FAST SPIRAL FOURIER TRANSFORM FOR ITERATIVE MR IMAGE RECONSTRUCTION

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ABSTRACT

We present a fast and accurate Discrete Spiral Fourier Transform and its inverse. The inverse solves the problem of reconstructing an image from MRI data acquired along a spiral k-space trajectory. First, we define the spiral FT and its adjoint. These discrete operators allow us to efficiently compute the inverse using fast-converging conjugate gradient methods. Next, we developed a fast approximate spiral FT using the pseudo-polar FFT, to enhance the computational performances and numerical accuracy of the algorithm. Preliminary results demonstrate that the proposed algorithm is more accurate than existing iterative methods that use similar interpolation and grid size.

1. INTRODUCTION

Spiral MRI has received much attention because of its fast acquisition, efficient use of the hardware, low motion and flow sensitivity. One can regard the spiral MRI as a physical device, which computes a spiral FT of an image. Formalizing this mathematically, we define the spiral FT as the operator that takes the digital image into the frequency domain by evaluating the FT on a family of spirals. The problem of reconstructing the image is then the problem of inverting the spiral FT.

Heuristic approaches to approximately reconstructing and image from spiral MRI data have been well addressed in literature [1-4], though often without specifically identifying the problem as one of inverting the spiral FT operator. Most of the approaches rely on interpolating the non-uniform data samples from the spiral grid onto a usually over sampled Cartesian grid and then applying a 2D FFT to reconstruct the data (i.e. gridding algorithms). These methods differ mostly by the choice of interpolation kernel and grid size. A weighting is usually applied on the spiral data before the interpolation to compensate for the difference in the grids’ sampling density.

To obtain a more accurate inverse spiral FT, iterative methods have been developed [5]. As we will discuss later, inverting the spiral FT can be achieved iteratively by combined application of the forward spiral FT and its adjoint operations. Existing iterative approaches can be viewed as computing approximations to the forward and adjoint transforms using gridding. One can think of the gridding algorithm as approximating the adjoint of the spiral transform, and the density compensation in the gridding algorithm as preconditioning. Another way to think about it would be as a single iteration of the iterative approach.

The above methods are fast and efficient in providing an approximate operation for the inverse spiral FT. However, the interpolations in the methods above are done onto a uniformly sampled Cartesian grid whereas the spiral trajectories have a highly non-uniform density around the center of k-space. Reconstruction then becomes a tradeoff between extensive over sampling to resolve the low frequency components at the expense of complexity, or loosing valuable information in the low frequency region.

We have developed fast forward and inverse spiral FT procedures. Specifically, we are able to accurately and rapidly compute the forward spiral FT and its adjoint. In our proposed method we use re-sampling onto a variable density grid (The pseudo-polar grid) that is more natural to the spiral sampling characteristics. By doing so, we are able to accurately reconstruct the low frequency data without the need of extensive over sampling. One can think of the adjoint operation of our proposed algorithm as gridding onto a pseudo-polar grid and then taking the adjoint of the pseudo-polar FFT.
2. THEORY

The spiral Fourier transform can be written in matrix form as,

\[ s = \Psi f \]  

(1)

Where \( f \) is the image stored as a column vector, \( \Psi \) is the spiral transform operator, and \( s \) is the Fourier transform sampled on the spiral grid. The inversion of the transform can be obtained by solving

\[ \hat{f} = \Psi^* s \]  

(2)

This closed form is useless for any practical applications because of the high complexity involved in inverting the matrix. This set of linear equations, can nevertheless be solved iteratively by the following relation,

\[ f^{(k+1)} = f^{(k)} - D\Psi^*(\Psi f^{(k)} - s) \]  

(3)

Eq. (3) shows that by being able to rapidly and accurately compute the forward spiral FT and its adjoint transform, we can accurately and rapidly compute the inverse.

This operation of interpolating samples lying on a Cartesian grid onto a non-uniformly sampled spiral grid is also known as inverse gridding. Fast algorithms such as NUFFT [1] and other gridding-based algorithms are used in practice in this iterative reconstruction framework. However, as mentioned previously, they all suffer from the same problem of accuracy around the k-space origin. These inaccuracies can slow the convergence rate and place a limit on the final result.

We have developed an algorithm that can rapidly and accurately compute the forward spiral FT and its adjoint. Specifically, it takes special notice of the high density of the spiral trajectory near the k-space origin. Underlying our algorithm for fast Spiral FT are tools developed recently as part of efforts to obtain polar Fourier transforms in other disciplines. In effect, those ideas provide a fast, highly accurate technique that directly obtains the Fourier transform on a polar rather than a Cartesian grid [6]. We have specialized the method so that instead of trying to find the full polar grid, we only find points lying on spirals.

A key tool in our method is the pseudo-polar FFT [7]. The pseudo-polar grid provides an intermediate stage between the Cartesian image grid, and the spiral frequency grid.

2.1. The Pseudo-Polar FFT

The pseudo-polar FFT is a FFT where the evaluated frequencies lie in an over-sampled set of non-angularly equispaced points (Fig. 1). This polar-like 2D grid enables fast Fourier computation. This grid has been explored by many since the 1970s. The pioneers in this field are Mersereau and Oppenheim [8]. This grid has the following property, as shown by Edholm and Herman [9] in the mid 1980’s:

\[ \text{Eq. (3) shows that by being able to rapidly and accurately compute the forward spiral FT and its adjoint transform, we can accurately and rapidly compute the inverse.} \]

Given a \( N \times N \) signal the exact evaluation of the Fourier Transform on the over-sampled grid with NS concentric squares and 2NP rays can be done by 1D-FFT operations only and with complexity \( 120N^2\text{PSlog}(NS) \).

2.2. Pseudo-Polar to Spiral

As discussed previously, the inversion of the spiral FT problem is reduced to rapidly and accurately evaluating the forward spiral FT and its adjoint on a spiral grid. In the algorithm, we choose to take the same approach as gridding, i.e. choose a grid for which we can rapidly compute the frequency data. But instead of using the Cartesian grid, we use the pseudo-polar one. The pseudo-polar grid, like the spiral has increasing density towards the origin of k-space. The points on it are closer to the Spiral points than the points lying on the Cartesian grid. The motivation for choosing this grid is the highly accurate interpolation from the pseudo-polar grid onto the spiral, and the ability to rapidly calculate the frequency samples on that grid. Fig. 1 shows a close view on the pseudo-polar grid overlaid on a Cartesian grid. Clearly it can be seen that the points on the pseudo-polar are closer to the spiral points than the Cartesian ones.
The points on the spiral grid can be evaluated from the points on the pseudo-pol lar grid by various interpolation schemes. Fig. 2 shows the block diagram of computing the forward spiral FT. Fig. 3 shows the block diagram of the iterative algorithm to compute the inverse spiral FT.

3. METHOD

We have applied the proposed algorithm on a 128x128 Shepp-Logan phantom using a 10-fold interleaved 1470 points spiral trajectory. The trajectory was designed to cover a 24cm FOV with 1.9mm spatial resolution. As a standard for evaluating the reconstruction quality, we reconstructed the phantom image using direct Fourier summation; but, in order to avoid density compensation errors, we reconstructed the image iteratively, thus getting the best reconstruction possible from the data, to be used as a reference image.

The following experiment was performed to evaluate the algorithm. The spiral FT of the phantom was evaluated numerically using direct Fourier summation. The image was reconstructed once by using the proposed iterative algorithm, and then by an iterative algorithm with interpolation from a Cartesian grid. The algorithms were given 20 iterations. The interpolation scheme and grid size where the same for the two methods. The result was compared both in image and frequency domain, using the direct method reconstructed image as reference.

4. RESULTS

After 20 iterations, the mean squared error compared to the direct summation reconstruction of the proposed algorithm was 4 times smaller than the Cartesian based reconstruction. Figure 5 shows a cross section along the center of the phantom. We can see that the proposed algorithm follows the direct summation reconstruction closely, while the Cartesian based algorithm shows significant ringing. A closer look at Fig. 4 provides more insight. They show the difference images in frequency domain, compared with the direct method reconstructed image. The images show a large error in the low frequencies for the Cartesian based algorithm. This considerable amount of error in the low frequency region is the explanation for the artifacts shown in the cross sectional plot. Up to the time of submission, only the bilinear interpolation was implemented, and used in the experiment.
5. DISCUSSION

These preliminary results show that the reconstruction approach using the pseudo-polar grid achieves smaller errors than the Cartesian approach for the same order of computational complexity. It is reasonable to assume that this would still be true for more sophisticated methods of interpolation. The advantage of using the pseudo-polar grid for spirals comes from the highly over-sampled grid near the k-space origin.

The rate of convergence has not been addressed in this paper, since we have only implemented the bilinear interpolator at this time. The interpolator introduces errors that degrade the rate of convergence and the condition number of the inversion process. However, from results achieved in the Polar [6] and the discrete Radon transform [7], where by proper conditioning and more accurate scheme of interpolations are used, the method can be expected to converge in around 2–6 iterations, depending on how well the spiral transform is conditioned.

5. CONCLUSION

We have proposed an iterative algorithm to invert the spiral Fourier transform. The algorithm is more accurate than comparable Cartesian grid based iterative algorithms, when using the same grid size and interpolation scheme. The complexity of the algorithm is in the order of $O(n^2 \log n)$, as in the Cartesian FFT.

An important feature of the iterative approach is being able to easily incorporate regularization and prior knowledge in the reconstruction. Moreover, this algorithm can be plugged in as is, in more sophisticated iterative reconstruction problems such as phase corrected reconstructions and SENSE.

11. REFERENCES