Credible Mechanisms

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Second-Price Auctions in Ad Exchanges

‘A proverbial black box’: Open-exchange auctions have a transparency problem

MAY 8, 2017 by Yuyu Chen
In a second-price auction, raising the price floors after the bids come in allows [online auctioneers] to make extra cash off unsuspecting buyers [. . .]

Ross Benes, reporting for Digiday, Sep 13 2017
Second-Price Auctions in Ad Exchanges

In a second-price auction, raising the price floors after the bids come in allows [online auctioneers] to make extra cash off unsuspecting buyers [...]. This practice persists because neither the publisher nor the ad buyer has complete access to all the data involved in the transaction, so unless they get together and compare their data, publishers and buyers won’t know for sure who their vendor is ripping off.

Ross Benes, reporting for Digiday, Sep 13 2017
“Chandelier Bidding”

Under New York City regulations auctioneers can fabricate bids up to an item's reserve price. Because a reserve price is secret and not listed in the catalog, bidders have no way of knowing which offers are real.

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Auctions by Telephone

Winning Bids By Source
Christie’s, New York, Spring 2013

American Paintings
Post-War & Contemporary Art
Old Masters

Reported by the Wall Street Journal
Auctions by Telephone
Incentive compatibility - for the auctioneer?

Hurwicz (1972):

*In effect, our concept of incentive compatibility merely requires that no one should find it profitable to “cheat,” where cheating is defined as behavior that can be made to look “legal” by a misrepresentation of a participant’s preferences or endowment, with the proviso that the fictitious preferences should be within certain “plausible” limits.*
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Bending the Rules

In a second-price auction:

1. Receive sealed bids $b_1 > b_2 > \cdots$
2. Award object to bidder 1, charge $b_1 - \epsilon$. 
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3. Looks ‘legal’ to bidder 1. (maybe $v_2 = b_1 - \epsilon$ ?)
4. Strict profit.

Auctioneer would want to deviate. (Vickrey 1961)
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2. Invert bid function \( b_1^{-1}(b_1) = v_1 \).
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3. Make TIOLI offer (to bidder 1) of $v_1 - \epsilon$.
4. Bidder 1 brings a lawsuit and wins.

‘automatically self-policing’ (Vickrey 1961)
Taking the opposite benchmark

Standard approach: No room for misrepresentation.

This paper: Auctioneer can misrepresent any bidder’s actions to any other bidder.

Auctioneer as communication nexus
Private ‘telephone calls’ to bidders. No public announcements.
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Auctioneer as communication nexus
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Informal definition
Auctioneer may deviate in ways that no single bidder can detect.
**credible** ≡ incentive-compatible for auctioneer to follow the rules.
Optimal auctions

regular i.i.d. values
auctioneer wants revenue
only winners make transfers
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Result 1
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if and only if
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Result 2
G is credible & strategy-proof
if and only if
G is an ascending auction.
Benchmark model: Symmetric independent private values

Following Myerson (1981)

1. One object.
2. Set of (two or more) bidders $N$, representative element $i$.
3. Only winning bidders make payments.
4. Outcome specifies who gets the object, how much they pay.
5. Bidders have private values, quasilinear utility $u_i$.
6. $\theta_i$ i.i.d. with density $f : [0, 1] \to \mathbb{R}$.
7. $f$ continuous, positive, regular.
Benchmark model: Symmetric independent private values

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8. Auctioneer utility function. $u_0(\text{outcome}) = \text{revenue}$
9. $i$ observes whether he gets the object, and how much he pays.
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Paper has more general theorems that relax winner-paying, symmetry, regularity. (omitted today)
Implementation via Extensive Forms

\( G \) denotes an extensive-form mechanism. (Each terminal history specifies an outcome.)

1. Finite depth.
2. No chance moves.
3. Perfect recall.
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4. For every history \( h \), there exists \( \theta_N \) such that \( h \) is on the path-of-play.
5. Every infoset has at least two actions.
6. If \( i \) is called to play at \( h \), then \( i \) can affect the outcome.

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Restrict attention to **measurable** protocols.
Motivation
Summary
Model
Theorem 1
Theorem 2
Conclusion

1 observes: \[ \begin{align*}
1 & \text{ wins at } \$5 \\
2 & \text{ wins at } \$5,
\end{align*} \]

2 observes: \[ \begin{align*}
1 & \text{ wins at } \$5 \\
2 & \text{ wins at } \$5,
\end{align*} \]
1 observes: \{ 1 \text{ wins at } \$5 \}, \{ 2 \text{ wins at } \$5 \}, \{ 2 \text{ wins at } \$10 \}

2 observes: \{ 1 \text{ wins at } \$5 \}, \{ 2 \text{ wins at } \$5 \}, \{ 2 \text{ wins at } \$10 \}
Motivation

Summary

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Conclusion

The value tree represents a game with two players, 1 and 2, where each player has two choices: $5 or $15. The payouts are as follows:

- If 1 wins at $5, the payout is $10.
- If 2 wins at $5, the payout is $5.
- If 1 wins at $5, the payout is $5.
- If 2 wins at $10, the payout is $15.

Player 1 observes:

- 1 wins at $5
- 2 wins at $5

Player 2 observes:

- 1 wins at $5
- 2 wins at $5
- 2 wins at $10
1 observes: \{ 1 wins at $5 \} \{ 2 wins at $5 \}, 2 wins at $10 \\
2 observes: \{ 1 wins at $5 \} \{ 2 wins at $5 \}, 2 wins at $10
1 observes: \[
\{ 
\text{1 wins at $5} \\
\text{2 wins at $5} \\
\text{1 wins at $5} \\
\text{2 wins at $10}
\}
\]

2 observes: \[
\{ 
\text{1 wins at $5} \\
\text{2 wins at $5} \\
\text{2 wins at $10}
\}\]
Motivation

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Conclusion

The diagram illustrates a decision-making process with rewards and outcomes.

- **1 wins at $5**, 1 observes: \{ 1 wins at $5, 2 wins at $5, 2 wins at $10 \}

- **2 wins at $5**, 1 observes: \{ 2 wins at $5, 2 wins at $10 \}

- **1 wins at $5**, 2 observes: \{ 1 wins at $5, 2 wins at $5, 2 wins at $10 \}

- **2 wins at $10**, 2 observes: \{ 2 wins at $10 \}

The outcomes are determined by the choices and rewards at each node of the decision tree.
1 observes: \{ 1 wins at $5 \} \{ 2 wins at $5 , 2 wins at $10 \}

2 observes: \{ 1 wins at $5 \} \{ 2 wins at $5 \} \{ 2 wins at $10 \}
1 observes: \[
\begin{align*}
\text{1 wins} & \quad \text{at $5$} \\
\text{2 wins} & \quad \text{at $5$}, \\
\text{2 wins} & \quad \text{at $10$}
\end{align*}
\]

2 observes: \[
\begin{align*}
\text{1 wins} & \quad \text{at $5$} \\
\text{2 wins} & \quad \text{at $5$} \\
\text{2 wins} & \quad \text{at $10$}
\end{align*}
\]
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1 observes: \[
\begin{align*}
&\text{1 wins at } \$5 \\
&\text{2 wins at } \$5 \\
&\text{1 wins at } \$5
\end{align*}
\]

2 observes: \[
\begin{align*}
&\text{2 wins at } \$10 \\
&\text{1 wins at } \$5 \\
&\text{2 wins at } \$10
\end{align*}
\]
Motivation

Summary

Model

Theorem 1

Theorem 2

Conclusion

The diagram shows a decision tree with the following outcomes:

**Motivation**

**Summary**

**Model**

**Theorem 1**

**Theorem 2**

**Conclusion**

1. **1 wins at $5**
2. **2 wins at $5**
3. **1 wins at $5**
4. **2 wins at $10**

**1 observes:**

- 1 wins at $5
- 2 wins at $5, 2 wins at $10

**2 observes:**

- 1 wins at $5
- 2 wins at $5, 2 wins at $10
A Messaging Game

1. Auctioneer can:
   1.1 Either: Choose an outcome and end the game.
   1.2 Or: Go to Step 2.
2. Auctioneer chooses some agent $i$, sends message $m \in \{i$’s infosets in $G\}$
3. $i$ privately observes $m$, chooses reply $r \in \{actions available at $m\}$.
4. Auctioneer privately observes $r$. Go to Step 1.
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**Full commitment**: To ‘run’ $G$, auctioneer commits to $S_0^G$. 
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**Partial commitment:** Auctioneer can deviate to any $S_0$ that an individual agent cannot distinguish from $S_0^G$. 
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How the Auctioneer Can Deviate

Consider protocol \((G, S_N)\), and \(S_0^G\) that ‘runs’ \(G\).
How the Auctioneer Can Deviate

Consider protocol \((G, S_N)\), and \(S_0^G\) that ‘runs’ \(G\).

\[ o_i \text{ observation for } i = \]
communication sequence \((m_i^t, r_i^t)_{t=1}^T\)
& whether \(i\) wins, how much \(i\) pays
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resulting observation denoted \( o_i(S_0, S_N, \theta_N) \)
How the Auctioneer Can Deviate

Consider protocol \((G, S_N)\), and \(S_0^G\) that ‘runs’ \(G\).

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\(o_i(S_0, S_N, \theta_N)\) has an \textit{innocent explanation} if:

\[
\exists \theta'_i : o_i(S_0, S_N, \theta_N) = o_i(S_0^G, S_N, (\theta_i, \theta'_i))
\]
How the Auctioneer Can Deviate

Consider protocol \((G, S_N)\), and \(S_0^G\) that ‘runs’ \(G\).

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\(o_i(S_0, S_N, \theta_N)\) has an innocent explanation if:

\[ \exists \theta'_{-i}: o_i(S_0, S_N, \theta_N) = o_i(S_0^G, S_N, (\theta_i, \theta'_{-i})) \]

\(S_0\) is safe if \(\forall i: \forall \theta_N: o_i(S_0, S_N, \theta_N)\) has an innocent explanation.

Definition

\((G, S_N)\) is credible if \(S_0^G\) is a best-response to \(S_N\).
(yields at least as much expected utility as any safe deviation)
Auctioneer's deviation

1 observes: \[
\begin{align*}
1 & \text{ wins at } \$5 \\
2 & \text{ wins at } \$5
\end{align*}
\]

2 observes: \[
\begin{align*}
1 & \text{ wins at } \$5 \\
2 & \text{ wins at } \$5 \\
& \text{ wins at } \$10
\end{align*}
\]
Innocent explanation for 1’s observation

1 observes: \[
\begin{align*}
1 & \text{ wins at } $5 \\
2 & \text{ wins at } $5
\end{align*}
\]

2 observes: \[
\begin{align*}
1 & \text{ wins at } $5 \\
2 & \text{ wins at } $5 \\
2 & \text{ wins at } $10
\end{align*}
\]
Innocent explanation for 2’s observation

1 observes:  \[ \begin{cases} 1 \text{ wins at $5} \\ 2 \text{ wins at $5} \end{cases} \begin{cases} 2 \text{ wins at $5} \end{cases}, \begin{cases} 2 \text{ wins at $10} \end{cases} \]

2 observes:  \[ \begin{cases} 1 \text{ wins at $5} \\ 2 \text{ wins at $5} \end{cases} \begin{cases} 2 \text{ wins at $5} \end{cases}, \begin{cases} 2 \text{ wins at $10} \end{cases} \]
Safe deviations can involve ‘midway’ deception.

The auctioneer can send different messages during the mechanism. She may adopt any communication strategy that is safe. (For every type profile and every agent $i$, $i$’s observation has an innocent explanation.)
Related literature

As above, but restricted to direct mechanisms

Dequiedt & Martimort 2015

This talk: Extensive forms.
Related literature

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Commit to today's auction, not tomorrow's auction

This talk: Not a repeated game.
Related literature

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Auctions as bargaining games
McAdams & Schwarz 2007, Vartiainen 2013, Lobel & Paes Leme 2017

This talk: No ‘red-handed’ rule-breaking.
Definition

\((G, S_N)\) is **optimal** if it maximizes revenue subject to BIC and participation constraints.

\((G, S_N)\) is **static** if every agent has exactly one infoset and is always called to play.

Definition

\((G, S_N)\) is a **first-price auction** if it is static, and each \(i\) either chooses a bid in some feasible set \(B_i \subset \mathbb{R}\) or declines. The player with the highest bid wins, and pays his bid. (break ties arbitrarily)
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We represent a reserve price by restricting \(B_i\).

Notice: Bids defined by consequences, not by labels.
credible static optimal auctions

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**Definition**

(G, S_N) is a **first-price auction** if it is static, and each i either chooses a bid in some feasible set B_i ⊂ R or declines. The player with the highest bid wins, and pays his bid. (break ties arbitrarily)

**Theorem 1**

Assume (G, S_N) is optimal. If (G, S_N) is a first-price auction, then it is credible and static. If (G, S_N) is credible and static, then it is almost-everywhere a first-price auction.
credible static optimal auctions

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Assume \((G, S_N)\) is optimal. If \((G, S_N)\) is a first-price auction, then it is credible and static. If \((G, S_N)\) is credible and static, then it is almost-everywhere a first-price auction.
Proof Idea

first-price $\rightarrow$ credible and static

By inspection.
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credible and static $\rightarrow$ almost-everywhere first-price

Suppose after $i$ plays $a$, there are two prices that $i$ might pay. Deviate to charge the higher price.

(Real proof must ensure deviation is measurable.)
**Proof Idea**

First-price $\rightarrow$ credible and static

By inspection.

Credible and static $\rightarrow$ almost-everywhere first-price

Suppose after $i$ plays $a$, there are two prices that $i$ might pay. Deviate to charge the higher price.

(Real proof must ensure deviation is measurable.)

**Lemma:** Almost everywhere, for each action $i$ takes, there is a unique price he pays if he wins.
Proof Idea

first-price \rightarrow \text{credible and static}

By inspection.

credible and static \rightarrow \text{almost-everywhere first-price}

Suppose after \( i \) plays \( a \), there are two prices that \( i \) might pay. Deviate to charge the higher price.

(Real proof must ensure deviation is measurable.)

**Lemma**: Almost everywhere, for each action \( i \) takes, there is a unique price he pays if he wins.

Optimal allocation + envelope theorem pins down payments, so (almost everywhere) the winning bidder has highest bid. QED.
Dominant-strategy or credible?

Big Changes Coming To Auctions, As Exchanges Roll The Dice On First-Price

by Sarah Sluis // Tuesday, September 5th, 2017 – 8:00 am

The second-price auction is crumbling.

Buyers, publishers, and ad tech companies who advocate a switch to first-price auctions say it’s because fair second-price auctions don’t exist any more. [Online auctioneers] have polluted them with hidden fees and manipulative auction dynamics.
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March 2019: Google Ad Manager announces switch to first-price.
The story so far
Strategy-proof

Definition

\((G, S_N)\) is strategy-proof if \(\forall i : \forall S'_{N\setminus i} : S_i \text{ best responds to } S'_{N\setminus i}\).
Strategy-proof

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\((G, S_N)\) is **strategy-proof** if \(\forall i : \forall S'_N \setminus i : S_i \text{ best responds to } S'_N \setminus i\).

Goal: Characterize the set of optimal extensive game forms credible \(\cap\) strategy-proof.
Strategy-proof

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No revelation principle.

1. Auctioneer could make any queries in any order.
2. Agents may receive information when called to play.
Strategy-proof

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\((G, S_N)\) is strategy-proof if \(\forall i : \forall S'_{N\setminus i} : S_i \text{ best responds to } S'_{N\setminus i}\).

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No revelation principle.

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Problem: Extensive-form ascending auctions move in discrete steps, can’t be exactly optimal.
Solution: Discretize the distribution.
Discretize the distribution

Type Space \( \theta^0 \rightarrow \theta^1 \rightarrow \cdots \rightarrow \theta^K \subset \mathbb{R}^+_0 \)

i.i.d. probability mass function \( p : \Theta_i \rightarrow (0, 1] \)
Discretize the distribution

Motivation

Summary

Model

Theorem 1

Theorem 2

Conclusion

Discretize the distribution

Type Space

\[ \theta^0 \quad \theta^1 \quad \ldots \quad \theta^K \subset \mathbb{R}_0^+ \]

\[ \epsilon \]

i.i.d. probability mass function \( p : \Theta_i \rightarrow (0, 1] \)
Discretize the distribution

i.i.d. probability mass function \( p : \Theta \rightarrow (0, 1) \)

pseudo-pdf \( f(\theta^k) \equiv \frac{p(\theta^k)}{\epsilon} \)

cdf \( F(\theta^k) \equiv \sum_{j=1}^{k} p(\theta^j) \)

Assumption. \( F \) is regular, i.e. \( \eta(\cdot) \) is strictly increasing.
Discretize the distribution

\[ \Theta_i \mapsto (0, 1) \]

Assumption. \( F \) is regular, i.e. \( \eta(\cdot) \) is strictly increasing.
A credible strategy-proof auction
A credible strategy-proof auction

\[ \Theta^h_1 \quad \Theta^h_2 \]
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\[ \Theta^h_1 \]

why not raise 1’s price by \( \varepsilon \)?

\[ \Theta^h_2 \]

2 has quit.

optimal reserve
Q: Why not raise 1’s price to $b_1 + \epsilon$, even after bidder 2 has quit? 
A: 1’s virtual value is positive.
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‘The book’ requires that 1 pay $b_1$.

$$\underbrace{-\epsilon f(b_1)b_1}_{\text{expected loss from 1 quitting}} + \underbrace{(1 - F(b_1))\epsilon}_{\text{expected gain from raising price}}$$
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\[\text{expected loss from 1 quitting} \quad \text{expected gain from raising price}\]

divide through by $\epsilon f(b_1)$

\[-\left(b_1 - \frac{1 - F(b_1)}{f(b_1)}\right) < 0\]

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\[
\begin{align*}
-\epsilon f(b_1) b_1 & \quad + \quad (1 - F(b_1))\epsilon \\
\text{expected loss from 1 quitting} & \quad \text{expected gain from raising price}
\end{align*}
\]

divide through by $\epsilon f(b_1)$

\[
- \left[ b_1 - \frac{1 - F(b_1)}{f(b_1)} \right] < 0
\]

virtual value

Ceci n’est pas une proof.
How to deal with ties?

For technical convenience:

**Definition**

$(G, S_N)$ is **orderly** if, for some reserve $\rho \leq \theta^K$, and some strict order $\triangleright$ on $N$, bidder $i$ wins the object iff:

1. $\theta_i \geq \rho$, and
2. For all $j \neq i$, $\theta_i$ is more than $\theta_j$, breaking ties with $\triangleright$. 
How to deal with ‘long-winded’ auctioneers?

‘You’re the only bidder left, so you’ll win if you bid the reserve. But first, tell me is your type a prime number? Is it a Mersenne prime?’

For technical convenience:

**Definition**

*i faces a posted price at history h* if

1. *i is called to play at h.*

2. *∃ price* $\tau_h$ *such that* $\forall$ *successors of* $h$, *if* *i wins then* *i pays* $\tau_h$.

*($G, S_N$) is concise* if, *at any history* $h$ *at which* *i faces a posted price, then* $h$ *is the last time* *i is called to play, and the infoset containing* $h$ *is singleton.*
**Definition**

\( (G, S_N) \) is an **ascending auction** if:

1. *The induced allocation rule is orderly.*
2. *The induced payment rule has threshold pricing.*
Definition

\((G, S_N)\) is an **ascending auction** if:

1. The induced allocation rule is orderly.
2. The induced payment rule has threshold pricing.
3. Each bidder has an initial bid equal to 0.
4. At each non-terminal history:
   4.1 Some bidder \(i\) is called to play, and offered some price \(\geq\) his current bid.
   4.2 The available actions either accept the price or quit.
   4.3 If \(i\) quits, then \(i\) is not called to play again, does not win, and pays 0.
   4.4 If \(i\) accepts price \(p\), then his bid is updated to \(p\).
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   4.4 If \(i\) accepts price \(p\), then his bid is updated to \(p\).
5. At each infoset, either there is a unique accepting action, or any accepting action guarantees that \(i\) wins at current price.
6. At each terminal history, the winner pays his latest bid.
credible, strategy-proof $\leftrightarrow$ ascending

**Theorem 2**

Assume $(G, S_N)$ is optimal. If $(G, S_N)$ is an ascending auction then it is credible and strategy-proof.

Assume additionally that $(G, S_N)$ is orderly and concise. If $(G, S_N)$ is credible and strategy-proof, then $(G, S_N)$ is an ascending auction.
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**Theorem 2**

Assume $(G, S_N)$ is optimal. If $(G, S_N)$ is an ascending auction then it is credible and strategy-proof.

Assume additionally that $(G, S_N)$ is orderly and concise. If $(G, S_N)$ is credible and strategy-proof, then $(G, S_N)$ is an ascending auction.

Green-Laffont-Holmström, Theorem 1, and Theorem 2 $\rightarrow$
Proof: ascending $\rightarrow$ credible

$$\pi(G, S_N)$$

1. Ascending $(G, S_N)$ is optimal.
Proof: ascending $\rightarrow$ credible

\[ \pi(G, S_N) = \pi(S_0^G, S_N) \]

1. Ascending $\langle G, S_N \rangle$ is optimal.
2. Consider $S_0^G$ that runs $G$. 

Contradiction, QED.
Proof: ascending $\rightarrow$ credible

\[
\pi(G, S_N) = \pi(S_0^G, S_N) < \pi(S_0', S_N)
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1. Ascending $(G, S_N)$ is optimal.
2. Consider $S_0^G$ that runs $G$.
3. Suppose $S_0'$ is a profitable safe deviation.
Proof: ascending $\rightarrow$ credible

\[ \pi(G, S_N) = \pi(S_0^G, S_N) < \pi(S'_0, S_N) \]

1. Ascending \((G, S_N)\) is optimal.
2. Consider \(S_0^G\) that runs \(G\).
3. Suppose \(S'_0\) is a profitable safe deviation.
4. For all \(i\), \(S_i\) remains a best response to \((S'_0, S_{N\setminus i})\).
Proof: ascending → credible

\[ \pi(G, S_N) = \pi(S_0^G, S_N) < \pi(S'_0, S_N) = \pi(G', S_N) \]

1. Ascending \((G, S_N)\) is optimal.
2. Consider \(S_0^G\) that runs \(G\).
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4. For all \(i\), \(S_i\) remains a best response to \((S'_0, S_{N \setminus i})\).
5. \((G', S_N)\) is also BIC, yields more revenue than \((G, S_N)\).

Contradiction, QED.
Proof sketch: credible, SP $\rightarrow$ ascending

A key feature of ascending auctions:
All the types who might still win pool on the same action.

Suppose $(G, S_N)$ SP. not pooling $\rightarrow$ not credible.
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Hurdle #1: What if 1’s chosen action is not monotone?
Proof sketch: credible, \( \text{SP} \rightarrow \text{ascending} \)

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---

Hurdle #1: What if 1’s chosen action is not monotone?

Hurdle #2: What if 2’s type is too high to be worth exaggerating?

Exaggerate 2’s type
A deviating algorithm

(High-level description, omits fine details.)

Given an arbitrary extensive form, take some history where bidder 1’s winning types don’t pool:
A deviating algorithm

(High-level description, omits fine details.)

Given an arbitrary extensive form, take some history where bidder 1’s winning types don’t pool:

1. Check if 1’s type is high enough to exploit.
   - If not, sell to bidder 2.
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strategy-proof, not pooling $\rightarrow$ profitable safe deviation
credible, strategy-proof $\rightarrow$ pooling $\rightarrow$ ascending auction
What if we relax optimality and winner-paying?

Definition

$(G, S_N)$ is **contestable** if, almost everywhere, if $i$ wins at $(\theta_i, \theta_i)$, then there exists $\theta'_i$ such that $i$ loses at $(\theta_i, \theta'_i)$.

Definition

$(G, S_N)$ is a **twin-bid auction** if each bidder $i$ chooses bids in $B_i \subset \mathbb{R} \times \mathbb{R}$, such that:

1. $i$ pays $b^1_i$ if he wins and $b^2_i$ if he loses.
2. If $\max_i b^1_i - b^2_i > 0$, then some bidder wins.
3. If bidder $i$ wins, then $b^1_i - b^2_i \geq 0$ and $\forall j : b^1_i - b^2_i \geq b^1_j - b^2_j$. 
What if we relax optimality and winner-paying?

Definition

\((G, S_N)\) is **contestable** if, almost everywhere, if \(i\) wins at \((\theta_i, \theta'_i)\), then there exists \(\theta'_{-i}\) such that \(i\) loses at \((\theta_i, \theta'_{-i})\).

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Theorem

Assume \((G, S_N)\) is contestable. \((G, S_N)\) is credible and static if and only if it is a twin-bid auction almost everywhere.
Modeling auctioneer IC yields a simple explanation of real-world auction formats.

Of course, these are not the only desiderata.
Calendar time isn’t ‘built into’ extensive forms
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What about asymmetric distributions?

First-price auction (static, credible)

‘Robustly’ credible. May not be optimal. Sometimes impossible to restore optimality.
What about asymmetric distributions?

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Sometimes impossible to restore optimality.

Proposition

There exist asymmetric distributions such that no credible static 
\((G, S_N)\) is \(\epsilon\)-optimal.

Ascending auction (strategy-proof, credible)

May not be credible or optimal.
Easy to restore both.
The **virtual values** ascending auction.
Bidders seldom display types on placards.

In the English system bids are . . . usually transmitted by signal. Such signals may be in the form of a wink, a nod, scratching an ear, lifting a pencil, tugging at the coat of the auctioneer or even staring into the auctioneer’s eyes – all of them perfectly legal.

Cassady 1967

Public communication affects aftermarkets and thus incentives. Ausubel & Cramton 2004, Carroll & Segal 2016, Dworczak 2017. (Outside the model today.)
## A Menagerie

### Table: \( \epsilon \)-optimal auctions

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optimal ∩ first-price = ∅

\( N = \{1, 2\} \)
\( \Theta_i = \{4, 5, 6\} \)
Tie-breaking order: 1 \(\triangleleft\) 2

Optimal reserve = 4.

Optimality requires:
\( b_1(5) = 5 \)
\( b_2(5) = 4.5 \)

When type profile is \((5, 5)\), tie-breaking rule requires to sell to bidder 2, even though he bid less. Not first-price auction!