mathcal{Y}$ as above is self-transverse if, for each closed point $y \in \mathcal{Y}$ and $\{z_1, \ldots, z_r\} = f^{-1}(y)$, the image of the tangent spaces $T_{z_i}(\mathcal{X})$, $1 \leq i \leq r$ under $df$ are in general position in $T_y\mathcal{Y}$.

Self-transversality is the condition stated by Herbert to ensure the validity of his Multiple Point Formula, see [Fu, Ex. 9.1.14]. If $f$ is self-transverse and $f^{-1}(y)$ has cardinality $r$ for some point $y \in \mathcal{Y}$, then $f$ is $r$-generic of codimension $n = \dim \mathcal{Y} - \dim \mathcal{X}$ in the sense of [Kl, §4.5]. Each $\mathcal{X}(m)$ is nonempty and smooth of dimension $\dim \mathcal{Y} - mn$ for $m \leq r$ (so $r \leq \frac{\dim \mathcal{Y}}{n}$).

The following consequence of self-transversality will be of fundamental importance.

Proposition 1.2. Let $f : X \to Y$ be a proper, unramified morphism of smooth, irreducible complex varieties. Assume $\dim X = \dim Y - 1$ so that the image $f(X)$ of $X$ is a divisor.
Assume in addition \( f \) is self-transverse. Then
\[
\operatorname{ord}_y(f(X)) = \# f^{-1}(y),
\]
where \( \operatorname{ord}_y(f(X)) := \max\{n \mid g \in I^n_y \} \) for any local holomorphic equation \( g \in \hat{O}_{Y,y} \) of \( f(X) \).

**Proof.** Let \( V_1, \ldots, V_r \) denote the tangent spaces to \( X \) at the points \( p_1, \ldots, p_r \) over \( y \). Let \( U \subseteq \mathbb{C}^n \) be a small analytic neighbourhood of \( y = 0 \in \mathbb{C}^n \). Let \( n_i \) denote a unit normal vector to the hyperplane \( V_i \subseteq \mathbb{C}^n \) for each \( i \). As the \( V_i \) are in general position the \( n_i \) are linearly independent, so we may assume \( n_i \) is the \( i \)-th standard basis vector and \( V_i \) is defined by \( x_i = 0 \). As unramified morphisms are local-analytic closed immersions, we have \( g = g_1 \ldots g_r \) where \( g_i \) defines a hypersurface with tangent plane \( V_i \). Hence \( g_i = c_i x_i \mod I^2_p \) for nonzero constants \( c_i \) and thus \( g = c x_1 \ldots x_r \mod I^{r+1}_p \), for a nonzero constant \( c \). The claim follows. \( \square \)

Let \( X \) be a smooth, projective, complex variety and \( \beta \in H_2(X, \mathbb{Z}) \). Let \( P = \{p_1, \ldots, p_a\} \) be a collections of distinct points of \( X \). For any integer \( g \geq 0 \) we let
\[
\overline{M}_{g,\beta}(X, P; n)
\]
denote the stack of genus \( g \) stable maps in the class of \( \beta \) with base point \( P \) and \( n \) markings, [AK, §10]. Points of \( \overline{M}_{g,\beta}(X, P; n) \) consist of morphisms \( f : C \to X \) together with markings \( p'_1, \ldots, p'_\alpha, q_1, \ldots, q_n \) in the smooth locus of the genus \( g \), connected nodal curve \( C \) such that:
\begin{enumerate}
  \item \( f_*[C] = \beta \).
  \item \( f(p'_i) = p_i \) for \( 1 \leq i \leq \alpha \).
  \item The datum \( (f, p'_i, q_j) \) has finite automorphism group.
\end{enumerate}

When \( n = 0 \) we set \( \overline{M}_{g,\beta}(X, P) := \overline{M}_{g,\beta}(X, P; 0) \). If \( X = \mathbb{P}^1 \) we write \( \overline{M}_{g,k}(\mathbb{P}^1, P; n) \) for \( \overline{M}_{g,k}[\mathbb{P}^1](\mathbb{P}^1, P; n) \) and for \( m \geq 2 \) we write
\[
\overline{M}_{g,k}((\mathbb{P}^1)^m, P; n)
\]
for \( \overline{M}_{g,k}[\Delta](((\mathbb{P}^1)^m, P; n) \) where \( \Delta \) is the class of the small diagonal \( \{(x, \ldots, x) \mid x \in \mathbb{P}^1\} \). Setting \( P = \{(0)^m, (1)^m, (\infty)^m\} := \{(0, \ldots, 0), (1, \ldots, 1), (\infty, \ldots, \infty)\} \subseteq (\mathbb{P}^1)^m \), we write
\[
\overline{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\}; n)
\]
for \( \overline{M}_{g,k}((\mathbb{P}^1)^m, \{(0)^m, (1)^m, (\infty)^m\}; n) \). We have a proper morphism
\[
\pi_k : \overline{M}_{g,k}(\mathbb{P}^1, \{0, 1, \infty\}; n) \to \overline{M}_{g,3+n}
\]
given by mapping a stable marked map to (the stabilization of) its base. We let
\[
\psi_i(m) : \overline{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\}; n) \to \overline{M}((\mathbb{P}^1)^{m-1}, \{0, 1, \infty\}; n)
\]
for \( i = 1 \) respectively \( i = 2 \) be the map induced from the projection \((\mathbb{P}^1)^m \to (\mathbb{P}^1)^{m-1}\) away from the last respectively the first factor of \((\mathbb{P}^1)^m\). We let
\[
\mathcal{M}^s_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\}; n) \subseteq \overline{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\}; n)
\]
denote the open locus parametrising marked stable maps \([f : C \to (\mathbb{P}^1)^m]\) such that the base \( C \) has only non-separating nodes, and further, if \( f_i := p r_i \circ f \), for \( pr_i : (\mathbb{P}^1)^m \to \mathbb{P}^1 \) the \( i \)-th projection, then \( f_i \) is finite with \( h^0(C, f_i^*\mathcal{O}_{\mathbb{P}^1}(1)) = 2 \). We additionally demand that \( f_i \) be etale near the base points \((p, q, r) \in C \) over \((0, 1, \infty)\), for \( 1 \leq i \leq m \).

Denote by
\[
\mathcal{M}^s_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\}; n) \subseteq \overline{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\}; n)
\]
the closure of \( \mathcal{M}^s_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\}; n) \) in \( \overline{M}_{g,k}((\mathbb{P}^1)^m, \{0, 1, \infty\}; n) \). We let
\[
\pi_k : \mathcal{M}_{g,k}(\mathbb{P}^1, \{0, 1, \infty\}; n) \to \overline{M}_{g,3+n},
\]
denote the restriction of \( \pi_k \) to \( \mathcal{M}_{g,k}(\mathbb{P}^1, \{0, 1, \infty\}; n) \).
In the special case \( g = 2k - 1 \), which plays an important role, we let 
\[
\tilde{H}(m) := \mathcal{M}_{2k-1,k}^g((\mathbb{P}^1)^m, \{0, 1, \infty\}).
\]

Let \( \mathcal{H}(1) \) denote the largest open substack of \( \mathcal{M}_{2k-1,k}(\mathbb{P}^1, \{0, 1, \infty\}) \) such that for any point \( x \in \mathcal{H}(1) \), each point \( y = [f : (C, q_1, q_2, q_3)] \rightarrow \mathbb{P}^1 \in \mathcal{H}(1) \) with \( \pi_k(x) = \pi_k(y) \) satisfies the following conditions:

1. \( C \) is irreducible and \( \text{Aut}[C, q_1, q_2, q_3] = \{\text{id}\} \).
2. \( \pi_k \) is unramified and self-transverse in an open subset about \( \pi_k(x) \).

Note that as \( C \) is irreducible, \( \mathcal{H}(1) \) is smooth of dimension \( 3g - 1 \) and clearly
\[
\mathcal{H}(1) \subseteq \tilde{\mathcal{H}}(1).
\]

Further, self-transversality of \( \pi_k \) is an open condition (cf. the proof of Proposition 3.11). By definition of \( \mathcal{H}(1) \), there is an open subset
\[
\mathcal{H}(0) \subseteq \mathcal{M}_{2k-1,3}
\]
with \( \pi_k^{-1}(\mathcal{H}(0)) \simeq \mathcal{H}(1) \). We continue to denote the restriction \( \pi_k : \mathcal{H}(1) \rightarrow \mathcal{H}(0) \) by \( \pi_k \). By the assumption \( \text{Aut}[C, q_1, q_2, q_3] = \{\text{id}\} \), both \( \mathcal{H}(0) \) and \( \mathcal{H}(1) \) are schemes, \([AC1]\).

Denote by \( \mathfrak{hur} \in A^1(\mathcal{M}_{2k-1,3}, \mathbb{Q}) \) the pullback of the Hurwitz divisor on \( \mathcal{M}_{2k-1,3} \), \([HM]\). Let
\[
\mathcal{A}_{g,k} = \mathcal{M}_{0,2g+2k-2}(\mathcal{B} \mathcal{G}_k)
\]
be the moduli space of degree \( k \) admissible covers of genus \( g \), with ordered branch points. We have a natural projection \( \pi_k : \mathcal{A}_{g,k} \rightarrow \mathcal{M}_g \) as well as the branch morphism
\[
q : \mathcal{A}_{g,k} \rightarrow \mathcal{M}_{2g-2k-2}.
\]

Let \( B_j \) denote the boundary divisors of \( \mathcal{M}_{0,n} \) with general point corresponding to a curve with two rational components, one of which has precisely \( j \) marked points. Let \( T \) be the divisor in \( \mathcal{A}_{g,k} \) corresponding to line bundles \( l \in W^1_k(C) \) on a smooth curve with a base point. Consider the open substack
\[
\mathcal{A}_{g,k}^o := q^*(\mathcal{M}_{0,2g+2k-2} \setminus \bigcup_{j \geq 3} B_j) \setminus T.
\]

The image of \( \mathcal{A}_{g,k}^o \) under \( \pi_k \) lies in the locus \( \mathcal{M}_{g}^{\text{irr}} \) of irreducible curves.

We have three boundary divisors \( E_0, E_2, E_3 \) on \( \mathcal{A}_{g,k}^o \). Firstly, \( E_0 \) denotes the pullback of the boundary \( \delta \) of \( \mathcal{M}_g^{\text{irr}} \). The general point of \( E_3 \) is the admissible cover corresponding to a finite cover \( C \rightarrow \mathbb{P}^1 \) from a smooth curve \( C \) and with a ramification profile \( (3, 1^{2g+2k-3}) \) over some branch point, and simple branching over all other branch points. The general point of \( E_2 \) corresponds to a finite cover \( C \rightarrow \mathbb{P}^1 \) with ramification profile \( (2, 2, 1^{2g+2k-4}) \). Denote by
\[
\mathcal{B}_{g,k} := \mathcal{A}_{g,k}^o/\mathcal{G}_{2g+2k-2},
\]
the space of admissible covers with unordered branching. We set \( \mathcal{B}_{g,k}^o := \mathcal{A}_{g,k}^o/\mathcal{G}_{2g+2k-2} \) and let \( D_0, D_2 \) resp. \( D_3 \) denote the reduced images of \( E_0, E_2 \) resp. \( E_3 \) in \( \mathcal{B}_{g,k}^o \). We write \( \lambda \) for the hodge class on both \( \mathcal{A}_{g,k}^o \) and \( \mathcal{B}_{g,k}^o \). For later use, recall the following computation \([FR, \text{Prop. 11.1}]\):

**Proposition 1.3** (Farkas–Rimányi). We have the following canonical bundle formula
\[
K_{\mathcal{B}_{g,k}} = \frac{1}{2} \left[ -\frac{2g + 2k - 1}{2g + 2k - 3} D_0 - \frac{4}{2g + 2k - 3} D_2 + \frac{2g + 2k - 9}{2g + 2k - 3} D_3 \right] \\
= 8\lambda + \frac{D_3}{6} - \frac{3D_0}{2}.
\]
We make a remark about the comparison between $B_{2k-1,k}^0$ and $\mathcal{H}(1)$. Let $B' \subseteq B_{2k-1,k}^0$ be the open locus of admissible covers $f : C \to T$ such that the stabilization $\hat{C}$ of $C$ is irreducible. Consider the open subset $\mathcal{M}_{2k-1,3}^{\text{irr}}$ of irreducible marked curves and let $B'' \subseteq B_{2k-1,k}^0 \times \mathcal{M}_{2k-1}^{\text{irr}}$ denote the locus where the markings $p, q, r \in \hat{C}$ avoid the image of unstable components and where $f(p), f(q), f(r)$ are distinct points in the image $T$. There is a morphism $B'' \to \mathcal{H}(1)$ extending to an isomorphism outside a codimension two set, cf. [P, §3.5].

2. Cycle Computations

2.1. The Brill–Noether cycles. Starting with $\mathcal{H}(1)$ as in the previous section, we have schemes $\mathcal{H}(m)$ and projective immersions $p_i^{(m)} : \mathcal{H}(m) \to \mathcal{H}(m-1)$, $m \geq 1$. Considering the fibre product

$$
\begin{array}{c}
\mathcal{H}(m+1) \xrightarrow{p_2^{(m+1)}} \mathcal{H}(m) \\
p_1^{(m+1)} \downarrow \quad \downarrow p_1^{(m)} \\
\mathcal{H}(m) \xrightarrow{p_1^{(m)}} \mathcal{H}(m-1),
\end{array}
$$

we have $\mathcal{H}(m+1) := \mathcal{Z}(m+1) \setminus \Delta$, where $p_i^{(m)}$ is defined by restriction to $\mathcal{H}(m)$.

**Definition 2.1.** Define the Brill–Noether cycles as

$$BN_m = p_1^{(2)} \cdots p_1^{(m)}[\mathcal{H}(m)] \in A^m(\mathcal{H}(1)).$$

2.2. The Eagon–Northcott cycles. Recall the kernel bundle description of Koszul cohomology, [AN]. Let $X$ be a projective variety, $L \in \text{Pic}(X)$ be globally generated. Define $M_L$ via the exact sequence

$$0 \to M_L \to H^0(X, L) \otimes O_X \xrightarrow{\text{ev}} L \to 0.$$ 

Then

$$K_{p,q}(X, L) \simeq \text{Coker}(\bigwedge^{p+1} H^0(L) \otimes H^0(L^{q-1}) \to H^0(\bigwedge^p M_L \otimes L^q))$$

$$\simeq \text{Ker}(H^1(\bigwedge^p M_L \otimes L^{q-1}) \to \bigwedge^{p+1} H^0(L) \otimes H^1(L^{q-1}))$$

The universal stable map gives a universal cover

$$
\begin{array}{c}
C \xrightarrow{f} \mathcal{P} \\
\nu \downarrow \quad \downarrow \mu \\
\tilde{\mathcal{H}}(1),
\end{array}
$$

where $\mathcal{P} := \mathcal{P}_{\tilde{\mathcal{H}}(1)}$. We have

$$0 \to \mathcal{E}_f \to f_*\omega_f \to O_{\mathcal{P}} \to 0,$$

where $\mathcal{E}_f$ is the universal Tschirnhausen bundle, and further have the projective bundle $\varphi : \mathcal{X} := \mathcal{P}(\mathcal{E}_f \otimes \omega_\mu) \to \mathcal{P}$ and a closed immersion $\iota : C \hookrightarrow \mathcal{X}$. Set $h := \mu \circ \varphi : \mathcal{X} \to \tilde{\mathcal{H}}(1)$. Define the universal kernel bundle $\mathcal{M}$ by

$$0 \to \mathcal{M}_{\mathcal{X}} \to h^*h_*(O_{\mathcal{X}(1)}) \to O_{\mathcal{X}(1)} \to 0.$$
For all integers $i, j$, define sheaves $A^{[i,j]}(m), B^{[i,j]}(m)$ inductively on $\tilde{H}(m)$. Set 
\[
A^{[i,j]}(1) := h_*(\bigwedge_i M_\mathcal{X}(j)), \quad B^{[i,j]}(1) := \bigwedge_i h_*(\mathcal{O}_\mathcal{X}(1)) \otimes h_*(\mathcal{O}_\mathcal{X}(j)).
\]
We define 
\[
A^{[i,j]}(m) := \psi^*_i(m)A^{[i,j]}(m-1) \oplus \psi^*_2(m) \cdots \psi^*_2(2)A^{[i,j]}(1),
\]
\[
B^{[i,j]}(m) := \psi^*_1(m)B^{[i,j]}(m-1) \oplus \psi^*_2(m) \cdots \psi^*_2(2)B^{[i,j]}(1).
\]

Let $\nu_m : \mathcal{C}_m \to \tilde{H}(m)$ be the universal curve, which is given by the fibre product 
\[
\begin{array}{ccc}
\mathcal{C}_m & \xrightarrow{\mu_i} & \mathcal{C}_{m-1} \\
\nu_m \downarrow & & \downarrow \nu_{m-1} \\
\tilde{H}(m) & \xrightarrow{\psi_i(m)} & \tilde{H}(m-1),
\end{array}
\]
where $i$ can be either 1 or 2 in the horizontal arrows. Define kernel bundles $\mathcal{K}_m$ by 
\[
0 \to \mathcal{K}_m \to \nu^*_m \nu_{m*} (\omega_{\mathcal{K}_m}) \to \omega_{\mathcal{K}_m} \to 0.
\]
Notice that $\mathcal{K}_1 \simeq \iota^* \mathcal{M}_X$, whereas $\mu_i^{(m)} \mathcal{K}_{m-1} \simeq \mathcal{K}_m$. Define sheaves $C^{[i,j]}(m), D^{[i,j]}(m)$ 
\[
C^{[i,j]}(m) := \nu_{m*} \left( \bigwedge_i \mathcal{K}_m \otimes \omega_{\mathcal{K}_m}^{\otimes j} \right),
\]
\[
D^{[i,j]}(m) := \bigwedge_i \nu_{m*} (\omega_{\mathcal{K}_m}) \otimes \nu_{m*} (\omega_{\mathcal{K}_m}^{\otimes j}).
\]
Define restriction maps 
\[
\beta^{[i,j]}(m) : A^{[i,j]}(m) \to C^{[i,j]}(m),
\]
inductively. Set $\beta^{[i,j]}(1)$ to be the composition 
\[
h_* \bigwedge_i M_\mathcal{X}(j) \to h_* \nu_1 t^* \bigwedge_i M_\mathcal{X}(j) \simeq \nu_1 t^* \bigwedge_i M(j) \simeq C^{[i,j]}(1).
\]
For $m > 1$, $\psi^*_i(m)\beta^{[i,j]}(1)(m-1)$ gives a morphism $\psi^*_i(m)A^{[i,j]}(m-1) \to \psi^*_i(m)C^{[i,j]}(m-1)$. Composing this with the base change morphism yields a morphism $\psi^*_i(m)A^{[i,j]}(m-1) \to C^{[i,j]}(m)$. Secondly, composing $\psi^*_m(m) \cdots \psi^*_2(2)\beta^{[i,j]}(1)$ with the natural base change maps yields $\psi^*_m(m) \cdots \psi^*_2(2)A^{[i,j]}(1) \to C^{[i,j]}(m)$. Define $\beta^{[i,j]}(m)$ as the sum of these two maps. When there is no confusion, write $\beta^{[i,j]}$ for $\beta^{[i,j]}(m)$. Similarly, there are maps 
\[
\gamma^{[i,j]} : B^{[i,j]}(m) \to D^{[i,j]}(m)
\]
given as a sum of restriction maps.

The short exact sequence (1) induces 
\[
0 \to \bigwedge_i M_\mathcal{X}(j) \to (h_* \bigwedge_i h_* \mathcal{O}_\mathcal{X}(1)) \otimes \mathcal{O}_\mathcal{X}(j) \to \bigwedge_i M(j+1) \to 0.
\]
If $j \geq 1$, the fact that the scroll $\mathcal{X}$ has a 2-linear minimal free resolution given by the Eagon–Northcott complex implies $R^1 h_* \bigwedge_i M_\mathcal{X}(j) = 0$, [FK3, §4]. Hence we have exact sequences 
\[
0 \to A^{[i,j]}(1) \to B^{[i,j]}(1) \to A^{[i-1,j+1]}(1) \to 0,
\]
provided $j \geq 1$. Pulling this back under the relevant projections and summing up we obtain 
\[
0 \to A^{[i,j]}(m) \to B^{[i,j]}(m) \to A^{[i-1,j+1]}(m) \to 0,
\]
for \( j \geq 1 \). We have the commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & A^{[k-1,1]}(m) & \rightarrow & B^{[k-1,1]}(m) & \rightarrow & A^{[k-2,2]}(m) & \rightarrow & 0 \\
\downarrow \gamma^{[k-1,1]} & & \downarrow \gamma^{[k-1,1]} & & \downarrow \gamma^{[k-2,2]} & & \\
0 & \rightarrow & C^{[k-1,1]}(m) & \rightarrow & D^{[k-1,1]}(m) & \rightarrow & C^{[k-2,2]}(m).
\end{array}
\]

Define vector bundles \( \tilde{A}(m) \) and \( \tilde{C}(m) \) on \( \tilde{H}(m) \) by

\[
\begin{align*}
\tilde{A}(1) & := R^1h_* \bigwedge^k \mathcal{M}_X, \quad \tilde{A}(m) := \psi^*_1(m)\tilde{A}(m-1) \oplus \psi^*_2(m) \cdots \psi^*_2(2)\tilde{A}(1) \\
\tilde{C}(m) & := R^1\nu_{ms} \bigwedge^k \mathcal{K}_m.
\end{align*}
\]

There is a natural morphism \( \tilde{\beta} : \tilde{A}(m) \rightarrow \tilde{C}(m) \). We have the commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & B^{[k,0]}(m) & \rightarrow & A^{[k-1,1]}(m) & \rightarrow & \tilde{A}(m) & \rightarrow & 0 \\
\downarrow \gamma^{[k,0]} & & \downarrow \gamma^{[k-1,1]} & & \downarrow \beta^{[k-1,1]} & & \downarrow \tilde{\beta} & & \\
0 & \rightarrow & D^{[k,0]}(m) & \rightarrow & C^{[k-1,1]}(m) & \rightarrow & \tilde{C}(m).
\end{array}
\]

We now define two classes in the \( K \)-group of \( \tilde{H}(m) \). Let \( \mathcal{V}(m) \) be the cokernel of the composition \( \phi^{[k-1,1]} \) of the natural maps in the diagram below:

\[
\begin{array}{cccc}
\text{Ker} \gamma^{[k-1,1]} / \text{Ker} \gamma^{[k,0]} & \rightarrow & A^{[k-2,2]}(m) & \rightarrow \\
\downarrow & & \downarrow \phi^{[k-1,1]} & & \uparrow \\
\text{Ker} \gamma^{[k-1,1]} / \text{Ker} \beta^{[k-1,1]} & \rightarrow & \text{Ker} \beta^{[k-2,2]}.
\end{array}
\]

Consider the morphism

\[
\mathcal{F}(m) : \mathcal{V}(m) \rightarrow C^{[k-2,2]}(m)
\]

given by the restriction of \( \beta^{[k-2,2]} \). By relative duality

\[
\tilde{C}(m)^* \simeq \nu_{ms} \left( \bigwedge^k \mathcal{K}_m^* \otimes \omega_{\nu_m} \right) \simeq C^{[k-2,2]}(m) \otimes \lambda^*,
\]

for \( \lambda := c_1(\nu_{ms}(\omega_{\nu_m})) \).

**Lemma 2.2.** The composition

\[
\tilde{\beta}^* \otimes \lambda \circ \beta^{[k-2,2]} : \mathcal{V}(m) \rightarrow \tilde{A}(m)^* \otimes \lambda
\]

of vector bundles is zero.

**Proof.** By Grauert’s theorem, \( C^{[k-2,2]}(m) \) is indeed locally free, see [FK3, §4]. At a closed point \( p = [(f_i : C \rightarrow \mathbb{P}^1)] \in \tilde{H}(m), \)

\[
\text{Im} \beta^{[k-2,2]} \subseteq \text{Im} \left( \bigwedge^k H^0(\omega_C) \otimes H^0(\omega_C) \right) \subseteq H^0 \left( \bigwedge^k M_{\omega_C} \otimes \omega_C^{\otimes 2} \right)
\]
by diagram (2), where \( \beta_p^{[k-2,2]} := \beta^{[k-2,2]} \otimes k(p) \). So it suffices to show \( \text{Im}(\bigwedge^{k-1} H^0(\omega_C) \otimes H^0(\omega_C)) \subseteq \text{Ker} \tilde{\beta}^\ast \). From diagram (3),

\[
\text{Im} \tilde{\beta} \subseteq K_{k-1,1}(C, \omega_C) = \text{Ker}(H^1(\bigwedge^k M_{\omega_C}) \to \bigwedge^{k-1} H^0(\omega_C) \otimes H^0(\omega_C)),
\]

which completes the proof. \( \square \)

We define:

\[ \mathcal{W}(m) := \text{Ker} \tilde{\beta}^\ast \otimes \lambda \] where \( \lambda := c_1(\nu_{m\ast} \omega_{\nu_m}) \).

By the above lemma, we have a morphism of sheaves

\[
\mathcal{F}(m) : \mathcal{V}(m) \to \mathcal{W}(m).
\]

A point \( p \in \tilde{\mathcal{H}}(m) \) defines an \( m \)-tuple \( \{ \gamma_i : C \to \mathbb{P}^1 \} \). Over \( p \) we have, a commutative diagram

\[
\begin{array}{ccccccc}
\cdots & \longrightarrow & \bigoplus_{i=1}^m \bigwedge^k H^0(X_i, \mathcal{O}(1)) & \longrightarrow & \bigoplus_{i=1}^m H^0(X_i, \bigwedge^{k-1} M_{X_i}(1)) & \longrightarrow & \bigoplus_{i=1}^m K_{k-1,1}(X_i, \mathcal{O}(1)) & \longrightarrow & 0 \\
\gamma_p^{[k,0]} & | & f_1 & | & f_2 \\
\cdots & \longrightarrow & \bigwedge^k H^0(C, \omega_C) & \longrightarrow & H^0(C, \bigwedge^{k-1} M_{\omega_C}(C)) & \longrightarrow & K_{k-1,1}(C, \omega_C) & \longrightarrow & 0
\end{array}
\]

where \( X_1, \ldots, X_m \) are the scrolls associated to the \( m \) minimal pencils of \( C \), and each component of the vertical maps are induced from the closed immersions \( C \hookrightarrow X_i \) and \( \gamma_p^{[k,0]} := \gamma^{[k,0]} \otimes k(p) \).

**Proposition 2.3.** Assume the map \( f_2 \) as above is injective at \( p \). Then \( \mathcal{V}(m) \) and \( \mathcal{W}(m) \) are vector bundles of the same rank near \( p \), and \( \mathcal{F}(m) : \mathcal{V}(m) \to \mathcal{W}(m) \) is an isomorphism at \( p \) if and only if \( b_{k-1,1}(C, \omega_C) = m(k-1) \).

**Proof.** By Grauert’s theorem, \( B^{[k-1,1]}(m), D^{[k-1,1]}(m) \) are vector bundles and \( \text{Ker} \gamma^{[k-1,1]} \) is locally free of rank \( (m-1)(2k-1)(2k-1) \). Likewise, \( \text{Ker} \gamma^{[k,0]} \) is locally free of rank \( (m-1)(2k-1) \), and \( \text{Ker} \gamma^{[k-1,1]} / \text{Ker} \gamma^{[k,0]} \) is locally free of rank \( (m-1)(2k-2)(2k-1) \). The morphism \( \phi^{[k-1,1]} \) is a map between vector bundles.

As \( \gamma_p^{[k,0]} \) is surjective, surjectivity of \( f_1 \) is equivalent to that of \( f_2 \) by the snake lemma. Further, as we are assuming \( f_2 \) is injective, \( \text{Ker} \gamma_p^{[k,0]} = \text{Ker} f_1 \). In particular, \( \phi_p^{[k-1,1]} := \phi^{[k-1,1]} \otimes k(p) \) is injective and \( \mathcal{V}(m) := \text{Coker} \phi^{[k-1,1]} \) is locally free of rank

\[
(2k-2)(2k-1) - m(k-1)
\]

near \( p \) (use [FK3, Lemma 4.5]). The morphism \( \tilde{\beta}^\ast \otimes \lambda : \tilde{\mathcal{C}}(m)^\ast \otimes \lambda \to \tilde{\mathcal{A}}^\ast(m) \otimes \lambda \) is a map of vector bundles, and from

\[
\begin{array}{ccccccc}
\cdots & \longrightarrow & \bigoplus_{i=1}^m \bigwedge^k H^0(X_i, \mathcal{O}(1)) & \longrightarrow & \bigoplus_{i=1}^m H^0(X_i, \bigwedge^{k-1} M_{X_i}(1)) & \longrightarrow & \bigoplus_{i=1}^m H^1(X_i, \bigwedge^k M_{X_i}) & \longrightarrow & 0 \\
\gamma_p^{[k,0]} & | & f_1 & | & f_p \\
\cdots & \longrightarrow & \bigwedge^k H^0(C, \omega_C) & \longrightarrow & H^0(C, \bigwedge^{k-1} M_{\omega_C}(C)) & \longrightarrow & H^1(C, \bigwedge^k M_{\omega_C}) & \longrightarrow & 0
\end{array}
\]

plus the snake lemma, injectivity of \( f_2 \) at \( p \) implies \( (\tilde{\beta}^\ast \otimes \lambda)_p \) is surjective and thus \( \mathcal{W}(m) \) is locally free of rank

\[
(4k-2)(2k-2) - m(k-1) = (2k-2)(2k-1) - m(k-1)
\]
near \(p\). We have a commutative diagram
\[
\begin{array}{c}
\bigoplus_{i=1}^m H^0(\wedge^{k-1} M_X(1)) \\ \downarrow f_i
\end{array}
\begin{array}{c}
\bigoplus_{i=1}^m \wedge^{k-1} H^0(\mathcal{O}(1)) \otimes H^0(\mathcal{O}(1)) \\ \downarrow \gamma_{p[i,k-1,1]}
\end{array}
\begin{array}{c}
\bigoplus_{i=1}^m H^0(\wedge^{k-2} M_X(2)) ,
\end{array}
\begin{array}{c}
H^0(\wedge^{k-1} M_{\omega_C}(\omega_C)) \\ \downarrow \beta_{p[k-2,2]}^\gamma
\end{array}
\begin{array}{c}
\wedge^{k-1} H^0(\omega_C) \otimes H^0(\omega_C) \\ \downarrow \beta_{p[k-2,2]} \\
H^0(\wedge^{k-2} M_{\omega_C}(\omega_C^\otimes 2)).
\end{array}
\]
with exact rows, cf. [FK3, Lemma 4.3]. As \(\gamma_{p[k-1,1]}\) is surjective, surjectivity of \(f_1\) is equivalent to surjectivity of the (injective) map
\[
\text{Ker} \gamma_{p[k-1,1]}/\text{Ker} \gamma_{p[k,0]} \to \text{Ker} \beta_{p[k-2,2]},
\]
or equivalently that the composition
\[
\mathcal{V}_p(m) \to A_p^{[k-2,2]}(m)/\text{Ker} \beta_{p[k-2,2]} \to H^0(C, \wedge^{k-2} M_{\omega_C}(\omega_C^\otimes 2))
\]
is injective. By Lemma 2.2, the image of \(\mathcal{V}_p(m)\) lies in \(W_p(m) \subseteq H^0(C, \wedge^{k-2} M_{\omega_C}(\omega_C^\otimes 2))\), and this completes the proof. \(\square\)

Make the following definitions
\[
\mathbf{V}(m) := c_1(A^{[k-2,2]}(m) - \text{Ker} \gamma_{p[k-1,1]} + \text{Ker} \gamma_{p[k,0]})
\]
\[
\mathbf{W}(m) := c_1(C^{[k-2,2]}(m) + \tilde{A}(m) \cdot \lambda^*)
\]
As in the proof of the proposition above, if the morphism \(f_2\) is injective at a point \(p\) then \(\mathbf{V}(m)\) resp. \(\mathbf{W}(m)\) agrees with \(c_1(\mathcal{V}(m))\) resp. \(c_1(\mathcal{W}(m))\) about \(p\).

The proposition above justifies the following definition

**Definition 2.4.** We define the Eagon-Northcott cycles
\[
\mathbf{EN}_m := \mathbf{W}(m) - \mathbf{V}(m) \in A^1(\mathcal{H}(m))
\]
\[
\mathbf{EN}_m := \psi_{14}(2) \ldots \psi_{14}^*(m)(\mathbf{EN}_m)_{|u(1)} \in A^{m+1}(\mathcal{H}(1))
\]

2.3. Computations. The following lemma is useful for induction arguments.

**Lemma 2.5.** The following formulae hold
\[
\begin{align*}
\mathbf{V}(m + 1) - \psi^*_1(m + 1)\mathbf{V}(m) & = c_1(\psi^*_2(m + 1) \ldots \psi^*_2(2)A^{[k-2,2]}(1) - \psi^*_2(m + 1) \ldots \psi^*_2(2)B^{[k-1,1]}(1) \\
& \quad + \psi^*_2(m + 1) \ldots \psi^*_2(2)B^{[k,0]}(1)) \\
\mathbf{W}(m + 1) - \psi^*_1(m + 1)\mathbf{W}(m) & = c_1(\psi^*_2(m + 1) \ldots \psi^*_2(2)\tilde{A}(1) \cdot \lambda^*)
\end{align*}
\]

*Proof.* These follow from the obvious identities
\[
A^{[k,j]}(m + 1)/\psi^*_1(m + 1)A^{[i,j]}(m) = \psi^*_2(m + 1) \ldots \psi^*_2(2)A^{[i,j]}(1)
\]
\[
B^{[i,j]}(m + 1)/\psi^*_1(m + 1)B^{[i,j]}(m) = \psi^*_2(m + 1) \ldots \psi^*_2(2)B^{[i,j]}(1)
\]
\[
\tilde{A}(m + 1)/\psi^*_1(m + 1)\tilde{A}(m) = \psi^*_2(m + 1) \ldots \psi^*_2(2)\tilde{A}(1)
\]
as well as the identities \(\psi^*_1(m + 1)D^{[i,j]}(m) = D^{[i,j]}(m + 1), \psi^*_1(m + 1)C^{[k-2,2]}(m) = C^{[k-2,2]}(m + 1), \psi^*_1(m + 1)\lambda = \lambda\) following from Grauert’s theorem. \(\square\)

Recall that \(\mathcal{H}(1)\) is isomorphic in codimension two to \(B_{2k-1,k}^o \times \mathcal{M}_{2k-1} \mathcal{M}_{2k-1,3}\). We continue to write \(D_0, D_2, D_3 \in A^1(\mathcal{H}(1), \mathbb{Q})\) for the pullback of the corresponding divisor classes from \(A^1(B_{2k-1,k}^o, \mathbb{Q})\).
Lemma 2.6. The following formulae hold in $A^1(\mathcal{H}(1), \mathbb{Q})$:

$$c_1(C^{[k-2,2]}(1)) = \frac{k+1}{(2k-3)(2k-1)}((2k-1)(4k-3)\binom{2k-2}{k-3} + (8k-3)\binom{2k-1}{k-2})\lambda - \frac{k(k+1)}{2(2k-1)(2k-3)}\binom{2k-1}{k-2}D_0,$$

$$c_1(A^{[k-2,2]}(1) + \widetilde{A}(1)) = 2k\binom{2k-2}{k-2}\lambda.$$

Proof. For $j \geq 0$, we have the short exact sequence:

$$0 \to C^{[k-2-j,2+j]}(1) \to D^{[k-2-j,2+j]}(1) \to C^{[k-3-j,2+j+1]}(1) \to 0.$$

Indeed, $R^1_{\nu_1}(\mathcal{A}^{[k-2-j]}_1 \otimes \mathcal{O}^{2+j}_{\nu_1}) = 0$, using kernel bundles and the fact that $K_{k-3-j,2+j}(C, \omega_C) = 0$ for any $[(C, q_1, q_2, q_3) \to \mathbf{P}^1] \in \mathcal{H}(1)$. Thus

$$c_1(C^{[k-2,2]}(1)) = \sum_{j=0}^{k-2} (-1)^j c_1(D^{[k-2-j,2+j]}(1)).$$

For $n \geq 2$,

$$\nu_{1*}(\omega^{n}_{\nu_1}) = (6n^2 - 6n + 1)\lambda - \frac{n^2 - n}{2}D_0,$$

see [ACG, Ch. 13]. By definition

$$c_1(D^{[k-2-j,2+j]}(1)) = c_1(\bigwedge^{k-2-j} \nu_{1*}\omega_{\nu_1} \otimes \nu_{1*}(\omega^{2+j}_{\nu_1}))$$

$$= \binom{2k-1}{k-2-j}((6(2+j)^2 - 6(2+j) + 1)\lambda - \frac{1}{2}(2+j)^2 - (2+j))D_0)$$

$$+ (2j+3)(2k-2)\binom{2k-2}{k-3-j}\lambda,$$

the first formula now follows, using any computer algebra package.

The second formula is an immediate consequence of the short exact sequences:

$$0 \to A^{[k-1,1]}(1) \to B^{[k-1,1]}(1) \to A^{[k-2,2]}(1) \to 0,$$

$$0 \to B^{[k,0]}(1) \to A^{[k-1,1]}(1) \to \widetilde{A}(1) \to 0$$

together with the fact that $h_*\mathcal{O}_X(1) \simeq \nu_{1*}\omega_{\nu_1}$.

Putting these facts together yields:

Lemma 2.7. The following identities hold in $A^1(\mathcal{H}(1), \mathbb{Q})$

(i) $c_1(C^{[k-2,2]}(1) - A^{[k-2,2]}(1) - \widetilde{A}(1)) = (k-1)\pi^*_k\mathfrak{hur}$,

(ii) $c_1(\widetilde{A}(1) \cdot (1 + \lambda^*)) = -(k-1)c_1(N_{\pi_k})$, where $N_{\pi_k}$ is the relative normal bundle of $\pi_k$.

Proof. The first claim follows from the previous lemma together with the computation of $\mathfrak{hur}$ in [HM]. Note that the only boundary component in $\mathcal{M}_g$ with nontrivial pullback to $\mathcal{H}(1)$ is $\delta_0$.

Lemma 1.3 plus the canonical bundle for $\mathcal{M}_g$ [HM] gives:

$$N_{\pi_k} = -5\lambda - \frac{1}{2}D_0 + \frac{D_3}{6}.$$

We need to show

$$c_1(\widetilde{A}(1)) = (k-1)(3\lambda + \frac{D_0}{4} - \frac{D_3}{12}),$$
We prove the claim by induction. By the double point formula [Fu, §9.3]
\[ \psi_1^*(m + 1)[H(m + 1)] = \psi_1^*(m)[H(m)] - c_1(N_{\psi_1}(m)), \]
where \( N_{\psi_1}(m) = \psi_1^*(m)T_{H(m-1)} - T_{H(m)} \) is the relative normal bundle of \( \psi_1(m) \). When \( m = 1 \),
\[ \psi_1^*(2)[H(1)] = \pi_k^*\text{hur} - c_1(N_{\pi_k}). \]
To begin the induction, we need to show
\[ W(1) - V(1) = (k - 1)(\pi_k^*\text{hur} - c_1(N_{\pi_k})). \]
From the definitions, we have
\[ W(1) - V(1) = c_1(C^{[k-2,2]}(1) + A(1) \cdot \lambda^* - A^{[k-2,2]}(1)) \]
as \( \gamma^{[k-1,1]}, \gamma^{[k,0]} \) are isomorphisms for \( m = 1 \). The claim follows from Lemma 2.7.

Assume \( (k - 1)\psi_{1s}(m)[H(m)] = W(m - 1) - V(m - 1) \). Then
\[
(k - 1)\psi_{1s}(m + 1)[H(m + 1)] = \psi_{1s}^*(m)(W(m - 1) - V(m - 1)) - (k - 1)c_1(\omega_{\psi_{1s}(m)})
\]
\[
= W(m) - V(m) - c_1(\psi_2^*(m)\psi_2^*(2)A(1)) \cdot \lambda^* - \psi_2^*(m)\psi_2^*(2)A^{[k-2,2]}(1))
\]
\[
- (k - 1)c_1(N_{\psi_{1s}(m)}),
\]
by Lemma 2.5. Observe that \( N_{\psi_{1s}(m)} = \psi_2^*(m)N_{\psi_{1s}(m - 1)} \) [Kl, Prop.4.7], so
\[
N_{\psi_{1s}(m)} = \psi_2^*(m)\psi_2^*(2)N_{\pi_k}.
\]
Hence it suffices to show
\[
c_1(\tilde{A}(1) \cdot \lambda^* - A^{[k-2,2]}(1) + B^{[k-1,1]}(1) - B^{[k,0]}(1)) = -(k - 1)N_{\pi_k}.
\]
Or, \( c_1(\tilde{A}(1) \cdot (\lambda^* + 1)) = -(k - 1)N_{\pi_k} \), which follows from Lemma 2.7. \( \square \)

3. GEOMETRY OF CURVES WITH MULTIPLE PENCILS

We start by finding a geometric condition to ensure the syzygies of the scrolls swept out by minimal pencils independently contribute to the extremal linear syzygies of a canonical curve.

3.1. Pencils in Geometrically General Position. Let \( C \) be a connected, nodal curve of genus \( g \) with no non-separating nodes. Let \( f_1, \ldots, f_m : C \to \mathbb{P}^1 \) be finite morphisms of degree \( k \) and assume that if \( L_i := f_i^*\mathcal{O}_{\mathbb{P}^1}(1) \), then \( h^0(\mathcal{O}(L_i)) = 2 \) for \( 1 \leq i \leq m \). We denote by \( X_{f_i} := \mathbb{P}(\mathcal{E}_{f_i}(-2)) \) the scroll associated to \( f_i \), and let \( \tilde{X}_{f_i} \subseteq \mathbb{P}^{g-1} \) be the image of \( X_{f_i} \) under
\[
X_{f_i} \to \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{E}_{f_i}(-2))),
\]
see [FK3, §4]. Then \( \tilde{X}_{f_i} \) has rational singularities and
\[
K_{p,q}(\tilde{X}_{f_i}, (\mathcal{O}_{\mathbb{P}^{g-1}}(1))|_{\tilde{X}_{f_i}}) \simeq K_{p,q}(X_{f_i}, \mathcal{O}_{\mathbb{P}(\mathcal{E}_{f_i}(-2))}(1))
\]
see [Sch1, §1]. Then \( \tilde{X}_{f_i} \) is a (possibly singular) rational normal scroll of degree \( g - k + 1 \) in \( \mathbb{P}^{g-1} \) for \( 1 \leq i \leq m \). If \( \{u,v\} \) is a basis for \( H^0(\mathcal{O}(L_i)) \) and \( \{g_1, \ldots, g_{g+1-k}\} \) is a basis for \( H^0(\mathcal{O}(\mathcal{O}_{\mathbb{P}^1}) \mathcal{L}^{-1}) \), then \( \tilde{X}_{f_i} \) is defined by the two by two minors of the matrix of linear forms
\[
\begin{pmatrix}
uy_1 & \cdots & uy_{g+1-k} \\
v_1 \cdots & voy_{g+1-k}
\end{pmatrix}.
\]
Choose distinct points \( z_1, \ldots, z_{g-1-k} \in C \) such that \( h^0(L_i + \sum_{j=1}^{g-1-k} z_j) = 2 \) for \( 1 \leq i \leq m \), or, equivalently, \( h^0(\mathcal{O}(\mathcal{O}_{\mathbb{P}^1}) \mathcal{L}^{-1}(-\sum_{j=1}^{g-1-k} z_j)) = 2 \) (this holds for a general choice of \( g - 1 - k \) points). Let \( p : \mathbb{P}^{g-1} \to \mathbb{P}^k \) be the projection away from the points \( z_1, \ldots, z_{g-1-k} \). Then \( p \) induces a composition of \( g - 1 - k \) inner projections on each scroll \( \tilde{X}_{f_i} \), as each scroll contains \( C \). Since each inner projection lowers the degree by one, we obtain quadric hypersurfaces
\[
Q_{f_i} := p(\tilde{X}_{f_i}) \in |\mathcal{O}_{\mathbb{P}^k}(2)|.
\]
If \( \{s,t\} \) is a basis for \( H^0(\mathcal{O}(\mathcal{O}_{\mathbb{P}^1}) \mathcal{L}^{-1}(-\sum_{j=1}^{g-1-k} z_j)) \) and \( \{u,v\} \) is a basis for \( H^0(\mathcal{O}(L_i)) \), then \( Q_{f_i} \) is the rank 4 quadric in \( \mathbb{P}(H^0(\mathcal{O}(\mathcal{O}_{\mathbb{P}^1}) \mathcal{L}^{-1}(\sum_{j=1}^{g-1-k} z_j)))^* \) defined by the determinant of
\[
\begin{pmatrix}
us & ut \\
v & ut
\end{pmatrix}.
We make the following definition:

**Definition 3.1.** The degree \( k \) pencils \( f_1, \ldots, f_m : C \to \mathbb{P}^1 \), are in geometrically general position with respect to \( z_1, \ldots, z_{g-1-k} \in C \) if

\[
\dim(\langle Q_{f_1}, \ldots, Q_{f_m} \rangle) = m - 1
\]

for the span \( \langle Q_{f_1}, \ldots, Q_{f_m} \rangle \subseteq |\mathcal{O}_{\mathbb{P}^k}(2)| \simeq \mathbb{P}^{(k+2)(k+1)/2} \).

As was pointed out to us by G. Farkas, if \( C \) is smooth then the above condition can be rephrased in terms of double points of the theta divisor, which is rather natural in light of \([G2]\). Suppose \( C \) is smooth. Consider the Jacobian \( \text{Pic}^{g-1}(C) \) and the theta divisor \( \Theta \subseteq \text{Pic}^{g-1}(C) \). Choose \( g - 1 - k \) distinct points \( z_1, \ldots, z_{g-1-k} \) on \( C \) and let \( D = \sum_{i=1}^{g-1-k} z_i \). Since \( h^1(C, L_i) = g + 1 - k \) then for a general choice of the points \( z_j \), each pencil \( L_i + D \) is a \( g^{1}_{g-1} \) and hence defines a double point of the theta divisor by the Riemann Singularity Theorem. The theorem of Andreotti–Mayer and Kempf, \([AM]\), \([Kem]\) states that the projectivized tangent cones of the theta divisor at these double points are rank 4 quadrics \( \tilde{Q}_{f_1}, \ldots, \tilde{Q}_{f_m} \) containing \( C \). From the determinantal descriptions above, \( \tilde{Q}_{f_i} \) is the cone over \( Q_{f_i} \), for \( 1 \leq i \leq m \). Hence:

**Proposition 3.2.** Assume \( C \) is smooth, admitting distinct degree \( k \) pencils \( f_1, \ldots, f_m : C \to \mathbb{P}^1 \) with \( L_i := f_i^*\mathcal{O}_{\mathbb{P}^1}(1), 1 \leq i \leq m \). Let \( D \in C_{g-1-k} \) be a reduced, effective, divisor such that \( h^0(C, L_i(D)) = 2 \) for \( 1 \leq i \leq m \). Then \( \{f_1, \ldots, f_m\} \) are in geometrically general position with respect to \( D \) if and only if the set of projectivized tangent cones \( \{\tilde{Q}_{f_i}, i \leq i \leq m\} \subseteq |\mathcal{O}_{\mathbb{P}^{g-1}}(2)| \) to \( L_i + D \in \Theta \) satisfies

\[
\dim(\langle \tilde{Q}_{f_1}, \ldots, \tilde{Q}_{f_m} \rangle) = m - 1.
\]

We test the reader’s patience by giving one final description of the quadrics \( \tilde{Q}_{f_i} \). From \([Sch1, \S 2]\), the quadrics \( \tilde{Q}_{f_i} \subseteq \mathbb{P}^{g-1} \) can be described geometrically as the union

\[
\tilde{Q}_{f_i} = \bigcup_{D \in [L_i]} \langle D + \sum_{j=1}^{g-1-k} z_j \rangle
\]

of the spans \( \langle D + \sum_{j=1}^{g-1-k} z_j \rangle \subseteq \mathbb{P}^{g-1} \). We can construct this scroll by using a base-point free pencil on a curve stably equivalent to \( C \). Let \( \tilde{C} \) denote the connected, nodal curve of compact type obtained by attaching rational tails \( E_1, \ldots, E_{g-1-k} \) to \( C \), with \( E_j \cap C = z_j \) for \( 1 \leq j \leq g - 1 - k \). We have a finite morphism \( g_i : \tilde{C} \to \mathbb{P}^1 \) of degree \( g - 1 \), with \( (g_i)_C = f_i \), \( \deg(g_i)_{E_j} = 1, 1 \leq j \leq g - 1 - k \). Furthermore, let \( H \in \text{Pic}(\tilde{C}) \) denote the line bundle with

\[
H_{E_j} \simeq \mathcal{O}_{E_j}, \quad H_C \simeq \omega_C,
\]

and consider the vector bundle \( g_i^*H \). The Mayer–Vietoris sequence gives \( h^0(\tilde{C}, \omega_{\tilde{C}}(-H)) = 1 \). One sees

\[
H^0(\mathbb{P}^1, (g_i^*H)^\vee(-2)) \neq 0,
\]

and hence we may write \( g_i^*H = V_i \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \). Similar considerations show that \( V_i \) is globally generated, and we have a morphism \( \mathbb{P}(V_i) \to \mathbb{P}^{g-1} \), cf. \([FK3, \S 4]\). Since

\[
h^0(\tilde{C}, H) = h^0(C, \omega_C), \quad h^0(\tilde{C}, H \otimes g_i^*\mathcal{O}_{\mathbb{P}^1}(-n)) = h^0(C, \omega_C \otimes L_i^{\otimes -n}(-\sum_{j=1}^{g-1-k} z_j)),
\]

then, by comparing with \([Sch1, \S 2]\), one sees that \( \mathbb{P}(V_i) \) has the same type as the resolution of \( \tilde{Q}_{f_i} \). The image of \( \mathbb{P}(V_i) \to \mathbb{P}^{g-1} \) is the quadric \( \tilde{Q}_{f_i} \), and further we have a natural morphism
\( \tilde{C} \to \mathbf{P}(V_i) \) induced by \( g_i^* V_i \to \mathcal{O}_{\tilde{C}} \). The image of the composition \( \tilde{C} \to \mathbf{P}(V_i) \to \mathbf{P}^{g-1} \) is precisely the canonically embedded curve \( C \).

Recall now Ehbauer’s notion of the projection of syzygies, [E], [Ap1]. Let \( V \) be a vector space and \( X \subseteq \mathbb{P}(V^*) \) be a projective variety. Choose \( x \in X \), which, viewed as a point in projective space, corresponds to an exact sequence of vector spaces

\[ 0 \to W_x \to V \to \mathbb{C} \to 0. \]

Let \( p_x : \mathbb{P}(V^*) \to \mathbb{P}(W_x^*) \) denote the projection centered in \( x \), let \( Y \subseteq \mathbb{P}(W_x^*) \) be the projection of \( X \), and let \( S_X, S_Y \) be the corresponding homogeneous coordinate rings. The sequence

\[ 0 \to \bigwedge^p W_x \to \bigwedge^p V \to \bigwedge^{p-1} W_x \to 0 \]

induces a map \( p_x : K_{p,1}(S_X, V) \to K_{p-1,1}(S_X, W_x) \) on Koszul cohomology of the coordinate ring \( S_X \). There is an injective morphism \( S_Y \hookrightarrow S_X \) which induces an injective map \( K_{p-1,1}(S_Y, W_x) \to K_{p-1,1}(S_Y, W_x) \). Ehbauer’s Lemma states that the image of \( p_x \) lies in \( K_{p-1,1}(S_Y, W_x) \), so we have a map

\[ p_x : K_{p,1}(S_X, V) \to K_{p-1,1}(S_Y, W_x). \]

Further, if \( X \) is connected, reduced, and \( L \) is base-point free then there is a natural isomorphism

\[ K_{p,1}(S_Y', V) \cong K_{p,1}(X, L) \]

for \( V = H^0(X, L) \) and \( X' := \phi_L(X) \). Hence we may talk about projection of syzygies in the case in the case of line bundles which are merely base-point free rather than very ample.

In the case of a nodal curve \( C \), with \( x \in C \) a smooth point and \( L \) any base-point free line bundle, then the projection map

\[ p_x : K_{p,1}(C, L) \to K_{p-1,1}(C, L(-x)) \]

can alternatively be defined using kernel bundles.

We now relate the assumption that minimal pencils are in geometrically general position to syzygies of the nodal curve \( C \). Recall that we have a natural restriction map

\[ j : \bigoplus_{i=1}^m K_{g-k,1}(X_{f_i}, \mathcal{O}_{X_{f_i}}(1)) \to K_{g-k,1}(C, \omega_C), \]

where \( X_{f_i} \) are the scrolls associated to the minimal pencils \( f_i \) as above.

**Proposition 3.3.** Let \( C \) be nodal, connected, with non-separating nodes. Assume we have a set of distinct, degree \( k \) pencils \( f_1, \ldots, f_m : C \to \mathbf{P}^1 \) with \( h^0(C, f_i^* \mathcal{O}_{\mathbf{P}^1}(1)) = 2 \) for all \( 1 \leq i \leq m \). Let \( D \) be a general, reduced, degree \( g - 1 - k \) divisor on the smooth locus of \( C \) with the property that \( h^0(f_i^* \mathcal{O}_{\mathbf{P}^1}(1)(D)) = 2 \) for all \( i \). Assume \( \{ f_1, \ldots, f_m \} \) is geometrically in general position with respect to \( D \). Then the linear map \( j \) as above is injective.

**Proof.** Let \( D = z_1 + \ldots + z_{g-1-k} \). It is easy to see that the map \( j \) is injective on each factor \( K_{g-k-1}(X_{f_i}, \mathcal{O}_{X_{f_i}}(1)), \) cf. [FK3, Lemma 4.3]. Suppose \( \lambda_1 j(v_1) + \ldots + \lambda_m j(v_m) = 0 \) for \( v_i \in K_{g-k-1}(X_{f_i}, \mathcal{O}_{X_{f_i}}(1)) \) for \( 1 \leq i \leq m \) not all zero. Composing the projection maps from each \( z_i \) we have a projection map \( p : \mathbf{P}^{g-1} \to \mathbf{P}^k \) as well as the projection map \( p : K_{g-k,1}(C, \omega_C) \to K_{1,1}(C, \omega_C(-D)) \) on Koszul cohomology. As \( D \) is general, we have that \( v_i \neq 0 \) implies that \( p(j(v_i)) \neq 0 \) is nonzero for \( 1 \leq i \leq m \), [AN, Prop. 2.14]. Each \( p(j(v_i)) \) may be considered as the equation of a quadric in \( \mathbf{P}^k \), and we have

\[ \lambda_1 p(j(v_1)) + \ldots + \lambda_m p(j(v_m)) = 0. \]

Each projection \( Q_{f_i} := p(X_{f_i}) \) is a quadric hypersurface in \( \mathbf{P}^k \) containing the image \( \phi_{\omega_C(-D)}(C) \).
It follows from the determinantal description that we may treat $Q_{f_i}$ as an element of $$\text{Ker}(\text{Sym}^2(H^0(\omega_C(-D))) \to H^0(2\omega_C(-D))) = K_{1,1}(C, \omega_C(-D)),$$
whereas $K_{1,1}(Q_{f_i}, \mathcal{O}_{Q_{f_i}}(1))$ is the space of quadrics containing $Q_{f_i}$, and hence is spanned by $[Q_{f_i}] \in K_{1,1}(Q_{f_i}, \mathcal{O}_{Q_{f_i}}(1))$. We then have a commuting diagram

$$\begin{align*}
K_{g-k,1}(\tilde{X}_{f_i}, \mathcal{O}_{\tilde{X}_{f_i}}(1)) &\cong K_{g-k,1}(X_{f_i}, \mathcal{O}_{X_{f_i}}(1)) \\
&\longrightarrow K_{g-k,1}(C, \omega_C)
\end{align*}$$

$$\begin{array}{c}
\downarrow \\
K_{1,1}(Q_{f_i}, \mathcal{O}_{Q_{f_i}}(1)) \\
\longrightarrow K_{1,1}(C, \omega_C(1))
\end{array}$$

Hence $p(j(v_j))$ is the equation of the quadric $Q_{f_i}$. Thus the assumption that the pencils are geometrically in general position implies $\lambda_1 = \ldots = \lambda_m = 0$ as required. \hfill $\square$

### 3.2. The Key Construction.

We now generalise a construction from [FK3, §3]. Let $C$ be an integral curve of genus $g \geq 3$ and gonality $k \leq \frac{g+1}{2}$, and choose a nonnegative integer $n \leq g + 1 - 2k$. Choose pairs of distinct points $(x_i, y_i)$ on $C$ for $1 \leq i \leq n$, and let $D$ be the semistable curve of genus $g + n$ obtained by adjoining smooth rational curves $R_i$ to $C$ at $x_i, y_i$. Mark $C$ at three general points $p, q, r$ (in particular, $\text{Aut}[C, p, q, r] = \{id\}$). Let $f : C \to \mathbb{P}^1$ be a morphism of degree $k$ with $(f(p), f(q), f(r)) = (0, 1, \infty)$ and $f(x_i) \neq f(y_i)$ for all $1 \leq i \leq n$.

**Definition 3.4** (The Key Construction). Let $C, D$ be as above. Construct a stable map $[h : D \to \mathbb{P}^1] \in \mathcal{M}_{g+n,k+n}^{ns}(\mathbb{P}^1, \{0,1,\infty\})$ by setting $h|_C = f : C \to \mathbb{P}^1$ and choosing $h|_{R_i} : R_i \to \mathbb{P}^1$ to be an isomorphism.

The stable map $[h]$ as above is a smooth point of $\mathcal{M}_{g+n,k+n}^{ns}(\mathbb{P}^1, \{0,1,\infty\})$, cf. Section 3.3.

Following [FK3], a smooth curve $C$ of genus $g \geq 3$ and gonality $k \leq \frac{g+1}{2}$ satisfies bpf-linear growth if

$$\dim G_{k+l}^1(C) \leq l, \text{ for } 0 \leq l \leq g + 1 - 2k$$

and, further, $$\dim G_{k+l}^{1,\text{bpf}}(C) < l, \text{ for } 0 < l \leq g + 1 - 2k$$

where $G_{d}^{1,\text{bpf}}(C) \subseteq G_{d}^{1}(C)$ is the open locus of base point free pencils of degree $d$ on $C$.

Let $C$ be a smooth curve of genus $g$ and gonality $k \leq \frac{g+1}{2}$, set $n = g + 1 - 2k$ and let $(D, p, q, r) \in \overline{\mathcal{M}}_{g+n,3}$ be the marked, stable curve of genus $g + n$ constructed above. Assume $C$ satisfies bpf-linear growth and that the underlying set of $W_k^1(C)$ consists of $m$ points $A_1, \ldots, A_m$.

To each minimal pencil $A_i$ we have a stable map $f_j : C \to \mathbb{P}^1$ for $1 \leq i \leq m$. As in definition 3.4, we extend $f_j$ to stable maps $h_j : D \to \mathbb{P}^1$. Let $\hat{D}$ denote the stabilization of $D$.

**Proposition 3.5.** Let $C, D$ be as above and set $n = g + 1 - 2k$. Assume the smooth curve $C$ satisfies bpf-linear growth and that the underlying set of $W_k^1(C)$ consists of $m$ pencils $A_1, \ldots, A_m$, each of which has ordinary ramification. Then the underlying set of the fibre of

$$\pi_{k+n} : \mathcal{M}_{g+n,k+n}(\mathbb{P}^1, \{0,1,\infty\}) \to \overline{\mathcal{M}}_{2(k+n)-1,3}$$

over $(\hat{D}, p, q, r)$ consists of $m$ points, namely the maps $\{h_j, 1 \leq j \leq m\}$ as defined above.

**Proof.** Let $\mathcal{B}_{g+n,k+n} \to \overline{\mathcal{M}}_{g+n}$ denote the space of degree $k+n$ admissible covers with unordered branch points. Arguing as in [FK3, Prop. 3.5], each admissible cover $F : B \to T$ over $\hat{D}$ contains a unique component of genus $g \geq 1$, which must be isomorphic to $C$. Further, $\text{deg} F|_C = k$, so $F|_C$ is amongst the $f_i$. All nodes of $B$ which lie on $C$ must be either above branch points of $F|_C$ or one of the points $x_j, y_j$ for some $j$. As the points $p, q, r$ are general, they are not nodes in the base of any admissible cover over $\hat{D}$. There is a well-defined, proper, birational morphism
Let the degree $k + n$ stable map $\phi : D' \to \mathbb{P}^1$ be a point of $\mathcal{M}_{g+n,k+n}(\mathbb{P}^1, \{0, 1, \infty\})$ lying over $(\hat{D}, p, q, r)$. Then $D'$ has a component $C'$ birational to $C$, and we have a morphism $\phi_C$ given by composing the normalization $C \to C'$ with $\phi$. Suppose $C'$ has $\delta$ nodes, and up to relabelling we may assume $x_i, y_i \in C$ lie over these nodes, for $1 \leq i \leq \delta$. By the above, $\phi_C$ is finite of degree $k$. Further we must have $D' = C' \cup R'_{\delta+1} \cup \cdots \cup R'_n$ for unstable components $R'_i$, and $\phi$ must have degree at least one on each $R'_i$ (by stability). The morphism $\phi$ identifies $x_i$ to $y_i$ for $1 \leq i \leq \delta$, and thus $h^0(\phi^*_C \mathcal{O}_{\mathbb{P}^1}(1)(-x_i - y_i)) \geq 1$ for all $1 \leq i \leq \delta$. As $(x_i, y_i)$ are general, $\dim G^1_k(C) \geq \delta$. Hence $\delta = 0$ by bpf-linear growth. In particular, $D' = D$.

To conclude, it suffices to show that $\phi$ does not contract any rational component $R_i$, as then $\phi$ must have degree 1 on $R_i$ (for degree reasons). But if $\phi$ contracts $R_i$, then by the generality of the $x_i, y_i$, $\dim G^1_k(C) \geq 1$, which again contradicts bpf-linear growth. \qed

Choose $0 \leq n \leq 2k - 5$ and, for any integer $m$, consider

$$
\overline{\mathcal{M}}_{2k-1-n,k-n}(\mathbb{P}^1)^m, \{0, 1, \infty\}; 2n) \times \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1)^m; 2)^{\times n}.
$$

Denote the markings on $[f] \in \overline{\mathcal{M}}_{2k-1-n,k-n}(\mathbb{P}^1)^m, \{0, 1, \infty\}; 2n)$ as $x_1, y_1, \ldots, x_n, y_n$. We have $ev_{x_i}, ev_{y_i} : \overline{\mathcal{M}}_{2k-1-n,k-n}(\mathbb{P}^1)^m, \{0, 1, \infty\}; 2n) \to (\mathbb{P}^1)^m$, $1 \leq i \leq n$ by evaluating at these markings. Let

$$
U_n(m) \subseteq \overline{\mathcal{M}}_{2k-1-n,k-n}(\mathbb{P}^1)^m, \{0, 1, \infty\}; 2n)
$$

denote the open locus of points $\alpha$ such that

$$
pr_j(ev_{x_i}(\alpha)) \neq pr_j(ev_{y_i}(\alpha)) \quad \text{for all} \quad 1 \leq j \leq m, 1 \leq i \leq n,
$$

for $pr_j : (\mathbb{P}^1)^m \to \mathbb{P}^1$ the projection to the $j$th factor. Further denote by

$$
ev_{1,j}, ev_{2,j} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1)^m; 2)^{\times n} \to (\mathbb{P}^1)^m, \quad 1 \leq j \leq n
$$

the evaluation morphisms from the $j$th factor of $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1)^m; 2)^{\times n}$. Let

$$
\mathcal{M}^{sm}_{0,1}(\mathbb{P}^1)^m; 2) \subseteq \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1)^m; 2)
$$

denote the open locus of morphisms $\mathbb{P}^1 \to (\mathbb{P}^1)^m$ with smooth base. We denote by

$$
V_n(m) \subseteq U_n(m) \times \mathcal{M}^{sm}_{0,1}(\mathbb{P}^1)^m; 2)^{\times n}
$$

the closed subset of points $(\alpha, \beta)$ with $ev_{x_i}(\alpha) = ev_{1,i}(\beta), ev_{y_i}(\alpha) = ev_{2,i}(\beta)$ for all $1 \leq i \leq n$. We have a glueing morphism

$$
q_n(m) : V_n(m) \to \overline{\mathcal{M}}_{2k-1,k}(\mathbb{P}^1)^m, \{0, 1, \infty\},
$$

defined by glueing maps together in the obvious way.

**Lemma 3.6.** Fix integers $n, m$. Let $u = [f : B \to \mathbb{P}^1] \in \overline{\mathcal{M}}_{g,k}(\mathbb{P}^1)^m, \{0, 1, \infty\}; 2n)$ with $f$ finite. Each component of $\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1)^m, \{0, 1, \infty\}; 2n)$ containing $u$ has dimension at least

$$
3g - m(g + 2 - 2k) + 2n.
$$
Proof. Each component containing \( f : B \to \mathbb{P}^1 \) has dimension at least
\[
\dim T^1(B/(\mathbb{P}^1)^m) - \dim T^2(B/(\mathbb{P}^1)^m) + (3 - 3\dim(\mathbb{P}^1)^m) + 2n,
\]
where \( T^i(B/(\mathbb{P}^1)^m), i = 1, 2 \), are vector spaces fitting into the short exact sequence
\[
0 \to T^0(B) \to H^0(B, f^*T_{(\mathbb{P}^1)^m}) \to T^1(B/(\mathbb{P}^1)^m) \to T^1(B) \to H^1(B, f^*T_{(\mathbb{P}^1)^m}) \to T^2(B/(\mathbb{P}^1)^m) \to 0,
\]
for \( T^i(B) := \text{Ext}^i(\Omega_B, \mathcal{O}_B) \), see e.g. [K1, §2]. The term \( T^1(B/(\mathbb{P}^1)^m) \) represents first order deformations \( f \) (forgetting the base points and markings) whereas \( T^2(B/(\mathbb{P}^1)^m) \) contains the obstructions. Thus
\[
\dim T^1(B/(\mathbb{P}^1)^m) - \dim T^2(B/(\mathbb{P}^1)^m) = \chi(f^*T_{(\mathbb{P}^1)^m}) + \dim T^1(B) - \dim T^0(B)
\]
To compute \( \dim T^1(B) - \dim T^0(B) \), we proceed as in [K1, Prop. 2.2.6]. As \( f \) is finite and hence stable, some power of \( \omega_B \otimes f^*\mathcal{O}_{(\mathbb{P}^1)^m}(1) \) is very ample and gives an embedding \( j : B \hookrightarrow \mathbb{P}^N \). Then \( (j, f) : B \hookrightarrow \mathbb{P}^N \times (\mathbb{P}^1)^m \) is a closed immersion and one computes
\[
\dim T^0(B) - \dim T^1(B) = \chi(f^*T_{(\mathbb{P}^1)^m}) + \chi(T_{\mathbb{P}^N}) - \chi(N_{B, \mathbb{P}^N \times (\mathbb{P}^1)^m}).
\]
Using the Euler sequence, Riemann–Roch on \( B \), the sequence
\[
0 \to f^*T_{(\mathbb{P}^1)^m} \to N_{B, \mathbb{P}^N \times (\mathbb{P}^1)^m} \to N_{B, \mathbb{P}^N} \to 0
\]
and the formula \( \deg N_{B, \mathbb{P}^N} = (N + 1) \deg B + 2g - 2 \), one computes that each component about \([f : B \to \mathbb{P}^1]\) has dimension at least \( 3g - m(g + 2 - 2k) + 2n \).

We now estimate the dimension of \( V_n(m) \).

**Lemma 3.7.** Fix \( g, k, n, m \). Let \( u = [(f, g_1, \ldots, g_n)] \in V_n(m) \) where \( f : B \to \mathbb{P}^1 \) is finite.

1. Every component \( J \) of \( V_n(m) \) containing \( u \) has dimension at least \( 6k - 2n - m - 3 \).
2. Assume in addition that \( \dim [f] \mathcal{M}_{2k-1-n,k-n}((\mathbb{P}^1)^m, \{0, 1, \infty\}; 2n) = 6k - m(n+1) - n - 3 \).
   Then \( \dim J = 6k - 2n - m - 3 \) at \( u \).

**Proof.** Part (1): Let \( J \subseteq V_n(m) \) be a component containing \( u \). The locus \( \mathcal{M}_{0,n}^m((\mathbb{P}^1)^m; 2) \) is irreducible of dimension
\[
2 + h^0(g_i^*T_{(\mathbb{P}^1)^m}) - \dim \text{PGL}(2) = 3m - 1.
\]
Each component of \( \mathcal{M}_{2k-1-n,k-n}((\mathbb{P}^1)^m, \{0, 1, \infty\}; 2n) \) containing \([f : B \to \mathbb{P}^1]\) has dimension at least \( 6k - m(n+1) - n - 3 \) from Lemma 3.6. Each condition \( ev_x(\alpha) = ev_{1,i}(\beta), ev_y(\alpha) = ev_{2,i}(\beta) \) reduces the dimension by at most \( m \). Thus \( J \) has dimension at least
\[
(6k - m(n+1) - n - 3) + n(3m - 1) - 2nm = 6k - 2n - m - 3.
\]
Part (2): Consider \( pr_1 : V_n(m) \to U_n(m) \subseteq \mathcal{M}_{2k-1-n,k-n}((\mathbb{P}^1)^m, \{0, 1, \infty\}; 2n) \). All fibres of \( pr_1 \) are pure of dimension \( n[3m - 1) - 2m] = n(m - 1) \). The claim follows.

### 3.3. Infinitesimal General Position.

If \( f : C \to X \) is a stable map from a nodal curve to a smooth projective variety, then first order deformations of \( f \) are described by \( \text{Ext}^2_C(\Omega_f^*, \mathcal{O}_C) \) and obstructions are given by \( \text{Ext}^2_C(\Omega_f^*, \mathcal{O}_C) \), where \( \Omega_f^* \) is the complex
\[
f^*\Omega_X \xrightarrow{df} \Omega_C,
\]
supported in degrees \(-1, 0\). If \( f \) is unramified at the generic point of each component of \( C \), \( \mathbb{R}\text{Hom}_{\mathcal{O}_C}(\Omega_f^*, \mathcal{O}_C) \) is quasi-isomorphic to \( N_f[-1] \), where \( N_f \) is the normal sheaf, [BHT, §4].
Let $C$ be a nodal curve and $T = \{p, q, r\} \subseteq C$ a marking in the smooth locus of $C$. Let $F : C \to (\mathbb{P}^1)^m \in \mathcal{M}_{g,k}((\mathbb{P}^1)^m,\{0,1,\infty\})$ with base points $T$. For any $\sigma = \{\sigma_1, \ldots, \sigma_j\} \subseteq \{1, \ldots, m\}$, we have a projection

$$pr_\sigma : (\mathbb{P}^1)^m \to (\mathbb{P}^1)^{|\sigma|}.$$ 

Let $F_\sigma := pr_\sigma \circ F$. We say that the set $F$ is successively unobstructed if

$$\text{Ext}^2_C(\Omega^*_F, \mathcal{O}_C(-T)) = 0$$

for all subsets $\sigma \subseteq \{1, \ldots, m\}$. This is equivalent to requiring $h^1(N_{F_\sigma}(-T)) = 0$ for all $\sigma$, provided each $f_i := pr_i \circ F$ is generically unramified. In the next few lemmas we will explore the deformation theoretic meaning of this definition.

**Lemma 3.8.** Let $B$ be a connected, nodal curve of genus $g$. Let

$$F : B \to (\mathbb{P}^1)^m \in \mathcal{M}_{g,k}((\mathbb{P}^1)^m,\{0,1,\infty\}),$$

with base points in $T$, which we assume avoid all ramification of $f_i$, $1 \leq i \leq m$. Assume $f_1, \ldots, f_m$ are finite. Let $\nu : \tilde{B} \to B$ be the normalization, with components $\tilde{B}_1, \ldots, \tilde{B}_r$, and let $\delta$ be the number of nodes of $B$. Then $F$ is successively unobstructed if and only if

$$h^0(N_{F_\sigma}(-T)) = 3(\delta + 1) - |\sigma|(g + 2 - 2k) - \sum_{i=1}^r (3 - 3g(\tilde{B}_i))$$

for all $\sigma \subseteq \{1, \ldots, m\}$. In particular, assuming either:

1. $B$ is integral, or
2. $B = C \bigcup_{i=1}^{r-1} R_i$ for $C$ integral nodal, $R_i \simeq \mathbb{P}^1$, $C \cap R_i = \{x_i, y_i\}$, $1 \leq i \leq r - 1$.

Then $F$ is successively unobstructed if and only if

$$h^0(N_{F_\sigma}(-T)) = 3g - |\sigma|(g + 2 - 2k).$$

**Proof.** We compute $\chi(N_{F_\sigma}(-T))$. Firstly, $\chi(N_{F_\sigma}(-T)) = \chi(N_{F_\sigma}) - 3(|\sigma| - 1)$, so it suffices to compute $\chi(N_{F_\sigma})$. There is a short exact sequence

$$0 \to F_\sigma^* T_{(\mathbb{P}^1)^m}/T_B \to N_{F_\sigma} \to \mathcal{E}xt^1_{\mathcal{O}_B}(\Omega^*_B, \mathcal{O}_B) \to 0,$$

where $j = |\sigma|$, see [BHT, Pg. 541]. The sheaf $\mathcal{E}xt^1_{\mathcal{O}_B}(\Omega^*_B, \mathcal{O}_B)$ is a skyscraper sheaf supported at the nodes of $B$, whereas

$$T_B \simeq \nu_* T_{\tilde{B}}(-\sum_{i=1}^\delta r_i + q_i),$$

where $\nu : \tilde{B} \to B$ is normalization and $r_i, q_i$, lie over the nodes of $B$, [ACG, §11.3]. The proof now follows by Riemann–Roch. \hfill $\Box$

The following is very similar to [LT, Prop. 1.4] and the proof is left to the reader.

**Proposition 3.9.** Let $C$ be a nodal curve, let $T \subseteq C$ be a marking and $f : (C,T) \to X$ a stable map to a smooth projective variety. Then the space of first order deformations $F : (C,T) \to X$ of $f$ such that $F|_T$ is constant is given by $\text{Ext}^2_C(\Omega^*_C, \mathcal{O}_C(-T)).$

Combining Lemma 3.6, Lemma 3.8 and Proposition 3.9 we obtain:

**Corollary 3.10.** Let $B$ be a connected nodal curve of genus $g$ and assume either:

1. $B$ is integral, or
2. $B = C \bigcup_{i=1}^{r-1} R_i$ for $C$ integral nodal, $R_i \simeq \mathbb{P}^1$, $C \cap R_i = \{x_i, y_i\}$, $1 \leq i \leq r - 1$. 


Let $F : B \to (\mathbb{P}^1)^m \in \overline{\mathcal{M}}_{g,k}((\mathbb{P}^1)^m, \{(0, 1, \infty)\})$, with finite components $f_1, \ldots, f_m$ and base points $T$ avoiding the ramification of $\{f_i\}$. Then $F$ is successively unobstructed with respect to $T$ if and only if, for any $\sigma \subseteq \{1, \ldots, m\}$, $\overline{\mathcal{M}}_{g,k}((\mathbb{P}^1)[\sigma], \{(0, 1, \infty)\})$ is smooth at $[F_{\sigma}]$ of dimension $3g - |\sigma|(g + 2 - 2k)$.

Recall from Section 1 the morphism
\[
\pi_k : \mathcal{M}_{g,k}(\mathbb{P}^1, \{(0, 1, \infty)\}) \to \overline{\mathcal{M}}_{g,3}.
\]
Let $C$ be an irreducible nodal curve of genus $g$ admitting a degree $k$ morphism to $\mathbb{P}^1$.

**Proposition 3.11.** Let $C$ be an irreducible nodal curve of genus $g$ admitting a degree $k$ morphism to $\mathbb{P}^1$. Assume that for a general marking $T$ of degree three and any $f : \tilde{C} \to \mathbb{P}^1 \in \pi_k^{-1}[(C, T)]$, the base $\tilde{C}$ is irreducible. Assume $\pi_k^{-1}[(C, T)] = \{f_1, \ldots, f_m\}$ is zero-dimensional of cardinality $m \geq 2$ and that $h^0(f_i^*\mathcal{O}_{\mathbb{P}^1}(2)) = 3$ for $1 \leq i \leq m$. Set
\[
[F := (f_i) : C \to (\mathbb{P}^1)^m].
\]
Then $\pi_k : \mathcal{M}_{g,k}(\mathbb{P}^1, \{(0, 1, \infty)\}) \to \overline{\mathcal{M}}_{g,3}$ is self-transverse in an open subset about $[f]$ if and only if $F$ is successively unobstructed (with respect to $T$).

**Proof.** Let $\sigma = \{\sigma_1, \ldots, \sigma_j\} \subseteq \{1, \ldots, m\}$ and for any $\ell$ set $\sigma\setminus\ell := \sigma \setminus \{\sigma_\ell\}$. Denote by
\[
d\pi_{\sigma} : \text{Ext}_C^1(\Omega_{F_{\sigma}}^* \mathcal{O}_C(-T)) \to \text{Ext}_C^1(\Omega_{C}^* \mathcal{O}_C(-T))
\]
the map taking a first order deformation of $F_{\sigma} : (C, T) \to (\mathbb{P}^1)^j$ preserving $F_{\sigma}(T)$ to a deformation of the marked curve $(C, T)$. Then $d\pi_{\sigma}$ is induced from the triangle
\[
\Omega_{C} \to \Omega_{F_{\sigma}}^* \to F_{\sigma}^* \Omega_{(\mathbb{P}^1)^j}[1].
\]
In particular, since
\[
\text{Hom}_C(F_{\sigma}^* \Omega_{(\mathbb{P}^1)^j}, \mathcal{O}_C(-T)) \cong \bigoplus_{i=1}^j H^0(f_i^* \mathcal{O}_{\mathbb{P}^1}(2)(-T)) = 0
\]
for $T$ general (as $h^0(f_i^* \mathcal{O}_{\mathbb{P}^1}(2)) = 3$), $d\pi_{\sigma}$ is injective. Set $V_{\sigma} := d\pi_{\sigma}(\text{Ext}_C^1(\Omega_{F_{\sigma}}^* \mathcal{O}_C(-T)))$.

The isomorphism $F_{\sigma\setminus\ell}^* \Omega_{(\mathbb{P}^1)^j[1]} \cong f_{\sigma\setminus\ell}^* \mathcal{O}_{\mathbb{P}^1} \to F_{\sigma}^* \Omega_{(\mathbb{P}^1)^j}$ induces a triangle
\[
\Omega_{C} \to \Omega_{F_{\sigma\setminus\ell}}^* \oplus \Omega_{f_{\sigma\setminus\ell}}^* \to \Omega_{F_{\sigma}}^*.
\]
Applying the Ext functor, we obtain
\[
V_{\sigma} = V_{\sigma\setminus\ell} \cap V_{\{\sigma_\ell\}} = V_{\{\sigma_1\}} \cap \ldots \cap V_{\{\sigma_j\}}.
\]
Hence, $\{V_{\{1\}}, \ldots, V_{\{m\}}\}$ is in general position in $\text{Ext}_C^1(\Omega_{C}^* \mathcal{O}_C(-T)) = T_{(C,T)} \overline{\mathcal{M}}_{g,3}$ if and only if, for any $\sigma \subseteq \{1, \ldots, m\}$, $V_{\sigma}$ has codimension $|\sigma|(g + 2 - 2k)$ in $T_{(C,T)} \overline{\mathcal{M}}_{g,3}$, which is equivalent to requiring that $h^1(N_{F_{\sigma}}(-T)) = 0$ by Lemma 3.8.

To conclude, we observe that if $h^1(N_{F_{\sigma}}(-T)) = 0$ for all subsets $\sigma$, then the same holds for an open subset about $(C, T)$. This is immediate from the fact that $\pi_k$ is unramified over $[(C, T)]$, as shown above, together with Corollary 3.10 (as the smooth locus is open).

**Definition 3.12.** Let $C$ be an irreducible nodal curve of gonality $k$ and let $T \subseteq C$ be a marking of degree three in the smooth locus. We say the minimal pencils of $C$ are “infinitesimally in general position” with respect to $T$ if:

1. For any $[f : \tilde{C} \to \mathbb{P}^1] \in \pi_k^{-1}[(C, T)]$, the base $\tilde{C}$ is irreducible, $h^0(f^* \mathcal{O}_{\mathbb{P}^1}(2)) = 3$ and $f$ is etale in an open set about $T$.
2. $\pi_k^{-1}[(C, T)] = \{f_1, \ldots, f_m\}$ is zero-dimensional and $F = (f_i) : C \to (\mathbb{P}^1)^m$ successively unobstructed with respect to $T$. 

We now return to the key construction (Section 3.2). Let \( C \) be an integral curve of genus \( g \) and gonality \( k \leq \frac{g+1}{2} \). We let \( f_1, \ldots, f_m : C \to \mathbb{P}^1 \) be degree \( k \) morphisms, with \( (f_j(p), f_j(q), f_j(r)) = (0, 1, \infty) \) for fixed points \( p, q, r \in C \). Set \( T = p + q + r \) and assume \( n \leq g + 1 - 2k \). Let
\[
F = (f_j) : C \to (\mathbb{P}^1)^m.
\]
For each \( 1 \leq i \leq n \), choose distinct points \( (x_i, y_i) \) of \( C \) with \( f_j(x_i) \neq f_j(y_i) \) for all \( i, j \). Choose
\[
\beta_i : (\mathbb{P}^1, (x_i, y_i)) \to (\mathbb{P}^1)^m, \quad 1 \leq i \leq n
\]
such that \( \beta_i(x_i) = (f_j(x_i))_{1 \leq j \leq m}, \beta_i(y_i) = (f_j(y_i))_{1 \leq j \leq m} \) and let
\[
H : D \to (\mathbb{P}^1)^m
\]
be the stable map of degree \( k + n \) and genus \( g + n \) obtained by glueing \( F \) and \( \beta_i \), §3.2.

**Proposition 3.13.** With notation as above, assume \( F \) is successively unobstructed with respect to \( T \). Assume that \( (x_i, y_i) \) lies outside the ramification locus for \( f_j \) as well as \( H \sigma \) for each \( \sigma \subseteq \{1, \ldots, m\} \) with \( |\sigma| \geq 2 \). Then \( H \sigma \) is successively unobstructed with respect to \( T \).

**Proof.** Let \( \sigma \subseteq \{1, \ldots, m\} \) and consider \( H \sigma = (h_{\sigma,i})_{1 \leq i \leq |\sigma|} \). We need \( H^1(D, N_{H\sigma}(-T)) = 0 \). When \( |\sigma| = 1 \), \( N_{H\sigma} \) has zero-dimensional support, so assume \( |\sigma| \geq 2 \). There is an inclusion
\[
0 \to N_{F\sigma} \to N_{H\sigma}|_{C},
\]
[\cite{GHS}, Lemma 2.6]. The cokernel of \( \alpha \) has zero dimensional support, so \( H^1(C, N_{F\sigma}(-T)) = 0 \) implies \( H^1(D, N_{H\sigma}|_{C}(-T)) = 0 \). By the Mayer-Vietoris sequence, it suffices that \( H^1(N_{H\sigma}|_{R_i}(-2)) = 0 \) for each unstable component \( R_i \) of \( D \). It is enough to show \( H^1(N_{\Delta_\sigma}(-2)) = 0 \) where \( \Delta_i := pr_\sigma \circ \beta_i : \mathbb{P}^1 \cong R_i \to (\mathbb{P}^1)^{|\sigma|} \). This follows from the short exact sequence
\[
0 \to \mathcal{O}_{\mathbb{P}^1}(2) \xrightarrow{d\Delta_i} \mathcal{O}_{\mathbb{P}^1}(2)^{|\sigma|} \to N_{\Delta_i} \to 0.
\]
\( \square \)

We end this section by proving a statement analogous to Proposition 3.13 with regards to the property of being geometrically in general position.

**Proposition 3.14.** With notation as above, assume that \( z_1, \ldots, z_{g-1-k} \) are distinct points of \( C \) such that \( h^0(C, f_j^*\mathcal{O}_{\mathbb{P}^1}(1)(\sum_{j=1}^{g-1-k} z_j)) = 2 \) for \( 1 \leq i \leq m \). Assume that \( z_j \notin \{x_i, y_i \mid 1 \leq i \leq n\} \) for all \( 1 \leq j \leq g - 1 - k \) and that \( \{f_1, \ldots, f_m\} \) are geometrically in general position with respect to \( \{z_i\} \). Then \( \{h_1, \ldots, h_m\} \) are geometrically in general position with respect to \( \{z_i\} \).

**Proof.** Using
\[
0 \to \bigoplus_j \mathcal{O}_{R_j}(-2) \to \mathcal{O}_D \to \mathcal{O}_C \to 0
\]
and twisting by \( h_i^*\mathcal{O}_{\mathbb{P}^1}(1)(\sum_\ell z_\ell) \), we see \( h^0(D, h_i^*\mathcal{O}_{\mathbb{P}^1}(1)(\sum_\ell z_\ell)) \approx h^0(C, f_j^*\mathcal{O}_{\mathbb{P}^1}(1)(\sum_\ell z_\ell)) \). In particular, \( h^0(h_i^*\mathcal{O}_{\mathbb{P}^1}(1)(\sum_\ell z_\ell)) = 2 \). From
\[
0 \to \mathcal{O}_C(-\sum_i x_i + y_i) \to \mathcal{O}_D \to \bigoplus_i \mathcal{O}_{R_i} \to 0
\]
we obtain \( h^0(C, \omega_C \otimes f_j^*\mathcal{O}_{\mathbb{P}^1}(1)(-\sum_\ell z_\ell)) \approx h^0(D, \omega_D \otimes h_i^*\mathcal{O}_{\mathbb{P}^1}(1)(-\sum_\ell z_\ell)) \). In particular, each \( s \in h^0(D, \omega_D \otimes h_i^*\mathcal{O}_{\mathbb{P}^1}(1)(-\sum_\ell z_\ell)) \) vanishes on each \( R_j \). Hence if \( u \in h^0(h_i^*\mathcal{O}_{\mathbb{P}^1}(1)) \),
\[
us \in h^0(D, \omega_D(-\sum_\ell z_\ell))
\]
vanishes on each \( R_j \) and hence lies in the subspace \( h^0(C, \omega_C(-\sum_\ell z_\ell)) \). From the determinantal description of the quadrics associated to pencils, \( Q_{h_i} \) is a cone over \( Q_{f_j} \). Hence \( \{h_1, \ldots, h_m\} \) are in geometrically general position with respect to \( \{z_i\} \). \( \square \)
4. Proof of Theorem 0.2

4.1. Outline of the proof. As our proof will be technical, it may be useful to first outline our argument. Suppose a smooth curve $C$ has genus $g$, gonality $k$ and $m$ minimal pencils $f_i : C \to \mathbb{P}^1$ satisfying the hypotheses of Theorem 0.2. Let $H = (h_i) : D \to (\mathbb{P}^1)^m$ be as in the Key Construction for $n = g + 1 - 2k$, see §3.2. Thus

$$D = C \cup \bigcup_{j=1}^n R_i, \quad R_i \simeq \mathbb{P}^1$$

with $h_i|_C \simeq f_i$ and $\deg(h_i|_{R_i}) = 1$. As is well-known, we have an injective map

$$K_{g-k,1}(C, \omega_C) \to K_{g-k,1}(D, \omega_D)$$

on syzygy spaces, and so it suffices to prove $b_{g-k,1}(D, \omega_D) \leq m(g-k)$ (we have $b_{g-k,1}(C, \omega_C) \geq m(g-k)$ as $\{f_i\}$ are geometrically in general position). For this, it suffices for

$$H = (h_i) : D \to (\mathbb{P}^1)^m \in \widetilde{\mathcal{H}}(m)$$

to not lie in the degeneracy locus $\text{Deg}(F(m))$ of the morphism

$$F(m) : \mathcal{V}(m) \to \mathcal{W}(m)$$

from Proposition 2.3. But $\text{Deg}(F(m))$ has codimension at most one and the dimension counts of §3.2, show that, if $[H] \in \text{Deg}(F(m))$, then there is a one-dimensional family

$$H_t = (h_{i,t}) : D_t \to (\mathbb{P}^1)^m \in \text{Deg}(F(m))$$

with $H_0 = H$ and $D_t$ irreducible for $t \neq 0$. For $t$ general, $H_t$ is successively unobstructed and $\{h_{i,t}\}$ is geometrically in general position for a general divisor of degree $g - 1 - k$. On the other hand, the forgetful map

$$\pi_{k+n} : \mathcal{M}_{g+n,k+n}(\mathbb{P}^1, \{0, 1, \infty\}) \to \overline{\mathcal{M}}_{g+n,3}$$

cannot be self-transverse with fibre of degree $m$ at $[D_t, (p_t, q_t, r_t)]$, for general points $p_t, q_t, r_t$, as otherwise Proposition 1.2 together with [HR] implies $b_{g-k,1}(D_t, \omega_{D_t}) \leq m(g-k)$, contradicting $[H_t] \in \text{Deg}(F(m))$. By Proposition 3.11, either $\pi_{k+n-1}([D_t])$ contains an $(m+1)^{th}$ pencil $h_{m+1,t}$ in addition to $h_{1,t}, \ldots, h_{m,t}$, or there is some $i \leq m$ such that $\dim(h^0(i, O_{\mathbb{P}^1}(1)) \geq 3$. Furthermore, in the first case

$$\lim_{t \to 0} h_{m,t} = h_i$$

for some $1 \leq i \leq m$, by Proposition 3.5. In this case, set $L_t = h_{i,t}^* O_{\mathbb{P}^1}(1) \otimes h_{m+1,t}^* O_{\mathbb{P}^1}(1)$ and in the second case set $L_t = h_{i,t}^* O_{\mathbb{P}^1}(2)$. Observe $h^0(D_t, L_t) \geq 4$.

Let $D \to \Delta$ be a fibred surface over a curve with fibre over $t$ equal to $D_t$, let $\Delta^s \subseteq \Delta$ be the locus $t \neq 0$ and $D^s$ the restriction of the fibred surface to $\Delta^s$ and suppose we have a line bundle $L^0 \in \text{Pic}(D^s)$ with $L^t_t = L_t$. Assume that $L^0$ can be extended to a line bundle $L \in \text{Pic}(D)$ with

$$L_0 \simeq h_{i,t}^* O_{\mathbb{P}^1}(2).$$

Further suppose $D$ is smooth. The components $R \subseteq D$ define Cartier divisors. Define

$$\tilde{L} := L \left( \sum_{j=1}^n R_j \right).$$

Assuming $\{x_i, y_i\} := R_i \cap C$ are chosen generally, one expects $h^0(D, \tilde{L}_0) = 3$, contradicting $h^0(D_t, \tilde{L}_t) = h^0(D_t, L_t) \geq 4$, for $t \in \Delta$ general\(^2\).

In the full proof below we work in a more general set-up, allowing us to bypass the assumptions that $D$ is smooth and that $L^0$ can be extended. We construct the twist $L_0$ in a modular way,
taking inspiration from the theory of twisted canonical divisors, [FarP]. More precisely, we work with stable maps $f_i : D \to \mathbf{P}^1$, together with markings $T \subseteq D$, such that $T$ defines a twisted differential, and use this to twist $f_i^* \mathcal{O}_{\mathbf{P}^1}(2)$ to produce $\mathcal{L}_0$. We then prove that $h^0(D, \mathcal{L}_0) = 3$ for general $\{x_i, y_i\}$ by induction on $n$.

4.2. The proof in full. Recall the space $V_n(m)$, §3.2. A point corresponds to $f : C \to (\mathbf{P}^1)^m$, with marking $(x_i, y_i)$, $1 \leq i \leq n$, together with $n$ marked maps $\beta_i : (\mathbf{P}^1, (x_i, y_i)) \to (\mathbf{P}^1)^m$ such that $f$ and $\beta_1, \ldots, \beta_n$ glue to produce a map

$$h : D \to (\mathbf{P}^1)^m.$$ Performing this procedure in families, we have a morphism

$$q_n(m) : V_n(m) \to \mathcal{M}_{2k-1, k}((\mathbf{P}^1)^m, \{0, 1, \infty\}).$$

Recall from §2 the morphism

$$\mathcal{F}(m) : \mathcal{V}(m) \to \mathcal{W}(m)$$
of sheaves, defined on the open locus $\mathcal{H}(m) \subseteq \mathcal{M}_{2k-1, k}((\mathbf{P}^1)^m, \{0, 1, \infty\})$. We let $\mathcal{H}^U(m) \subseteq \mathcal{H}(m)$ be the open set such that $\mathcal{V}(m)$ and $\mathcal{W}(m)$ are both locally free of rank of rank $(2k - 2)(2k-1) - m(k - 1)$, see Proposition 2.3. Let

$$\mathcal{K}(m) : = \text{Deg}(\mathcal{F}(m)) \cap \mathcal{H}^U(m) \subseteq \mathcal{H}^U(m)$$
be the degeneracy locus of $\mathcal{F}(m)|_{\mathcal{H}^U(m)}$, i.e. locus where $\mathcal{F}(m)$ is not of full rank. Define

$$Z_n(m) : = \text{pr}_1(q_n(m)^{-1}(\mathcal{K}(m)))$$
where $\text{pr}_1 : V_n(m) \to U_n(m) \subseteq \mathcal{M}_{2k-1-n, k-n}((\mathbf{P}^1)^m, \{0, 1, \infty\}; 2n)$ is the projection. Theorem 0.2 follows from the following, stronger result. Recall the morphism

$$\pi_k : \mathcal{M}_{g, k}((\mathbf{P}^1)^m, \{0, 1, \infty\}) \to \mathcal{M}_{g, 3}.$$ Theorem 4.1. Let $C$ be an integral, nodal curve of genus $2k - 1 - n$ and gonality $k - n$ for $k \geq 3$, $0 \leq n \leq 2k - 5$. Suppose

$$\pi_{k-1}^{-1}[C, \{p, q, r\}] = \{f_1, \ldots, f_m : C \to \mathbf{P}^1\}.$$ Let $p, q, r \in C$ be distinct points in the étale locus of each $f_i$, $1 \leq i \leq m$. For $1 \leq i \leq n$, let $(x_i, y_i) \in C$ be distinct pairs of points which are in the étale locus of $f_j$ and with $f_j(x_i) \neq f_j(y_i)$ for $1 \leq j \leq m$. Assume:

1. $\pi_{k-1}^{-1}([B(x_i, y_i), \{p, q, r\}])$ is zero-dimensional of cardinality $m$, where $B(x_i, y_i)$ is the curve obtained from $C$ by glueing $x_i$ to $y_i$ for $1 \leq i \leq n$.
2. For all $S \subseteq \{x_j, y_j \mid 1 \leq j \leq n\}$ of cardinality at most $n$, $h^0(C, f_i^* \mathcal{O}_{\mathbf{P}^1}(2)(\sum_{s \in S} s)) = 3$ for $1 \leq i \leq m$.
3. $\{f_1, \ldots, f_m\}$ are infinitesimally in general position with respect to $\{p, q, r\}$.
4. The line bundles $\{f_i^* \mathcal{O}_{\mathbf{P}^1}(1) \mid 1 \leq i \leq m\}$ are geometrically in general position with respect to a general divisor of degree $k - 2$ on $C$.

Then if $F = (f_i) : C \to (\mathbf{P}^1)^m$, $[F, (x_i, y_i)] \notin Z_n(m)$.

Proof of Theorem 0.2 assuming Theorem 4.1. Let $C$ be a smooth curve of genus $2k - 1 - n$ and non-maximal gonality $k - n$, $0 \leq n \leq 2k - 5$, satisfying the assumptions of Theorem 0.2. In particular, assumptions (3), (4) hold by hypothesis. For $1 \leq i \leq n$, let $(x_i, y_i) \in C$ be general pairs of distinct points. As $p, q, r \in C$ are general, and in particular $\text{Aut}[C, \{p, q, r\} = \{\text{id}\}$,

$$\pi_{k-1}^{-1}([C, \{p, q, r\}]) = \{f_1, \ldots, f_m : C \to \mathbf{P}^1\},$$
with $L_i \simeq f_i^* \mathcal{O}_{\mathbf{P}^1}(1)$. By Proposition 3.5, assumption (1) holds. Next, $h^0(f_i^* \mathcal{O}_{\mathbf{P}^1}(2)) = 3$ implies

$$h^0(\omega_C \otimes f_i^* \mathcal{O}_{\mathbf{P}^1}(-2)) = n + 1,$$
by Riemann–Roch. Hence, if \( \{x_j, y_j \mid 1 \leq j \leq n\} \) are sufficiently general and \( S \subseteq \{x_j, y_j \mid 1 \leq j \leq n\} \) has cardinality at most \( n \) (or even \( n + 1 \)),
\[
h^0(\omega_C \otimes f_i^* \mathcal{O}_{\mathbb{P}^1}(-2)(-\sum_{s \in S} s)) = n + 1 - |S|.
\]
By Riemann–Roch again, one thus sees that (2) holds for \( \{x_j, y_j\} \) general.

Thus \( F = (f_i) : C \to \mathbb{P}^1 \notin Z_n(m) \). This means that, for any
\[
\beta_i : (\mathbb{P}^1, (x_i, y_i)) \to (\mathbb{P}^1)^m, \ 1 \leq i \leq n
\]
of degree one and satisfying \( (\beta_i(x_i), \beta_i(y_i)) = (F(x_i), F(y_i)) \), the resulting map
\[
[g : D \to (\mathbb{P}^1)^m] \in \tilde{H}(m)
\]
as in Section 3.2 is not in \( \mathcal{K}(m) \). Note that \( [g] \in \mathcal{H}^{tf}(m) \) by Proposition 2.3, Proposition 3.3 and Proposition 3.14. Thus \( b_{k-1,1}(D, \omega_D) = m(k-1) \) by Proposition 2.3. By [V2, Corollary 1], \( b_{k-1,1}(C, \omega_C) \leq b_{k-1,1}(D, \omega_D) = m(k-1) \). By Proposition 3.3, \( b_{k-1,1}(C, \omega_C) \geq m(k-1) \) and
\[
j : \bigoplus_{i=1}^m K_{g-k,1}(X_{f_i}, \mathcal{O}_{X_{f_i}}(1)) \to K_{g-k,1}(C, \omega_C)
\]
is injective. So we must have \( b_{k-1,1}(C, \omega_C) = m(k-1) \) and \( j \) is an isomorphism. \( \square \)

We will prove Theorem 4.1 by induction on \( n \). We start with the base case.

**Lemma 4.2.** Theorem 4.1 holds in the case \( n = 0 \) (and arbitrary \( k \geq 3 \)).

**Proof.** Note \( [F = (f_i)] \in \mathcal{H}^{tf}(m) \) by Proposition 2.3 and Proposition 3.3. Further, \( \pi_k \) is self-transverse near each \( f_i \) by Proposition 3.11. Thus the local equation for the Hurwitz divisor \( \mathfrak{hur} \) vanishes to order precisely \( m \) at \( C \), by Proposition 1.2. It follows that the local equation for the Syzygy divisor \( \mathfrak{Sy} \) vanishes to order precisely \( m(k-1) \) and thus \( b_{k-1,1}(C, \omega_C) \leq m(k-1) \) by [HR], [FK3, Th. 3.1]. Thus \( [F] \notin \mathcal{K}(m) \) as required. \( \square \)

It remains to prove the induction step. Let \( 1 \leq \ell \leq 2k - 5 \) and let \( C \) be a smooth curve of genus \( 2k - 1 - \ell \) and gonality \( k - \ell \), together with points \( \{p, q, r\}, \{x_i, y_i \mid 1 \leq i \leq \ell\} \) satisfying all the hypotheses of Theorem 4.1 (for \( n = \ell \)). Assume Theorem 4.1 holds with \( n = \ell - 1 \) and arbitrary \( k = c \) such that \( 0 \leq \ell - 1 \leq 2c - 5 \).

The key technical tool is the following proposition.

**Proposition 4.3.** [FK3, Prop. 3.9, Prop. 3.10] Let \( (\Delta, 0) \) be an irreducible pointed variety and \( G : \mathcal{B} \to \mathbb{P}^1_\Delta \)
a family of stable maps of genus \( g \) and degree \( k \) with \( \mathcal{B}_t \) irreducible for \( t \in \Delta \) general and
\[
\mathcal{B}_0 \simeq C \cup R, \ R \simeq \mathbb{P}^1, \ R \cap C = \{u, v\}, \ \deg G_R = 1
\]
for irreducible \( C \). Then there is a morphism \( \nu : \overline{B} \to B \) between families of nodal curves of \( \Delta \), together with a line bundle \( \tau \in \text{Pic}(\overline{B}) \) such that one of the following cases occur:

1. \( \overline{B}_0 \simeq B_0 \) and further \( \tau_{0|C} \simeq \mathcal{O}_C(u + v) \), \( \deg \tau_{0|R} = -2 \).
2. \( \overline{B}_0 \) is a blow-up of \( B_0 \) at a node \( p \in \{u, v\} \) with exceptional component \( E \). Identifying \( R, C \) with their strict transforms, \( \tau_{0|C} \simeq \mathcal{O}_C(u + v) \), \( \deg \tau_{0|E} = \deg \tau_{0|E} = -1 \).

Furthermore, additionally assume \( h^0(G_0^* \mathcal{O}_{\mathbb{P}^1}(1)) = 2, \ \omega_C \otimes G_0^* \mathcal{O}_{\mathbb{P}^1}(-1) \) is base-point free and that the locus of \( t \in \Delta \) with \( B_t \) reducible has codimension two about \( 0 \in \Delta \). Then, after a base change, we may assume that we are in case (2).

We wish to prove that if \( \pi_{k-n}^{-1}[C, \{p, q, r\}] = \{f_1, \ldots, f_m\} \) and \( F = (f_i) : C \to (\mathbb{P}^1)^m \) then \( [F] \notin Z_\ell(m) \). We first prove a weaker statement.
Proposition 4.4. Let \( 1 \leq \ell \leq 2k - 5 \) and assume Theorem 4.1 holds for \( n = \ell - 1 \). With notation as above, further assume:

\[(2)' : \text{For all } S \subseteq \{x_j, y_j \mid 1 \leq j \leq \ell\} \text{ of cardinality at most } \ell + 1, \]
\[h^0(C, f^*_t \mathcal{O}_{\mathbb{P}^1}(2)(\sum_{s \in S} s)) = 3 \text{ for } 1 \leq i \leq m.\]

Then \([F] \notin \mathcal{Z}_\ell(m)\).

The proof of Proposition 4.4 is very similar to the proof of [FK3, Prop. 3.11]. By Riemann–Roch again, \((2)'\) holds for \( \{x_j, y_j\} \) general.

Proof of Proposition 4.4. Suppose \([F : C \to (\mathbb{P}^1)^m, (x_i, y_i)_{i \leq \ell}] \in \mathcal{Z}_\ell(m)\). Then there exist \( \ell \) two-marked maps \( \beta_i : (\mathbb{P}^1, (x_i, y_i)) \to (\mathbb{P}^1)^m \) which glue to produce a genus \( 2k - 1 \), degree \( k \) stable map \( h : D \to (\mathbb{P}^1)^m \) such that \([h] \in \mathcal{K}(m)\). By Propositions 3.14 and 2.3, this is equivalent to having \( b_{k-1,1}(D, \omega_D) > m(k - 1) \), which does not depend on the choice of maps \( (\beta_i) \) (provided they glue to \( F \)). For a suitable choice of \( (\beta_i) \) we may ensure that each \( (x_i, y_i) \) is outside the ramification locus of \( h_\sigma = pr_\sigma \circ h \) for each \( \sigma \subseteq \{1, \ldots, m\} \) with \(|\sigma| \geq 2\).

Let \( C_\ell := C \cup R_\ell, \ R_\ell \simeq \mathbb{P}^1, \ R_\ell \cap C = \{x_\ell, y_\ell\} \) and let \( F_\ell : C_\ell \to (\mathbb{P}^1)^m \) be obtained by glueing \( F \) and \( \beta_\ell \). Obviously \([F_\ell, (x_i, y_i)_{i \leq \ell - 1}] \in \mathcal{Z}_{\ell - 1}(m)\). Any component \( J \subseteq q_{\ell - 1}(m)^{-1}(\mathcal{K}(m)) \) has dimension at least
\[\dim V_{\ell - 1}(m) - 1 \geq 6k - 2\ell - m - 2\]
by Lemma 3.7. On the other hand, \( V_\ell(m) \) has dimension \( 6k - 2\ell - m - 3 \) by Lemma 3.7, Corollary 3.10 and Proposition 3.13. Thus the general point
\[[F' : C' \to (\mathbb{P}^1)^m, (\beta'_i)] \in J\]
has irreducible base \( C' \).

Thus we have a one dimensional, pointed variety \( (\Delta, 0) \) and marked families
\[\mathcal{G} : B \to (\mathbb{P}^1)^m_\Delta, \ (b_i : \mathbb{P}^1_\Delta \to (\mathbb{P}^1)^m)_{i \leq \ell - 1}\]
with \([\mathcal{G}_t, (b_{i,t})] \in J\) for all \( t \), \( (\mathcal{G}_t, (b_{i,t})) = (F_\ell, (\beta_i)_{i \leq \ell - 1}) \) and with \( B_t \) irreducible for general \( t \). By semicontinuity, \( \mathcal{G}_t \) is successively unobstructed and, if
\[f_{t,i} := pr_i \circ \mathcal{G}_t,\]
then \( \{f_{t,i}^* \mathcal{O}_{\mathbb{P}^1}(1) \mid 1 \leq i \leq m\} \) are geometrically in general position for general \( t \). By [FK3, Lemma 3.12], \( h^0(f_{t,i}^* \mathcal{O}_{\mathbb{P}^1}(2)(\sum_{s \in S} s)) = 3 \) for \( 1 \leq i \leq m \) and any \( S \subseteq \{x_{i,j}, y_{i,j} \mid 1 \leq j \leq \ell - 1\} \) of cardinality at most \( \ell - 1 \), where \( (x_{i,j}, y_{i,j}) \) are the markings on \( B_t \). Lastly, by the same argument as in [FK3, Lemma 3.13],
\[\pi_k^{-1}([C'_{(x_i, y_i), (p', q', r')}]\]
is zero-dimensional of cardinality \( m \), for \( C' := B_t \) and \( t \) general, where \( p', q', r' \) are the base points. By the induction hypothesis, \([\mathcal{G}_t] \notin \mathcal{Z}_{\ell - 1}(m)\), which is a contradiction.

\[\square\]

We can now use Proposition 4.4 to prove the full theorem.
Proof of Theorem 4.1. We follow verbatim the proof of [FK3, Thm. 3.6]. Proceeding by induction, one argues as in Proposition 4.4. Thanks to the conclusion of Proposition 4.4, one now has the additional information that the locus of reducible curves in the component \( J \subseteq q_{r-1}(m)^{-1}(\mathcal{K}(m)) \) has codimension two near \([F_t, (x_i, y_i)]_{i \leq t-1}\). As in [FK3, Thm. 3.6], we then may use [FK3, Lemma 3.12] (with Assumption (I)) as well an identical argument to [FK3, Lemma 3.13] (utilizing case (2) of Proposition 4.3).

5. The Case of Two Pencils

In this section, we consider the classically studied case of curves with two minimal pencils. We start by constructing such examples using K3 surfaces.

5.1. K3 Surfaces and Curves with Two Minimal Pencils. Consider the lattice \( \Lambda = \mathbb{Z}[C] \oplus \mathbb{Z}[E_1] \oplus \mathbb{Z}[E_2] \) and intersection matrix

\[
\begin{pmatrix}
(C \cdot C) & (C \cdot E_1) & (C \cdot E_2) \\
(E_1 \cdot C) & (E_1 \cdot E_1) & (E_1 \cdot E_2) \\
(E_2 \cdot C) & (E_2 \cdot E_1) & (E_2 \cdot E_2)
\end{pmatrix} = \begin{pmatrix}
2g - 2 & k & k \\
k & 0 & 2 \\
k & 2 & 0
\end{pmatrix}
\]

for fixed \( k \geq 3 \). Then \( \lambda \) has discriminant \( 4(k^2 + 2 - 2g) \).

Proposition 5.1. Assume \( g \leq \frac{k^2}{2} \), \( k \leq \frac{g+1}{2} \) and \( k \geq 3 \). There exists a K3 surface \( X_\Lambda \) with \( \text{Pic}(X_\Lambda) \simeq \Lambda \) and with \( E_1 + E_2 \) is big and nef. Further, \( E_i \) are the classes of smooth elliptic curves for \( i = 1, 2 \). If we further assume \( g < \frac{k^2}{2} \), then \( |C| \) is base-point free for \( i = 1, 2 \).

Proof. Since \( g \leq \frac{k^2}{2} \), \( \text{Disc}(\Lambda) > 0 \). The lattice \( \Lambda \) is not positive definite, since \((E_1 - E_2)^2 = -4\). Hence it is even with signature \((1, 2)\) and there exists a K3 surface \( X_\Lambda \) with \( \text{Pic}(X_\Lambda) \simeq \Lambda \), [Mo]. We may take \( E_1 + E_2 \) to be big and nef by [H, Cor. 8.2.9].

To show that \( E_1 \) is the class of a smooth elliptic curve, it suffices to show that \( E_1 \) is nef, [H, Prop. 2.3.10]. Since \( E_1 + E_2 \) is base point free, \((E_1 \cdot E_1 + E_1 + E_2) = 2 > 0 \), the class \( E_1 \) is effective. It suffices to show that there is no smooth rational curve \( R \) with \((R \cdot E_1) < 0 \) and \( E_1 - R \) effective. Suppose such an \( R \) exists. We have \((R)^2 = -2 \) and \( 0 \leq (E_1 - R \cdot E_1 + E_2) \leq 2 \). Suppose firstly that \((R \cdot E_1) \leq -2 \). Then \((E_1 - R)^2 = -(E_1 \cdot R_1) - 2 \geq 2 \). Applying the Hodge Index Theorem [H, Remark 1.2.2] to \((E_1 - R, E_1 + E_2) \) then leads to a contradiction. So we must have \((R \cdot E_1) = -1 \). Suppose \( R = aC + bE_1 + cE_2 \), \( a, b, c \in \mathbb{Z} \). Then \(-1 = (R \cdot E_1) = ak + 2c \). Thus

\[
(R - bE_1)^2 = (aC + cE_2)^2 = a^2(2g - 2) + 2c(ak) = a^2(2g - 2 - k^2) - ak \leq -2a^2 - ak.
\]

On the other hand \((R \cdot E_1 + E_2) \geq 0 \) so \( ak + 2b = (R \cdot E_2) \geq 1 \) and

\[
(R - bE_1)^2 = -2 + 2b \geq -1 - ak.
\]

This forces \( a = 0 \) and hence \(-1 = 2c \) which is impossible. Thus \( E_1 \) is the class of a smooth elliptic curve and likewise for \( E_2 \).

Assume \( g < \frac{k^2}{2} \). We claim that \( |C| \) is base-point free. We firstly claim that \( C \) is nef. If \( C \) is not nef, then there exists a smooth rational curve \( R \) with \((C \cdot R) < 0 \) and \( C - R \) effective. Write \( R = aC + b_1E_1 + b_2E_2 \). Then \( 2b_i \geq -ak \) for \( i = 1, 2 \) since \( E_1, E_2 \) are nef and \( R \) effective. Thus

\[
(R \cdot C) = a(2g - 2) + b_1k + b_2k \geq a(2g - 2 - k^2).
\]

As \( 2g - k^2 \leq 0 \), this implies \( a > 0 \). Next, since \( C - R \) is effective, \( 2b_i \leq (1 - a)k \) for \( i = 1, 2 \) and in particular \( b_i \leq 0 \). Now

\[
-2 = (R)^2 = a(C \cdot R) + b_1(E_1 \cdot R) + b_2(E_2 \cdot R)
\]
and all three terms on the right hand side of the above equation are non-positive. Thus $1 \leq a \leq 2$.

If $a = 2$, then $2b_i \leq -k$ and thus $b_i \neq 0$ for $i = 1, 2$. We then must have $(E_1 \cdot R) = (E_2 \cdot R) = 0$ and so $b_i = -k$ for $i = 1, 2$. But

$$(2C - kE_1 - kE_2)^2 = 4(2g - 2) - 4k^2 < -8,$$

which is a contradiction. So we have $a = 1$. If $(E_i \cdot R) = 0$ then $b_j = -\frac{k}{2} \leq 2$ for $j \neq i \in \{1, 2\}$ and then we are forced to have $(E_j \cdot R) = 0$ and $b_i = -\frac{k}{2}$. We clearly cannot have $b_1 = b_2 = 0$ nor $R = C - E_i$ for some $i = 1, 2$, so we must have $b_1 = b_2 = -\frac{k}{2}$. Then one computes

$$(R)^2 = (C - \frac{k}{2}(E_1 + E_2))^2 = 2g - 2 - k^2 < -2,$$

since $g < \frac{t^2}{2}$, which is a contradiction.

It remains to show that $|C|$ is base-point free. If $|C|$ is not base-point free, then, by Mayer’s Theorem [May] there exists a smooth elliptic curve $F$ and a smooth rational curve $\Gamma$ with $(F \cdot \Gamma) = 1$ and $C = gF + \Gamma$, see also [H, Cor 2.3.15]. As $E_i$ is nef and $(C \cdot E_i) = k < g$, we must have $(E_i \cdot F) = 0$ for $i = 1, 2$. This forces $E_1 = F = E_2$ which is impossible.

In the next few lemmas we check that curves in the class $|C|$ have gonality $k$ and precisely two minimal pencils, both of which are of type $I$ (under certain bounds).

**Lemma 5.2.** Assume $g \leq \frac{4k^2}{9}$, $k \leq \frac{g+1}{2}$ and $k \geq 6$ and let $X_{\Lambda}$ be as in Proposition 5.1 and let $C \in |C|$ be a smooth curve. Let $M$ be a line bundle on $X_{\Lambda}$ such that, if $N = C - M$, then $h^0(M) = h^0(M_C) \geq 2$, $h^0(N) = h^0(N_C) \geq 2$, $h^1(M) = h^1(N) = 0$ and $(M \cdot N) = \text{Cliff}(C) + 2$. Then either $M = E_i$ or $M = C - E_i$ for some $i \in \{1, 2\}$.

**Proof.** Firstly, the assumption $g \leq \frac{4k^2}{9}$ and $k \geq 5$ implies $g < \frac{k^2}{2}$ as needed for Proposition 5.1. We have $O_C(E_i) \in W^1_k(C)$, so gon$(C) \leq k$ and hence Cliff$(C) \leq k - 2$ and

$$(M \cdot N) \leq k.$$

Let $C \in |C|$ be a smooth curve. Riemann–Roch implies that $(M)^2, (N)^2 \geq 0$. Write $M = aC + b_1E_1 + b_2E_2$ for integers $a,b_1,b_2$. After interchanging $M$ and $N$, we may assume $a \leq 0$. We need to show $M = E_i$ for some $i \in \{1, 2\}$.

As both $M$ and $N$ are effective, by intersecting with $E_1, E_2$ we have $-ak \leq 2b_i \leq (1 - a)k$ for $i = 1, 2$. We will firstly show we must have $a = 0$. Suppose $a < 0$. We have

$$(M)^2 = a^2(2g - 2) + 2ak(b_1 + b_2) + 4b_1b_2 \leq a^2(2g - 2) + 2ak(b_1 + b_2) + (b_1 + b_2)^2.$$

Now if $f(t) = a^2(2g - 2) + (2ak)t + t^2$ then the zeroes of $f(t)$ are $-ak \pm a\beta$ for $\beta := \sqrt{k^2 - 2g + 2}$. As $b_1 + b_2 \geq -ak \geq -ak + a\beta$, then from the assumption $(M)^2 \geq 0$ we must have $b_1 + b_2 \geq -ak - a\beta$. Next

$$(N)^2 \leq (1 - a)^2(2g - 2) - 2(1 - a)k(b_1 + b_2) + (b_1 + b_2)^2$$

and, since $(N)^2 \geq 0$, we get $b_1 + b_2 \leq (1 - a)(k - \beta)$. So,

$$-a(k + \beta) \leq b_1 + b_2 \leq (1 - a)(k - \beta), \quad \beta := \sqrt{k^2 - 2g + 2}.$$

In particular, $(1 - a)(k - \beta) \geq -a(k + \beta) + \beta \leq \frac{k}{1 - 2a}$. Since we are assuming $a \leq -1$, we have

$$\beta \leq \frac{k}{3},$$

which gives $\frac{4k^2}{9} - g + 1 \leq 0$. This contradicts our assumption $g \leq \frac{4k^2}{9}$.

Lastly, suppose $a = 0$. Then

$$(M \cdot N) = (b_1 + b_2)k - 4b_1b_2 \geq (b_1 + b_2)(k - (b_1 + b_2))$$

and
and \(0 \leq b_i \leq \frac{k}{2}\) for \(i = 1, 2\) and further \(b_1 + b_2 \leq k - \beta\). If \(b_1 + b_2 = 1\), then we are done, so assume \(b_1 + b_2 \geq 2\). We have seen that the assumption \(g \leq \frac{4k^2}{9}\) implies \(\beta > \frac{k}{2}\) and so
\[
2 \leq b_1 + b_2 < \frac{2k}{3}.
\]

Set \(h(t) = t(k - t)\). Then
\[
(M \cdot N) \geq h(b_1 + b_2) \geq \min(h(2), h(\frac{2k}{3})) = \min(2(k - 2), \frac{2k^2}{9}) \geq k + 1,
\]
for \(k \geq 6\), which is a contradiction.

The previous lemma immediately lets us compute the Clifford index of \(C \in [C]\).

**Lemma 5.3.** Assume \(g \leq \frac{4k^2}{9}\), \(k \leq \frac{q+1}{2}\) and \(k \geq 6\). Let \(X_A\) be as in Proposition 5.1. Every smooth curve of class \([C]\) has Clifford index \(k - 2\).

**Proof.** By [GL1] and [G-Mar], there is a line bundle \(M\) on \(X_A\) such that, if \(N = C - M\), then \(h^0(M) = h^0(M_C) \geq 2\), \(h^0(N) = h^0(N_C) \geq 2\), \(h^1(M) = h^1(N) = 0\) and \((M \cdot N) = \text{Cliff}(C) + 2\). Then by Lemma 5.2, \((M \cdot N) = (E_i \cdot C - E_i) = k\) as required.

In particular, the above lemma implies that, for every smooth curve \(C\) of class \([C]\), \(\text{gon}(C) = k\).

**Lemma 5.4.** Assume \(g < \frac{k^2}{2}\), \(k \leq \frac{q+1}{2}\) and \(k \geq 6\) and let \(X_A\) be as in Proposition 5.1. Then \(h^1(X_A, C - 2E_i) = 0\) for \(i = 1, 2\).

**Proof.** It suffices to consider the case \(i = 1\). We have \((C - 2E_1)^2 = 2g - 2 - 4k \geq -4\). Note that \(H^0(2E_1 - C) = 0\) since \(E_1\) is nef. In case \(k = \frac{q+1}{2}\), it suffices by Riemann–Roch to show that \(C - 2E_1\) is not effective, whereas in case \(k = \frac{q}{2}\) it suffices to show that that \(C - 2E_1\) is the class of a smooth \(-2\) curve. If \(k \leq \frac{q}{2} - 1\), it suffices to show that \(C - 2E_1\) is nef. Thus, in all cases, it suffices to show that there is no smooth rational curve \(R\) with \((R \cdot C - 2E_1) < 0\) and \(C - 2E_1 - R\) effective and non-trivial.

Suppose such an \(R = aC + b_1E_1 + b_2E_2\) exists. Since \(2b_i \geq -ak\) for \(i = 1, 2\),
\[
(R \cdot C - 2E_1) = a(2g - 2 - 2k) + b_1k + b_2(k - 4) \geq a(2g - 2 - k^2).
\]
As \(2g - 2 - k^2 < 0\) we have \(a \geq 1\). Next, since \(C - 2E_1 - R\) is effective, \(2(b_1 + 2) \leq (1 - a)k\) and \(2b_2 \leq (1 - a)k\). Rewriting the first inequality as \(b_1 + 2a \leq (1 - a)(\frac{k}{2} - 2)\).

\[
-2 = (R)^2 = a(C - 2E_1 \cdot R) + (b_1 + 2a)(E_1 \cdot R) + b_2(E_2 \cdot R),
\]
and all three terms on the right are non-positive, so \(1 \leq a \leq 2\).

If \(a = 2\), we need \((E_1 \cdot R) = (E_2 \cdot R) = 0\) which forces \(b_1 = b_2 = -k\) and then \((R)^2 < -8\), which is impossible. So \(a = 1\). If \((E_1 \cdot R) = 0\), then \(b_2 = -\frac{k}{2} \leq -3\) for which in turn forces \((E_2 \cdot R) = 0\) and \(b_1 = -\frac{k}{2}\). If \((E_2 \cdot R) = 0\) then \(b_1 = -\frac{k}{2}\) and \(b_2 \in \{-\frac{k}{2}, \frac{1-k}{2}\}\). In all cases, \((R)^2 < -2\) using \(g < \frac{k^2}{2}\), which is a contradiction.

Thus \(a = 1\) and \((E_1 \cdot R) \neq 0\), \((E_2 \cdot R) \neq 0\). In this case, we must have either \(b_1 = -2\) or \(b_2 = 0\). We cannot have \((b_1, b_2) = (-2, 0)\) since \(C - 2E_1 - R\) is assumed to be nontrivial. If \(b_2 = 0\) then \((E_1 \cdot R) = k\) which contradicts the above formula for \((R)^2 = -2\). So we must have \(b_1 = -2\) giving the contradiction \((E_2 \cdot R) = k - 4 \geq 2\).

Putting everything together, we now obtain:
Proposition 5.5. Assume \( g \leq \frac{4k^2}{3} \), \( k \leq \frac{g+1}{2} \) and \( k \geq 6 \) and let \( X_\Lambda \) be as in Proposition 5.1. Let \( C \) be a smooth curve of class \([C]\) and let \( A_i = E_i|_C \) for \( i = 1, 2 \). Then \( A_1 \) and \( A_2 \) are not isomorphic, \( h^0(C, 2A_1) = h^0(C, 2A_2) = 3 \) and \( W_k^1(C) = \{A_1, A_2\} \). Further, the map \( g : C \to \mathbf{P}^1 \times \mathbf{P}^1 \) induced by \(|A_1| \times |A_2|\) is birational to its image. Lastly, \( C \) has gonality \( k \) and satisfies bpf-linear growth.

Proof. By Lemma 5.4 and the sequence
\[
0 \to 2E_i - C \to 2E_i \to 2A_i \to 0
\]
we see \( h^0(C, 2A_i) = h^0(X_\Lambda, 2E_i) = 3 \) for \( i = 1, 2 \). Next, observe that if \( M \) is any line bundle on \( X_\Lambda \) with \( h^1(M) = 0 \), \( (M \cdot E_i) > 0 \), then from
\[
0 \to M \to M + E_i \to (M + E_i)|_{E_i} \to 0,
\]
we have \( h^1(M + E_i) = 0 \). Thus \( h^1(C + E_1 - E_2) = h^1((C - 2E_2) + E_2 + E_1) = 0 \). From
\[
0 \to E_2 - E_1 - C \to E_2 - E_1 \to A_2 \otimes A_1^* \to 0
\]
we see \( h^0(A_2 \otimes A_1^*) = h^0(E_2 - E_1) = 0 \), since \( E_2 \) is nef and \((E_2 \cdot E_2 - E_1) < 0 \). Thus \( A_1 \) and \( A_2 \) are not isomorphic.

Suppose \( L \in W_k^1(C) \). Since \( \text{gon}(C) = k \) from Lemma 5.3, \( h^0(C, L) = 2 \) and \( L \) is basepoint free. By [DM, §4], there exists a line bundle \( M \) on \( X_\Lambda \) such that, setting \( N = C - M \),
\[
h^0(M) = h^0(M_C) \geq 2, \quad h^0(N) = h^0(N_C) \geq 2, \quad h^1(M) = h^1(N) = 0, \quad \deg(M_C) \leq g - 1 \text{ and } (M \cdot N) = k.
\]
Further, there is a reduced \( Z_0 \in |L| \) with \( Z_0 \subseteq M \cap C \). From the proof of Lemma 5.2, this implies \( M = E_i \) for some \( i = 1, 2 \). As \( Z_0 \subseteq E_i \cap C \), and \( k = \deg(Z_0) = (E_i \cdot C) \), we must have \( L = A_1 \). Lastly, \( C \) has Clifford index \( k - 2 \) and gonality \( k \) by Lemma 5.3 and satisfies bpf-linear growth by Lemma 5.2 and Theorem A.4.

It remains to show that the map \( g : C \to \mathbf{P}^1 \times \mathbf{P}^1 \) induced by \(|A_1| \times |A_2|\) is birational to its image. We claim that the morphism \( h : X_\Lambda \to \mathbf{P}^1 \times \mathbf{P}^1 \) induced by \(|E_1| \times |E_2|\) is finite. Indeed, otherwise there would be some effective class \( R \) with \((R)^2 = -2, (R \cdot E_1) = (R \cdot E_2) = 0 \). Thus \( R = aC - \frac{a^2}{2}(E_1 + E_2) \) and \(-2 = a^2(2g - 2 \times 2) \) which is impossible as \( 2g - 2 - k^2 \leq -3 \). Thus \( h \) is finite of degree two and \( g = h|_C \). Suppose \( g \) has degree two, i.e. \( h^{-1}(h(C)) = C \). As \( h_*[C] = (k, k) \) this implies \([C] \in \mathbb{Z}[E_1] \oplus \mathbb{Z}[E_2] \subseteq \Lambda \), which is a contradiction. Thus \( g \) is birational.

\[ \square \]

5.2. Coppens’ Construction and Bpf-Linear Growth. Recall a construction of Coppens, [C]. Fix \( g \geq 8 \) and \( k \geq 4 \). A necessary condition for the existence of a curve of genus \( g \) carrying two independent pencils of type \( I \) is that \( g \leq (k - 1)^2 \). Coppens proved that this bound is sufficient with an inductive argument, which we now recall.

Let \( C \) be a smooth curve of genus \( g - 1 \geq 8 \). Recall that two pencils \( f_1, f_2 : C \to \mathbf{P}^1 \) are said to be independent if there exists no automorphism \( i : C \to C \) such that \( f_2 = f_1 \circ i \). The pencil \( f_i \) is further said to be of type \( I \) if \( h^0(f_i^*\mathcal{O}_{\mathbf{P}^1}(2)) = 3 \).

Assume \( C \) is a sufficiently general curve with two independent pencils \( f_1, f_2 \) and genus \( g \leq (k - 1)^2 \). Then there are points \( p \neq q \in C \) such that \( f_i(p) = f_i(q) \) for \( i = 1, 2 \). Thus if \( D \) is the nodal curve of genus \( g \) obtained by identifying \( p \) and \( q \), we have two pencils \( f'_1, f'_2 : D \to \mathbf{P}^1 \). If \( L_i := f'_i^*\mathcal{O}_{\mathbf{P}^1}(1) \), one shows that \( L_1, L_2 \) are the unique rank one, torsion free sheaves on \( D \) with degree less than or equal to \( k \) and at least two sections. Thus \( D \) has gonality \( k \) and precisely two minimal pencils, and furthermore one checks that \( D \) can be deformed to a smooth curve with precisely two independent minimal pencils of degree \( k \), both of type \( I \).

We will show that, with the notation above, that if one in addition assumes \( k \leq \frac{g+9}{4} \) and that \( C \) satisfies bpf-linear growth, then the nodal curve \( D \) of genus \( g \) may be deformed to a smooth curve with precisely two minimal pencils which furthermore satisfies bpf-linear growth.
Let $\overline{\text{Pic}}^d(D)$ denote the compactified Jacobian of rank one, torsion free sheaves of degree $k$ on $D$ and let

$$W^1_d(D) := \{ M \in \overline{\text{Pic}}^d(D) \mid h^0(M) \geq 2 \}$$

be the closed subset of those sheaves with at least two sections.

**Lemma 5.6.** Let $C, D$ be as above. Assume $C$ satisfies bpf-linear growth. Assume $0 \leq n \leq g-2k$ and let $Z \subseteq W^1_{k+n}(D)$ be a component of $W^1_{k+n}(D)$ of dimension at least $n-1$. Then one of the following cases hold:

1. $Z$ has dimension precisely $n-1$.
2. The general point of $Z$ is a line bundle $[M] \in Z$ with $M = L_i(T)$ for $L_i \in W^1_k(D)$ and $T \subseteq D$ a reduced divisor in the smooth locus. In this case $\dim(Z) = n$.

**Proof.** We prove this by induction on $n$, the statement holding for $n = 0$ since $W^1_k(D) = \{ L_1, L_2 \}$. Suppose $\dim(Z) \geq n$ and let $[M] \in Z$ be a general point. We first of all claim that $M$ is locally free. Suppose otherwise. Then, letting $\nu : C \to D$ be the normalization morphism, we have $M = \nu_* N$ for some line bundle $N \in W^1_{k+n-1}(C)$. Thus $\dim W^1_{k+n-1}(C) \geq n$ contradicting that $C$ satisfies bpf-linear growth.

Thus the general point $M$ is locally free. Let $x \in D$ be the node. We firstly claim that $x$ is not a base point of $M$, for $[M] \in Z$ general. Suppose otherwise. Then, setting

$$M' := \text{Ker}(M \to M_x),$$

where $M \to M_x$ is the evaluation morphism, we have that $[M'] \in Z'$, where $Z' \subseteq W^1_{k+n-1}(D)$ is an irreducible closed subset. Note that, if $M'$ is locally free then $\dim \text{Ext}^1(M_x, M') = 1$ and otherwise $\dim \text{Ext}^1(M_x, M') = 2$, [K2, Lemma 5.10]. Thus, in the former case, $\dim(Z') \geq n$, which contradicts the induction hypothesis, and in the latter case $\dim(Z') \geq n-1$ and the general point of $Z'$ is not locally free, which also contradicts the induction hypothesis.

Thus the node $x$ is not a base point of $M$. Hence, setting

$$M' := \text{Im}(H^0(M) \otimes \mathcal{O}_D \to M)$$

to be the base-point free part of $M$, we have that $M' = M(-T_1)$ for some effective divisor $T_1$ of degree $t_1$. Assume firstly that $t_1 \geq 1$, that is, that $M$ is not base-point free. We have $[M'] \in Z' \subseteq W^1_{k+n-1}(D)$ where $Z'$ has dimension at least $n-t_1$ and thus, by induction, we must have $M' = L_i(T_2)$ for a general effective divisor $T_2$ of degree $n-t_1$ and $i \in \{ 1, 2 \}$. Further, we must have equality $\dim(Z') = n-t_1$ and so $T_1$ may to chosen to be a general effective divisor of degree $t_1$. This gives the claim.

We are left with the case where $M$ is a base-point free line bundle. But this case cannot occur. Indeed, applying $\nu^*$ would produce a component $Z' \subseteq G^1_{k+n}(C)$ of dimension $n$, which is impossible as we are assuming that $n \leq g-2k = (g-1) + 1 - 2k$ and $C$ satisfies bpf-linear growth. \hfill $\square$

Let $W^1_{d,lf}(D) \subseteq W^1_d(D)$ denote the open locus of locally free sheaves which we endow with the scheme structure of a determinantal variety in the usual way, [ACGH, Ch. IV].

**Lemma 5.7.** Let $C, D$ be as above with $W^1_k(D) = \{ L_1, L_2 \}$. Let $T \subseteq D$ be a general, reduced divisor in the smooth locus of degree $0 \leq n \leq g+2-2k$. Then $W^1_{k+n,lf}(D)$ is smooth of dimension $n$ at the point $[L_i(T)]$.

**Proof.** As already remarked above, $L_i$ are of type $I$, i.e. $h^0(D, L_i^\otimes 2) = 3$ for $i = 1, 2$. Further $h^1(L_i) = g+1-k$ by Riemann–Roch, so if $T$ is general $h^1(L_i(T)) = h^0(\omega_D \otimes L_i^*(-T)) = g+1-k-n$ and thus $h^0(L_i(T)) = 2$ by Riemann–Roch. Likewise, $h^0(D, L_i^\otimes 2(T)) = 3$. By
Proposition 4.2 of [ACGH, Ch. IV] (which goes through verbatim in the case of an integral, nodal curve), the tangent space to $W_{k+n}^{1,\text{bpf}}(D)$ is then $(\text{Im}(\mu))^\perp$, where

$$\mu : H^0(D, L_i(T)) \otimes H^0(D, \omega_D \otimes L_i^*(-T)) \to H^0(D, \omega_D)$$

is the Petri map. Thus $T_{[L_i(T)]}W_{k+n}^{1,\text{bpf}}(D)$ has dimension $g - 2(g + 1 - k - n) + \dim \ker(\mu)$. By the base-point free pencil trick [ACGH, Pg. 126] (which holds in our context), $\dim \ker(\mu) = h^0(\omega_D \otimes L_i^{g-2}(-T)) = g + 2 - 2k - n$, by Riemann–Roch. Thus $\dim T_{[L_i(T)]}W_{k+n}^{1,\text{bpf}}(D) = n$. But obviously $\dim W_{k+n}^{1,\text{bpf}}(D) \geq n$, so this finishes the proof.

Putting the above lemmas together we obtain the following result.

**Proposition 5.8.** Let $C, D$ be as above where $D$ has genus $g$ and gonality $k$. Assume $C$ satisfies bpf-linear growth and let $3 \leq k \leq \frac{2g+8}{3}$. Let $(\Delta, 0)$ be a smooth, pointed curve and $D \to \Delta$ a flat family of nodal curves such that $D_0 \simeq D$ and $D_t$ is smooth of gonality $k$ for $t \neq 0$. Assume $W_k^1(D_t) = \{L_{1,t}, L_{2,t}\}$, where $L_{1,t}$ and $L_{2,t}$ are not isomorphic and of type $I$. Then $D_t$ satisfies bpf-linear growth for $t \in \Delta$ general.

**Proof.** By Lemma A.3, it suffices to check $\dim W_{k+n}^{1,\text{bpf}}(D_t) < n$ for $1 \leq n \leq g - 2k$. Suppose $\dim W_{k+n}^{1,\text{bpf}}(D_t) \geq n$ for general $t$ and some $1 \leq n \leq g - 2k$. After a finite base change, we have a variety $W$, together with a proper morphism $W \to \Delta$, with fibre over $t$ equal to $W_{k+n}^{1,\text{bpf}}(D_t)$, cf. [ACG, Ch. XXI]. By assumption, we have a component $I \subseteq W$ of relative dimension at least $n$ over $\Delta$ whose general point is base-point free. By Lemma 5.6, the fibre $I_0$ over $0$ must contain the point $[L_i(T)]$ for some $i \in \{1, 2\}$ and $T$ any general Cartier divisor of degree $n$. But, by Lemma 5.7, $W$ is smooth at $[L_i(T)]$. But there is a closed subset $J \subseteq W$ of relative dimension $n$ containing all points of the form $L_{i,t}(T_t)$ for $T_t$ a general divisor of degree $d$ on $D_t$. As $[L_i(T)]$ lies in a unique component, we must have $J = I$. This contradicts the assumption that the general point of $I$ is base-point free.

Let $\mathcal{M}_{g,k}(2) \subseteq \mathcal{M}_g$ denote the moduli space of smooth curve of genus $g$ and gonality $k \leq \frac{g+1}{2}$ such that $W_k^1(C) = \{L_1, L_2\}$ where $L_1$ and $L_2$ are independent and of type $I$. Assume $k \geq 4$ and $g \geq 8$. Then $\mathcal{M}_{g,k}(2)$ is nonempty if and only if $g \leq (k - 1)^2$, [C], and furthermore is irreducible, [Ty].

**Theorem 5.9.** Assume $k \geq 6$, $g \geq 8$ and let $[C] \in \mathcal{M}_{g,k}(2)$ be general. Then $b_{g-k,1}(C, \omega_C) = 2(g - k)$.

**Proof.** We will firstly show that $[C]$ satisfies bpf-linear growth. Note that if $k \geq 5$ we always have either $g \leq \frac{4k^2}{9}$ or $k \leq \frac{g+8}{4}$. If $g \leq \frac{4k^2}{9}$, then the fact that a general $[C]$ satisfies bpf-linear growth follows from Proposition 5.5. The remaining cases now follow by Coppens’ inductive construction and Proposition 5.8. Note that the base case in Coppens’ construction is $g = 2k - 1$, which falls into the range of Proposition 5.5 (which in particular gives a new proof of [C, §2]).

It follows from Coppens’ construction that the two minimal pencils on a general point $[C]$ have only ordinary ramification. Let $[C] \in \mathcal{M}_{g,k}(2)$ be general and choose general points $p, q, r \in C$. We claim that the two pencils $f_1, f_2 : C \to \mathbb{P}^1$ are infinitesimally in general position with respect to $\{p, q, r\}$. This amounts to showing $H^1(C, N_F(-p - q - r)) = 0$, where $F = (f_1, f_2)$. As in [AC2], there is an exact sequence

$$0 \to \mathcal{O}_Z \to N_F \to N'_F \to 0,$$

where $Z$ has zero-dimensional support and $N'_F$ is a line bundle. By [AC2, Prop. 2.4], $N_F$ is a line bundle of degree $2g - 2 + 4k$ and so $N_F(-p - q - r)$ has degree greater than $2g - 1$, giving $H^1(C, N_F(-p - q - r)) = 0$. 


It remains to prove that \( f_1, f_2 \) are geometrically in general position. Let \( T \in C_{g-1-k} \) be general. We need to show that \([Q_{f_1}] \neq [Q_{f_2}] \in |O_{\mathbb{P}}(2)|\), in the notation of Section 3.1. Assume otherwise. Then \( \tilde{Q}_{f_1} = \tilde{Q}_{f_2} \) as quadrics in \( \mathbb{P}^{d-1} \) and hence their resolutions \( \mathbb{P}(V_1), \mathbb{P}(V_2) \) are isomorphic. Noting that \( C \) does not entirely lie in the vertex of \( \tilde{Q}_{f_1} \) (as \( C \subseteq \mathbb{P}^{d-1} \) is nondegenerate), we have an isomorphism \( \psi : \mathbb{P}(V_1) \to \mathbb{P}(V_2) \) preserving both the hyperplane class and the image of \( C \). Let \( R \) and \( H \) denote the ruling and hyperplane class of \( \mathbb{P}(V_1) \), and let \( R_2 \in \text{Pic}(\mathbb{P}(V_1)) \simeq \text{Pic}(\mathbb{P}(V_2)) \) denote the class of the ruling. So \( R_2 = \alpha R + \beta H \) for \( \alpha, \beta \in \mathbb{Z} \). Intersecting with \( H^\dim\mathbb{P}(V_1) - 1 \) produces the equation \( 1 = \alpha + \deg(\mathbb{P}(V_1))\beta = \alpha + 2\beta \). Restricting to \( C \), we obtain

\[
L_2 \simeq (1 - 2\beta)L_1 + \beta \omega_C,
\]

where \( L_i \simeq f_i^*O_{\mathbb{P}^1}(1) \). Taking degrees and using that \( k \neq g - 1 \), we see \( \beta = 0 \) and so \( L_1 \simeq L_2 \) which is a contradiction.

**Appendix A. Assorted Results on Bpf-Linear Growth**

In this Appendix we gather some results on bpf-linear growth which are implicit in the existing literature, in particular in works of Mumford, Keem and Aprodu–Farkas.

We begin with an easy observation about the bpf-linear growth condition. For a smooth curve \( C \), recall that

\[
W_d^r(C) := \{ L \in \text{Pic}^dC \mid \alpha(C, L) \geq r + 1 \}
\]

which can be given the structure of a determinantal variety, [ACGH]. We set \( W_k^{r,\text{bpf}}(C) \subseteq W_d^r(C) \) to be the open locus of base-point free line bundles.

**Lemma A.1.** A smooth curve \( C \) satisfies bpf-linear growth if and only if \( \dim W_k^1(C) = 0 \) and \( \dim W_k^{1,\text{bpf}}(C) < n \) for \( 1 \leq n \leq g + 1 - 2k \).

**Proof.** It is clear that the above condition is equivalent to

\[
\dim W_k^1(C) \leq m, \quad \text{for } 0 \leq m \leq g - 2k + 1
\]

\[
\dim W_k^{1,\text{bpf}}(C) < m, \quad \text{for } 0 < m \leq g - 2k + 1.
\]

From the proof of [FK3, Lemma 3.3], this is equivalent to the bpf-linear growth conditions

\[
\dim G_k^1(C) \leq m, \quad \text{for } 0 \leq m \leq g - 2k + 1
\]

\[
\dim G_k^{1,\text{bpf}}(C) < m, \quad \text{for } 0 < m \leq g - 2k + 1.
\]

The bpf-linear growth condition is implicit in the well-known work of Mumford and Keem on dimensions of Brill–Noether loci. In fact, we have:

**Theorem A.2** (Mumford–Keem). Let \( C \) be a smooth curve of genus \( g \) and gonality \( k \geq 3 \).

1. If \( k = 3 \), then \( C \) satisfies bpf-linear growth unless \( C \) admits a degree two morphism to an elliptic curve.

2. If \( k = 4 \) and \( g \geq 11 \), then \( C \) satisfies bpf-linear growth unless \( C \) admits a degree two morphism to a curve of genus \( \ell \) for \( 1 \leq \ell \leq 2 \).

3. If \( k = 5 \) and \( g \geq 15 \) then \( C \) satisfies bpf-linear growth unless \( C \) admits either a degree three morphism to an elliptic curve or a degree two morphism to a curve of genus three.

**Proof.** The first part of the Theorem is proven in the course of the proof of the theorem from [Mu, Appendix], whereas the second and third parts are proven in [Ke, Thm 2.1, 3.1].

The arguments of Mumford and Keem also show that, provided \( k \) is small enough, the condition \( \dim W_{g-k+1}^{1,\text{bpf}}(C) < g - 2k + 1 \) of bpf-linear growth is redundant.
Lemma A.3. Let $C$ be a smooth curve of gonality $3 \leq k \leq \frac{2g+8}{3}$. Suppose $\dim W_{k+n}^{1,bpf}(C) < n$ for $1 \leq n \leq g - 2k$ and $\dim W_k^1(C) = 0$. Then $\dim W_{g-k+1}^{1,bpf}(C) < g - 2k + 1$.

Proof. We follow an argument of Mumford and Keem. Suppose there is a component $Z \subseteq W_{g-k+1}^1(C)$ of dimension at least $g + 2 - 2k$ such that the general element $[L] \in Z$ is base-point free. By the assumption $\dim W_{k+n}^{1,bpf}(C) < n$ for $1 \leq n \leq g - 2k$ and $\dim W_k^1(C) = 0$, we must have $h^0(L) = 2$. By results of Kempf and Severi (see [Ke, §1]), this gives

$$\rho(g, 1, g + 1 - k) + h^0(\omega_C \otimes L^{-1}) = g - 2k + h^0(\omega_C \otimes L^{-1}) \geq g + 2 - 2k$$

and hence $h^0(\omega_C \otimes L^{-2}) \geq 2$. On the other hand, $\deg(\omega_C \otimes L^{-2}) = 2k - 4$, so $\dim W_{2k-4}^1(C) \geq g + 2 - 2k$. But this contradicts H. Martens’ Theorem [H-Mar] since $3 \leq k \leq \frac{2g+8}{3}$. □

Using the results of [AF], one may prove that, for curves of non-maximal gonality $k$ which lie on a K3 surface, a sufficient condition for bpf-linear growth is that the only line bundles $A$ achieving the Clifford index are elements of $W_k^1(C)$.

Theorem A.4. Let $C$ be a curve of gonality $k \leq \lfloor \frac{2g+1}{3} \rfloor$ and genus $g \geq 3$, abstractly embedded on a K3 surface $S$. Assume $C$ has Clifford dimension one. Suppose that for any line bundle $M$ on $S$ satisfying the properties

1. $h^0(M) = h^0(M_C) \geq 2$ and $h^1(M) = 0$,
2. Setting $N = C - M$, we have $h^0(N) = h^0(N_C) \geq 2$ and $h^1(N) = 0$,
3. $(M \cdot N) \leq k$

then either $M_C \in W_k^1(C)$ or $N_C \in W_k^1(C)$. Then $C$ satisfies bpf-linear growth.

Proof. Set $L = \mathcal{O}_S(C)$. For any base point free pencil $A$ of degree $d$ on $C$, with $k < d \leq g+1-k$, we have a corresponding short exact sequence

$$0 \to M \to E \to N \otimes I \to 0,$$

on $S$, where $E$ is a rank two Lazarsfeld–Mukai bundle satisfying $c_2(E) = d$, $L, M$ are line bundles and $I$ is the ideal sheaf of a zero-dimensional subscheme of $S$ of length $\ell \geq 0$, [DM, Lemma 4.4]. We further may assume $M - N$ is effective (possibly after swapping $M$ and $N$ if $\ell = 0$), $h^0(M), h^0(N) \geq 2$ and $N$ is base-point free, see [CP, Lemma 2.1].

From the results in [AF, Lemmas 3.9.3.10], it suffices to show we have the strict inequality $(M \cdot N) > k$. This is automatic if $\text{Cliff}(M_C) > k - 2$ or $h^1(S, M) \neq 0$, so assume $\text{Cliff}(M_C) = k - 2$, $h^1(M) = 0$ and $(M \cdot N) \leq k$.

From

$$0 \to N^* \to M \to M_C \to 0,$$

and the fact that $h^0(N^*) = 0$, $h^1(N) = 0$, we see $h^0(M) = h^0(M_C) \geq 2$. Further, $h^0(N) = h^1(M_C) = h^0(N_C) \geq 2$. Thus, by our assumptions, $M_C \in W_k^1(C)$ or $N_C \in W_k^2(C)$. As we are assuming $M - N$ is effective and since $C$ is nef, $(M - N \cdot C) \geq 0$ so $N \cdot C \leq M \cdot C$ and thus $N_C \in W_k^1(C)$. Thus $h^0(N) = 2$, $h^1(N) = h^2(N) = 0$ and the base-point free line bundle $N$ is the class of a smooth elliptic curve (by Riemann–Roch).

But, by [DM, Corollary 4.5], there is some divisor $N' \in |N|$ and some reduced $A' \in |A|$ such that $A' \subseteq N' \cap C$. As $C$ is irreducible of genus greater than one, this forces $d \leq k = (N \cdot C)$ which is contradiction. □

As a Corollary, we obtain the claim that a sufficient condition for bpf-linear growth is that the only line bundles $A$ achieving the Clifford index are elements of $W_k^1(C)$.

Corollary A.5. Let $C$ be a curve of gonality $k \leq \lfloor \frac{2g+1}{3} \rfloor$, abstractly embedded on a K3 surface $S$. Suppose that if $A$ is a line bundle with $h^0(A) \geq 2$, $h^1(A) \geq 2$, $\deg(A) \leq g - 1$ and $\text{Cliff}(A) = \text{Cliff}(C)$, then we have $A \in W_k^1(C)$. Then $C$ satisfies bpf-linear growth.
Proof. This is the same as the proof of Theorem A.4, with the assumptions once again forcing $N_C \in W^1_k(C)$. □

References


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