Credible Auctions: A Trilemma

Mohammad Akbarpour
Stanford GSB

Shengwu Li
Harvard

Simplicity and Complexity in Economic Theory
Stanford University
April 2020

Second-Price Auctions in Ad Exchanges

‘A proverbial black box’: Open-exchange auctions have a transparency problem

MAY 8, 2017 *by* Yuyu Chen

Second-Price Auctions in Ad Exchanges

In a second-price auction, raising the price floors after the bids come in allows [online auctioneers] to make extra cash off unsuspecting buyers [. . .]

Ross Benes, reporting for *Digiday*, Sep 13 2017

Second-Price Auctions in Ad Exchanges

In a second-price auction, raising the price floors after the bids come in allows [online auctioneers] to make extra cash off unsuspecting buyers [. . .] This practice persists because neither the publisher nor the ad buyer has complete access to all the data involved in the transaction, so unless they get together and compare their data, publishers and buyers won't know for sure who their vendor is ripping off.

Ross Benes, reporting for *Digiday*, Sep 13 2017

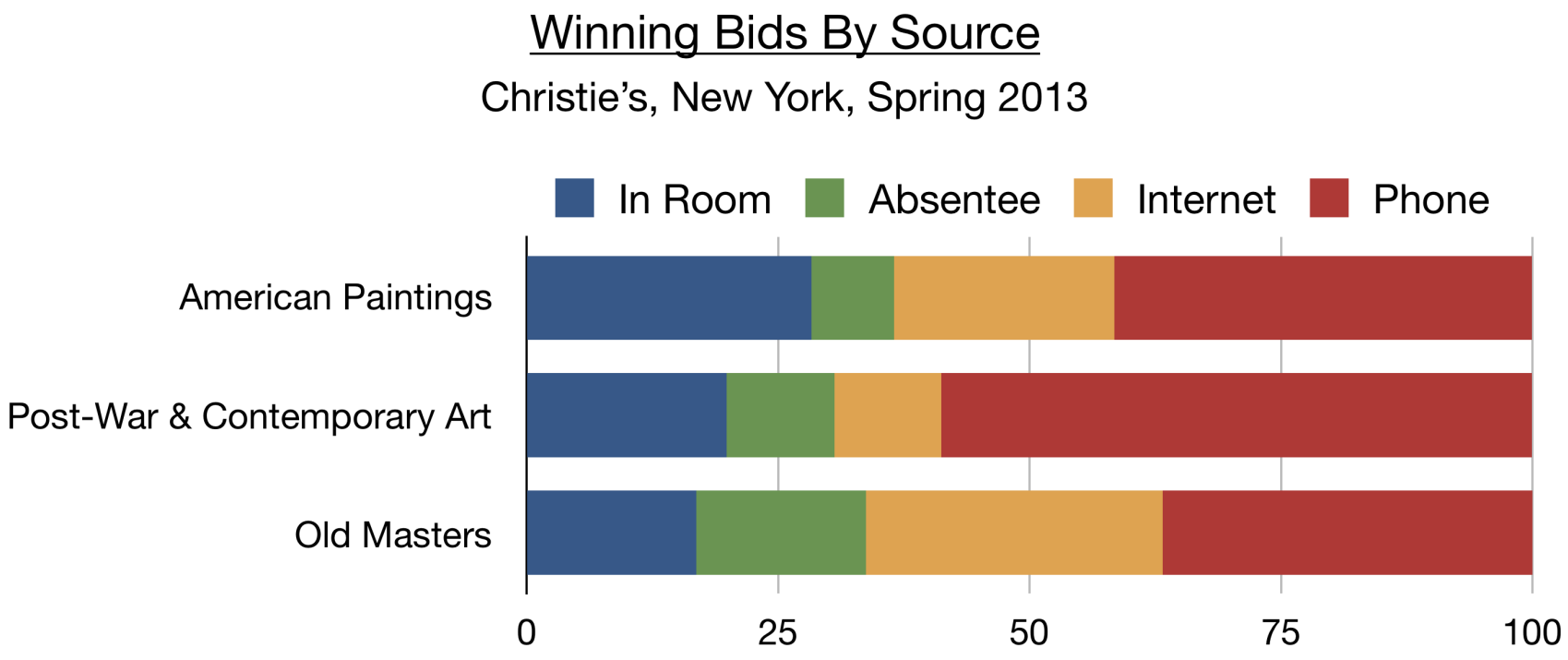
"Chandelier Bidding"

"Chandelier Bidding"

Under New York City regulations auctioneers can fabricate bids up to an item's reserve price. Because a reserve price is secret and not listed in the catalog, bidders have no way of knowing which offers are real.

NYT, April 24, 2000

Auctions by Telephone



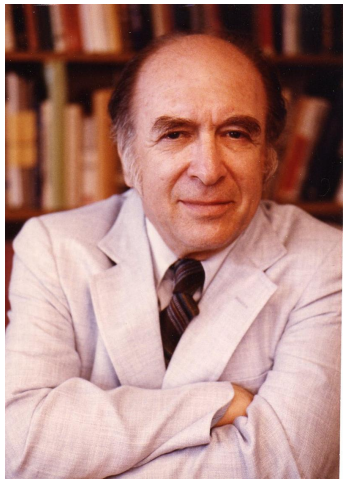
Reported by the Wall Street Journal

Auctions by Telephone

Incentive compatibility - for the auctioneer?

Hurwicz (1972):

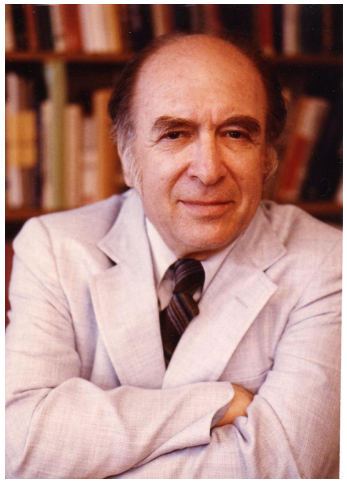
In effect, our concept of incentive compatibility merely requires that no one should find it profitable to "cheat," where cheating is defined as behavior that can be made to look "legal" by a misrepresentation of a participant's preferences or endowment, with the proviso that the fictitious preferences should be within certain "plausible" limits.



Incentive compatibility - for the auctioneer?

Hurwicz (1972):

In effect, our concept of incentive compatibility merely requires that no one should find it profitable to "cheat," where cheating is defined as behavior that can be made to look "legal" by a misrepresentation of a participant's preferences or endowment, with the proviso that the fictitious preferences should be within certain "plausible" limits.



Bending the Rules

In a second-price auction:

1. Receive sealed bids $b_1 > b_2 > \cdots$
2. Award object to bidder 1, charge $b_1 - \epsilon$.

Bending the Rules

In a second-price auction:

1. Receive sealed bids $b_1 > b_2 > \cdots$
2. Award object to bidder 1, charge $b_1 - \epsilon$.
3. Looks 'legal' to bidder 1. (maybe $v_2 = b_1 - \epsilon$?)
4. Strict profit.

Auctioneer would want to deviate. (Vickrey 1961)

Bending the Rules

In a second-price auction:

1. Receive sealed bids $b_1 > b_2 > \cdots$
2. Award object to bidder 1, charge $b_1 - \epsilon$.
3. Looks 'legal' to bidder 1. (maybe $v_2 = b_1 - \epsilon$?)
4. Strict profit.

Auctioneer would want to deviate. (Vickrey 1961)

In a first-price auction:

1. Receive bids $b_1 > b_2 > \cdots$

Bending the Rules

In a second-price auction:

1. Receive sealed bids $b_1 > b_2 > \cdots$
2. Award object to bidder 1, charge $b_1 - \epsilon$.
3. Looks 'legal' to bidder 1. (maybe $v_2 = b_1 - \epsilon$?)
4. Strict profit.

Auctioneer would want to deviate. (Vickrey 1961)

In a first-price auction:

1. Receive bids $b_1 > b_2 > \cdots$
2. Invert bid function $\mathbf{b}_1^{-1}(b_1) = v_1$.
3. Make TIOLI offer (to bidder 1) of $v_1 - \epsilon$.

Bending the Rules

In a second-price auction:

1. Receive sealed bids $b_1 > b_2 > \cdots$
2. Award object to bidder 1, charge $b_1 - \epsilon$.
3. Looks 'legal' to bidder 1. (maybe $v_2 = b_1 - \epsilon$?)
4. Strict profit.

Auctioneer would want to deviate. (Vickrey 1961)

In a first-price auction:

1. Receive bids $b_1 > b_2 > \cdots$
2. Invert bid function $\mathbf{b}_1^{-1}(b_1) = v_1$.
3. Make TIOLI offer (to bidder 1) of $v_1 - \epsilon$.
4. Bidder 1 brings a lawsuit and wins.

'automatically self-policing' (Vickrey 1961)

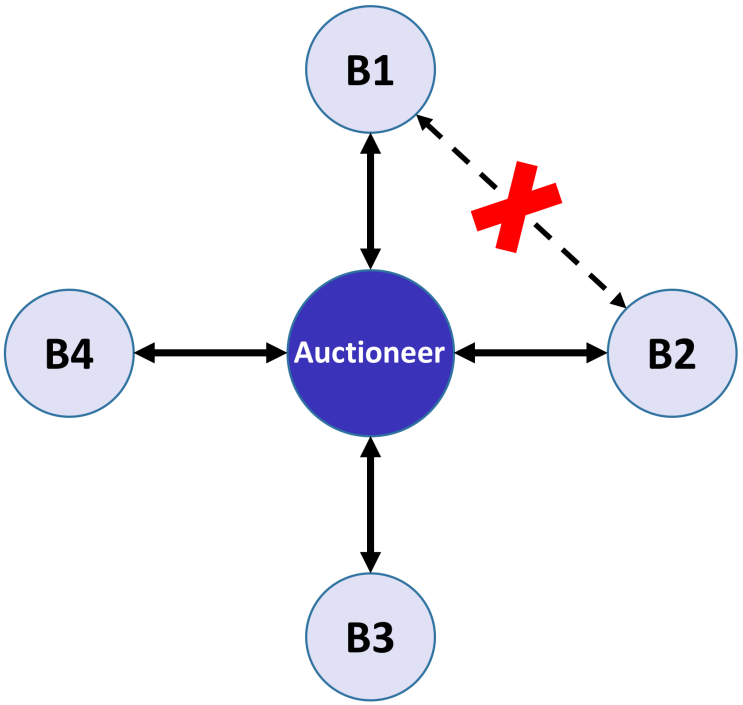
Taking the opposite benchmark

Standard approach: No room for misrepresentation.

This paper: Auctioneer can misrepresent any bidder's actions to any other bidder.

Auctioneer as communication nexus

Private 'telephone calls' to bidders. No public announcements.



Taking the opposite benchmark

Standard approach: No room for misrepresentation.

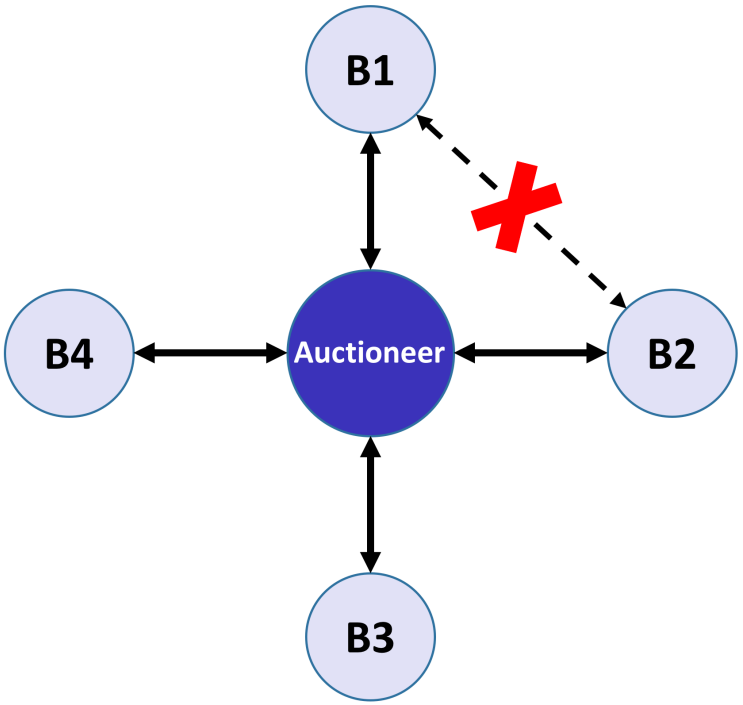
This paper: Auctioneer can misrepresent any bidder's actions to any other bidder.

Auctioneer as communication nexus

Private 'telephone calls' to bidders. No public announcements.

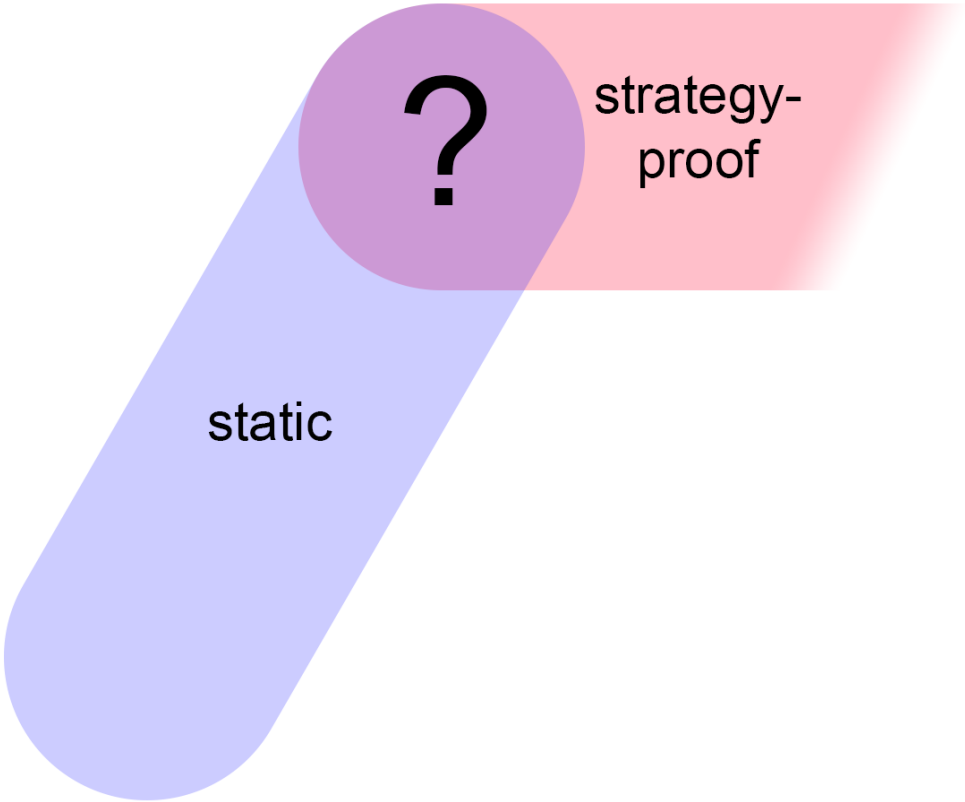
Informal definition

Auctioneer may deviate in ways that no single bidder can detect.
credible $\equiv$ incentive-compatible for auctioneer to follow the rules.



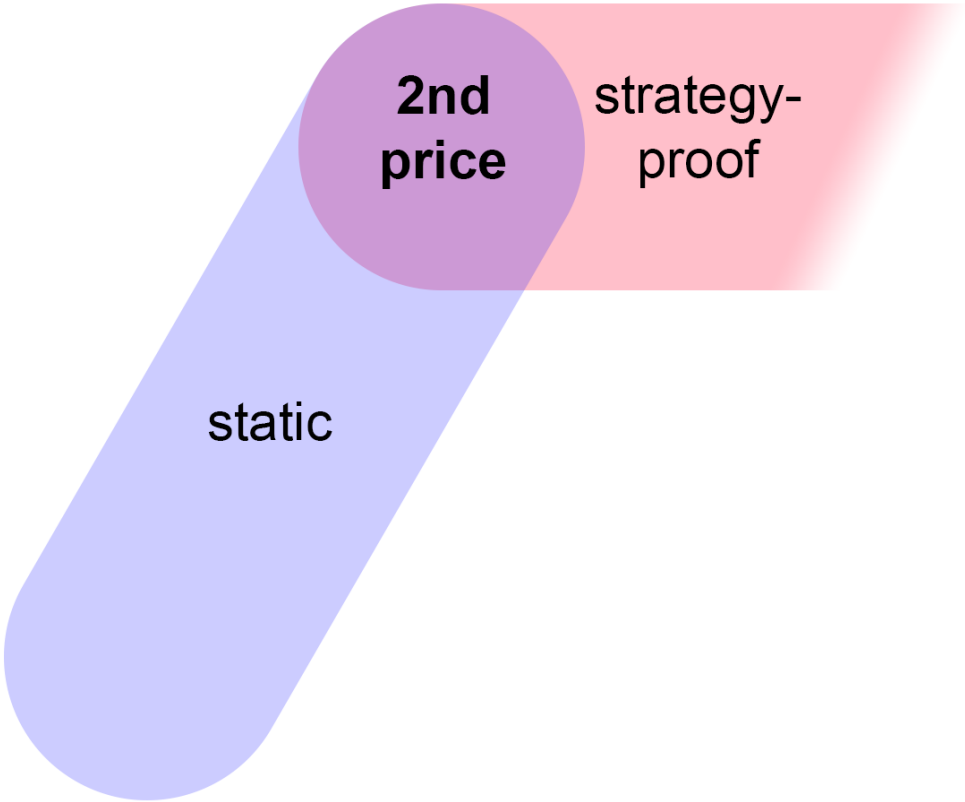
Optimal auctions

regular i.i.d. values
auctioneer wants revenue
only winners make transfers



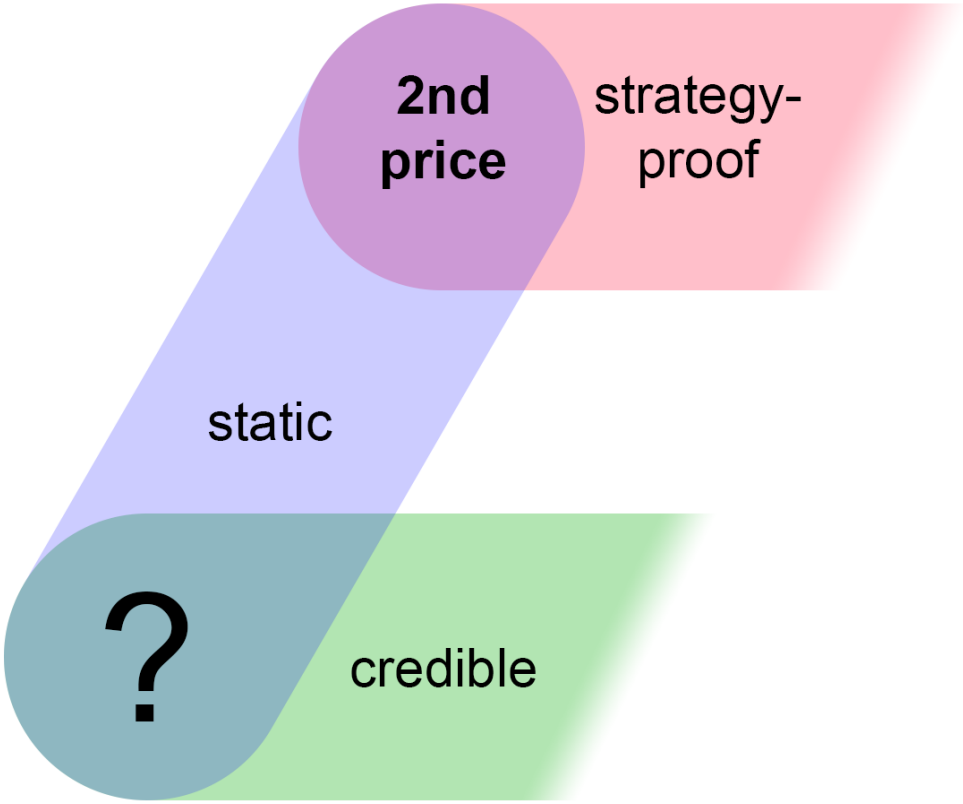
Optimal auctions

regular i.i.d. values
auctioneer wants revenue
only winners make transfers



Optimal auctions

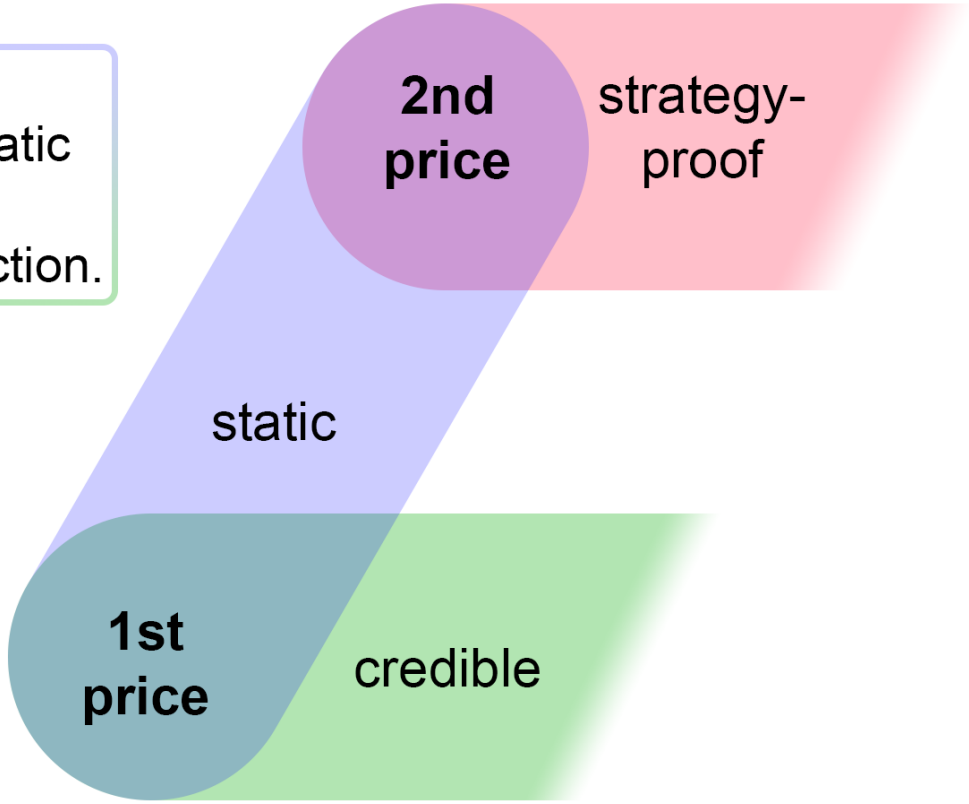
regular i.i.d. values
auctioneer wants revenue
only winners make transfers



Optimal auctions

regular i.i.d. values
auctioneer wants revenue
only winners make transfers

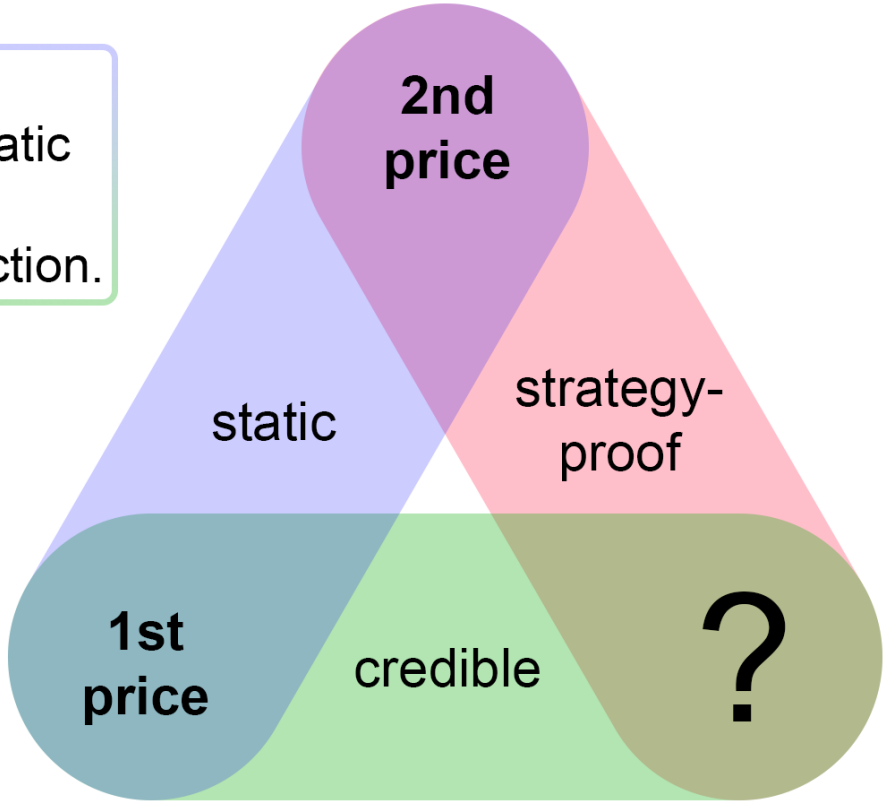
Result 1
G is credible & static
if and only if
G is a 1st price auction.



Optimal auctions

regular i.i.d. values
auctioneer wants revenue
only winners make transfers

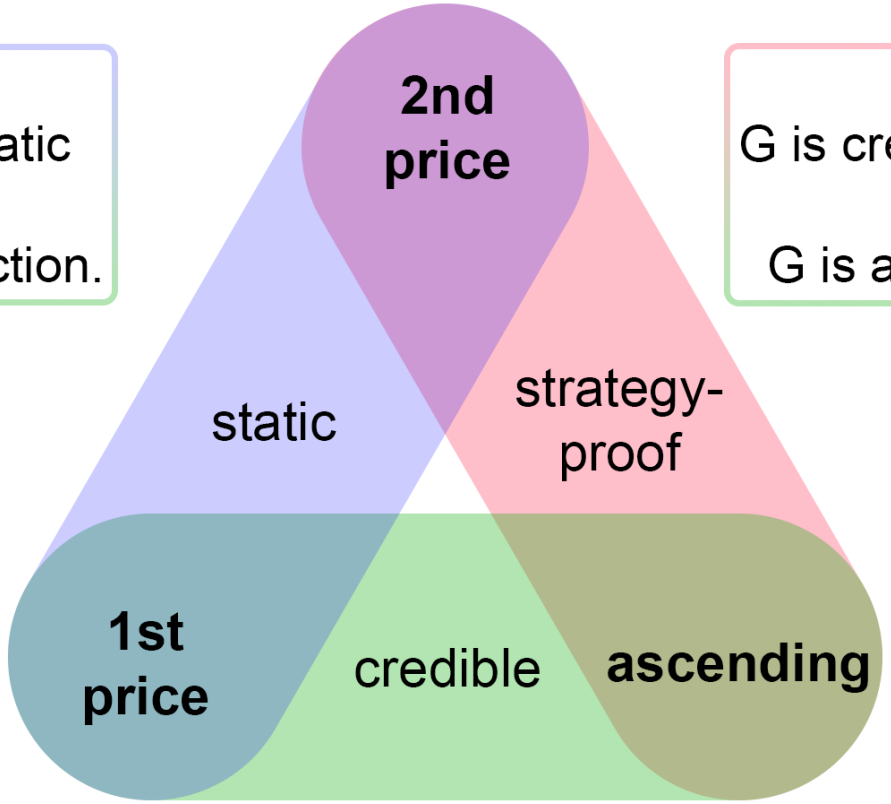
Result 1
G is credible & static
if and only if
G is a 1st price auction.



Optimal auctions

regular i.i.d. values
auctioneer wants revenue
only winners make transfers

Result 1
G is credible & static
if and only if
G is a 1st price auction.



Result 2
G is credible & strategy-proof
if and only if
G is an ascending auction.

Benchmark model: Symmetric independent private values

Following Myerson (1981)

1. One object.
2. Set of (two or more) bidders N, representative element i.
3. Only winning bidders make payments.
4. Outcome specifies who gets the object, how much they pay.
5. Bidders have private values, quasilinear utility u_i
6. θ_i i.i.d. with density $f : [0, 1] \to \mathbb{R}$
7. f continuous, positive, regular.

Benchmark model: Symmetric independent private values

Following Myerson (1981)

1. One object.
2. Set of (two or more) bidders N, representative element i.
3. Only winning bidders make payments.
4. Outcome specifies who gets the object, how much they pay.
5. Bidders have private values, quasilinear utility u_i
6. θ_i i.i.d. with density $f : [0, 1] \to \mathbb{R}$
7. f continuous, positive, regular.
8. Auctioneer utility function. $u_0(\text{outcome}) = \text{revenue}$
9. i observes whether he gets the object, and how much he pays.

Benchmark model: Symmetric independent private values

Following Myerson (1981)

1. One object.
2. Set of (two or more) bidders N, representative element i.
3. Only winning bidders make payments.
4. Outcome specifies who gets the object, how much they pay.
5. Bidders have private values, quasilinear utility u_i
6. θ_i i.i.d. with density $f : [0, 1] \to \mathbb{R}$
7. f continuous, positive, regular.
8. Auctioneer utility function. $u_0(\text{outcome}) = \text{revenue}$
9. i observes whether he gets the object, and how much he pays.

Paper has more general theorems that relax winner-paying, symmetry, regularity. (omitted today)

Implementation via Extensive Forms

G denotes an **extensive-form mechanism**. (Each terminal history specifies an outcome.)

1. Finite depth.
2. No chance moves.
3. Perfect recall.

Implementation via Extensive Forms

G denotes an **extensive-form mechanism**. (Each terminal history specifies an outcome.)

1. Finite depth.
2. No chance moves.
3. Perfect recall.

$$S_i : \text{infosets} \times \Theta_i \to \text{actions}$$

Implementation via Extensive Forms

G denotes an **extensive-form mechanism**. (Each terminal history specifies an outcome.)

1. Finite depth.
2. No chance moves.
3. Perfect recall.

$$S_i : \text{infosets} \times \Theta_i \to \text{actions}$$

A **protocol** consists of mechanism G and strategy profile S_N.

Implementation via Extensive Forms

G denotes an **extensive-form mechanism**. (Each terminal history specifies an outcome.)

1. Finite depth.
2. No chance moves.
3. Perfect recall.

$$S_i : \text{infosets} \times \Theta_i \to \text{actions}$$

A **protocol** consists of mechanism G and strategy profile S_N.

4. For every history h, there exists θ_N such that h is on the path-of-play.
5. Every infoset has at least two actions.
6. If i is called to play at h, then i can affect the outcome.

Implementation via Extensive Forms

G denotes an **extensive-form mechanism**. (Each terminal history specifies an outcome.)

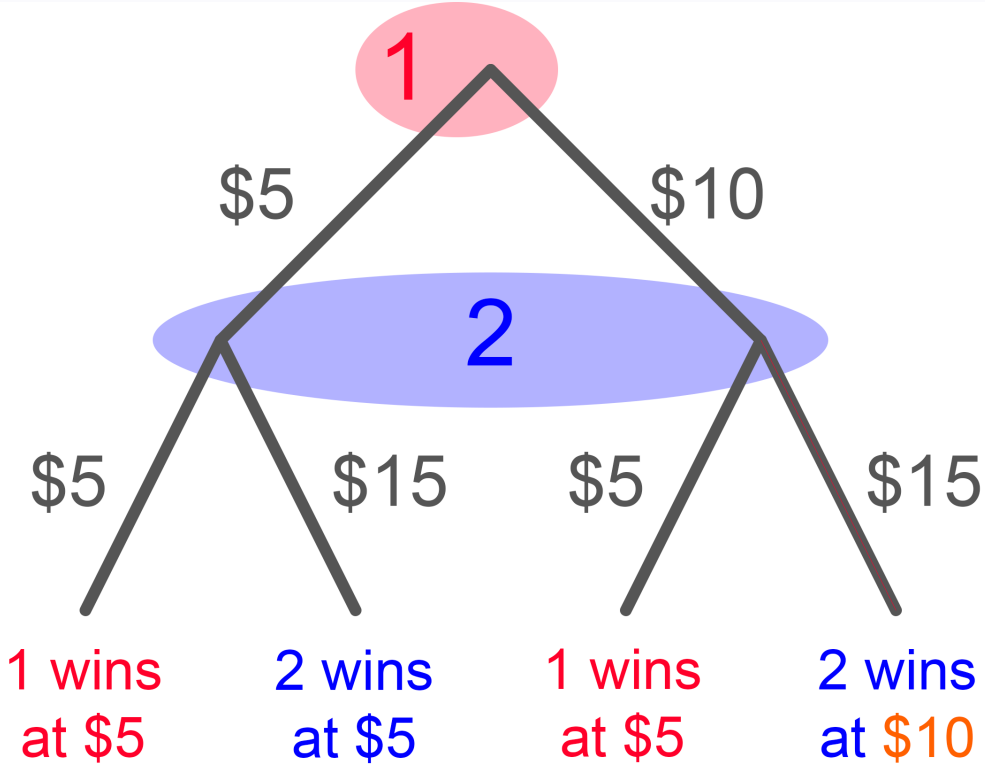
1. Finite depth.
2. No chance moves.
3. Perfect recall.

$$S_i : \text{infosets} \times \Theta_i \to \text{actions}$$

A **protocol** consists of mechanism G and strategy profile S_N.

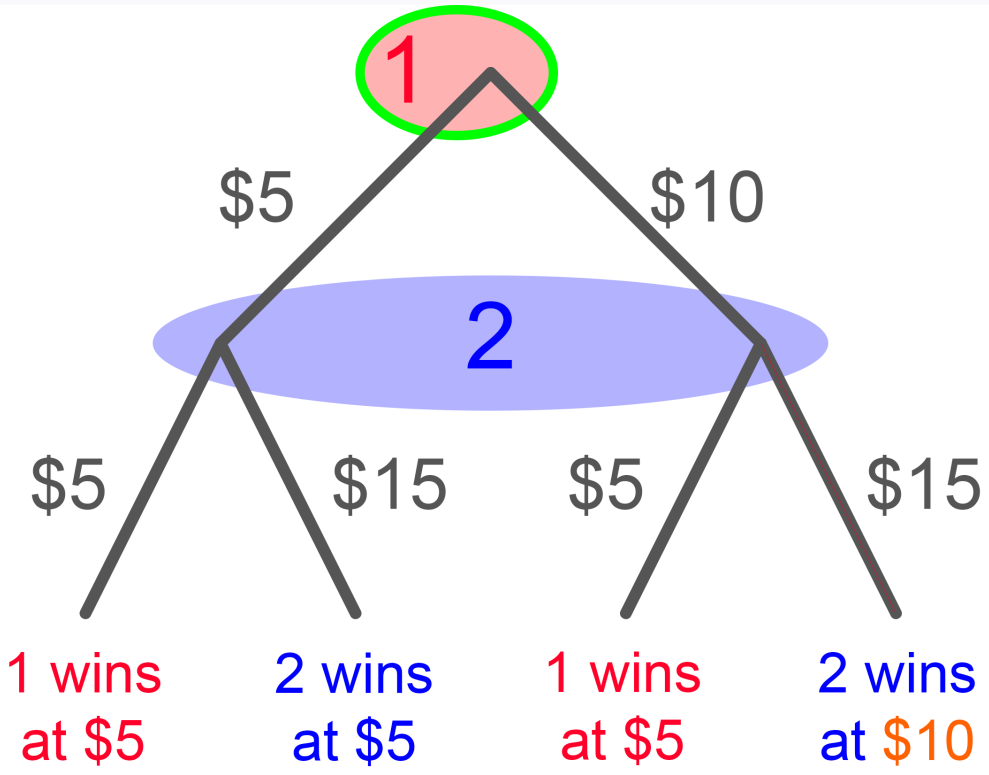
4. For every history h, there exists θ_N such that h is on the path-of-play.
5. Every infoset has at least two actions.
6. If i is called to play at h, then i can affect the outcome.

Restrict attention to **measurable** protocols.



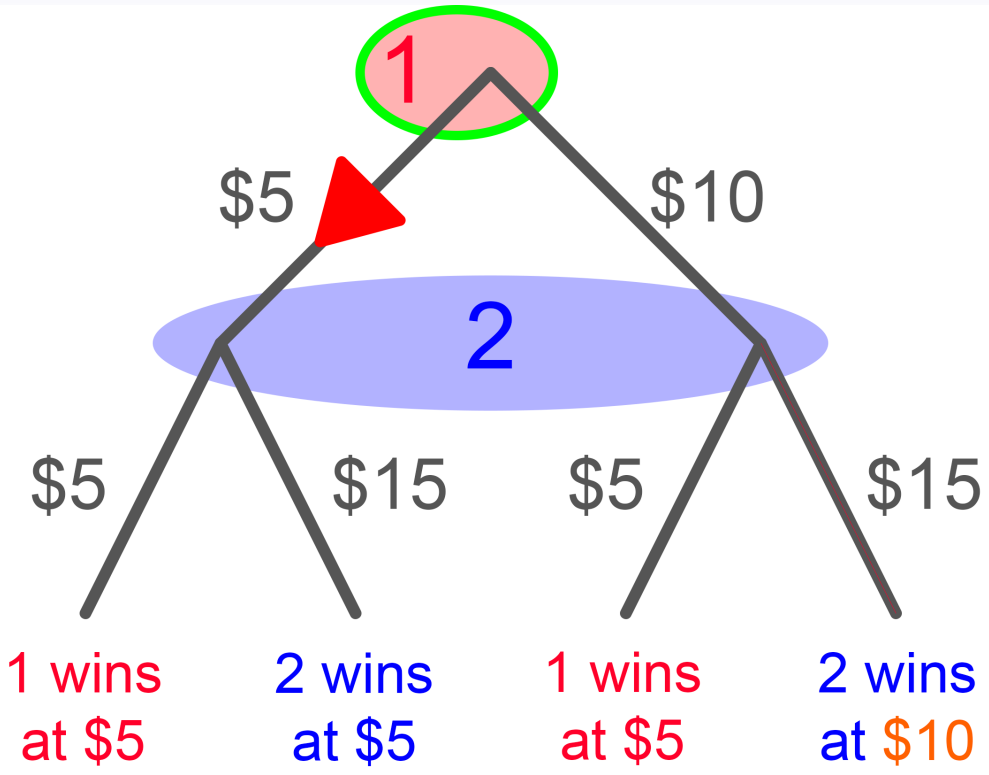
1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



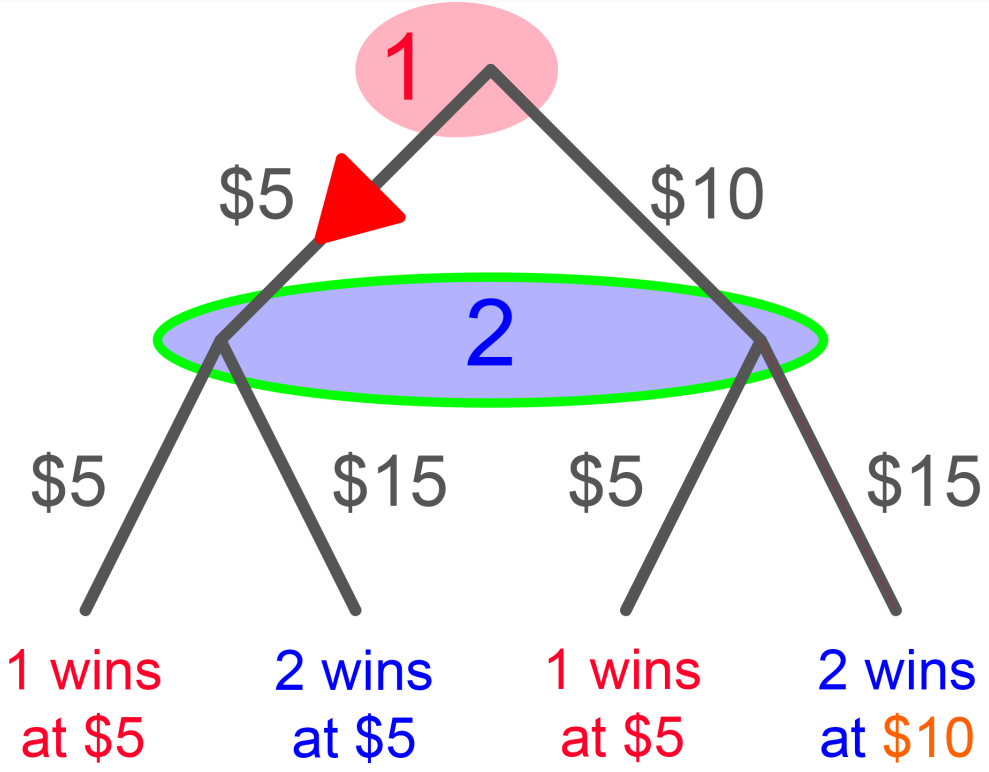
1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



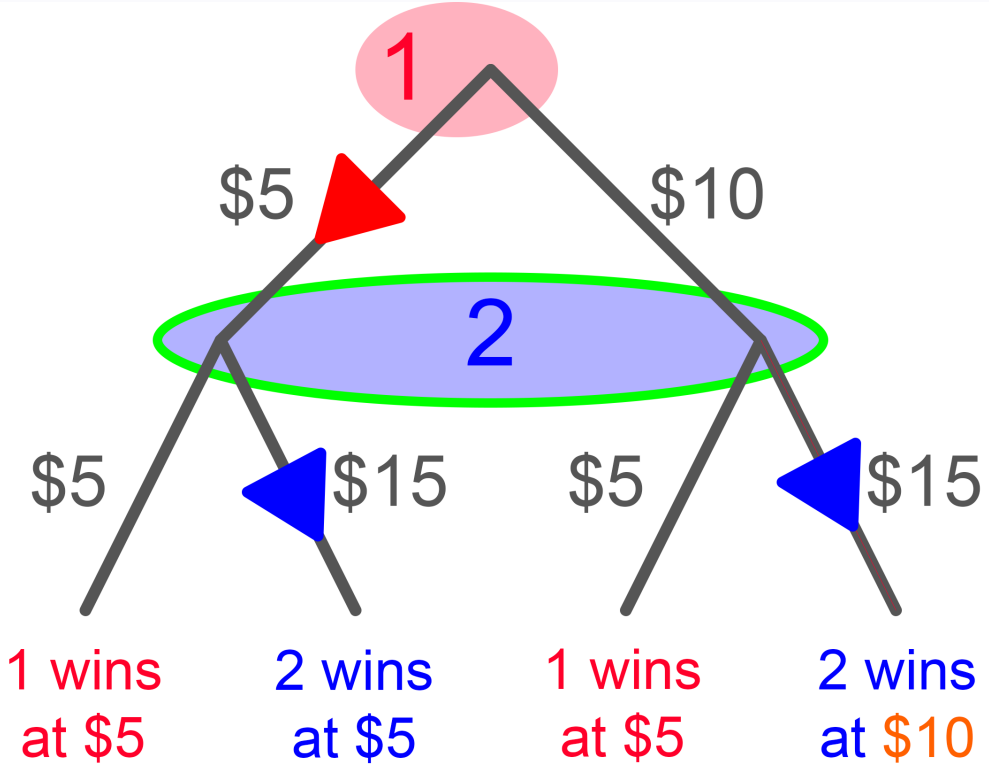
1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



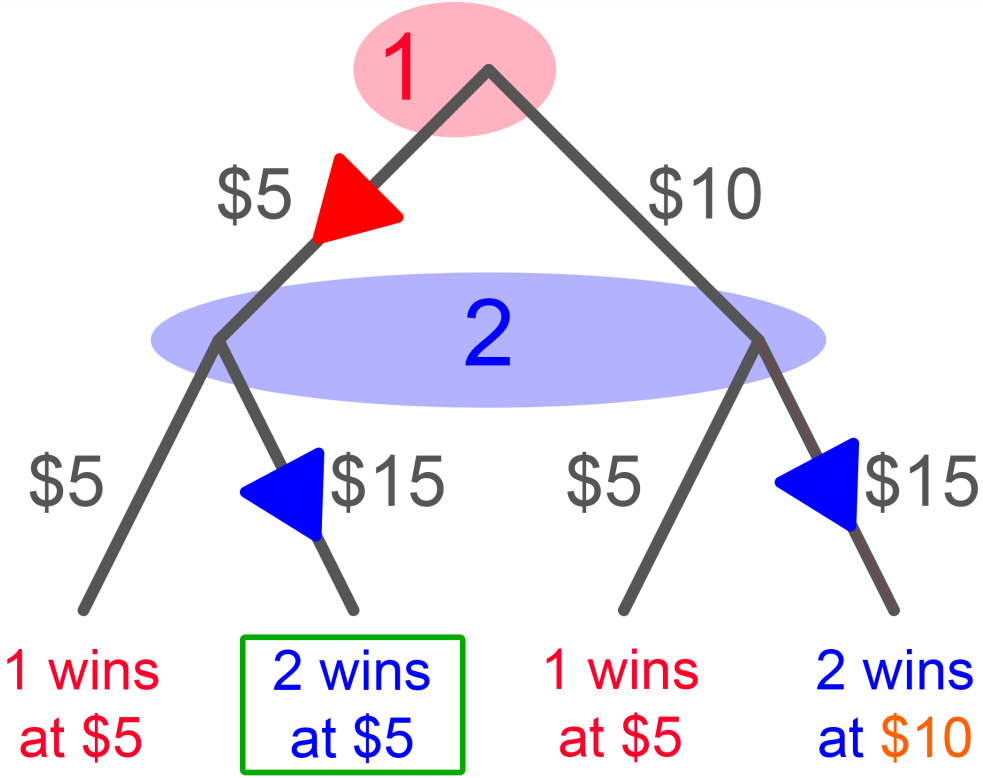
1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



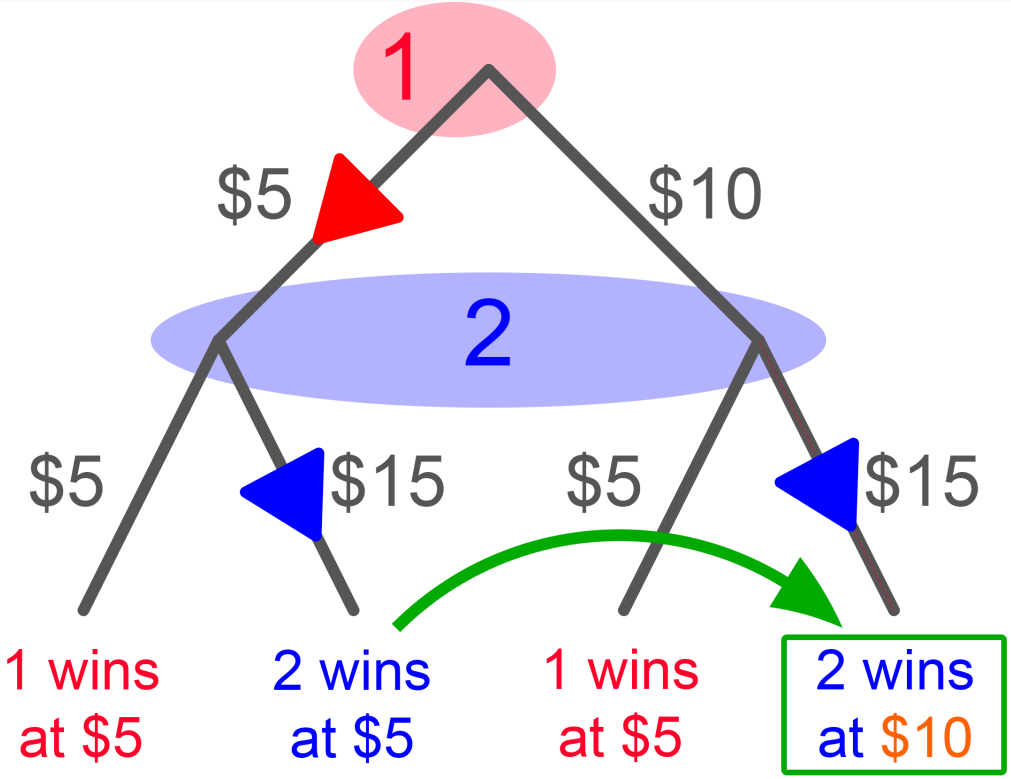
1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



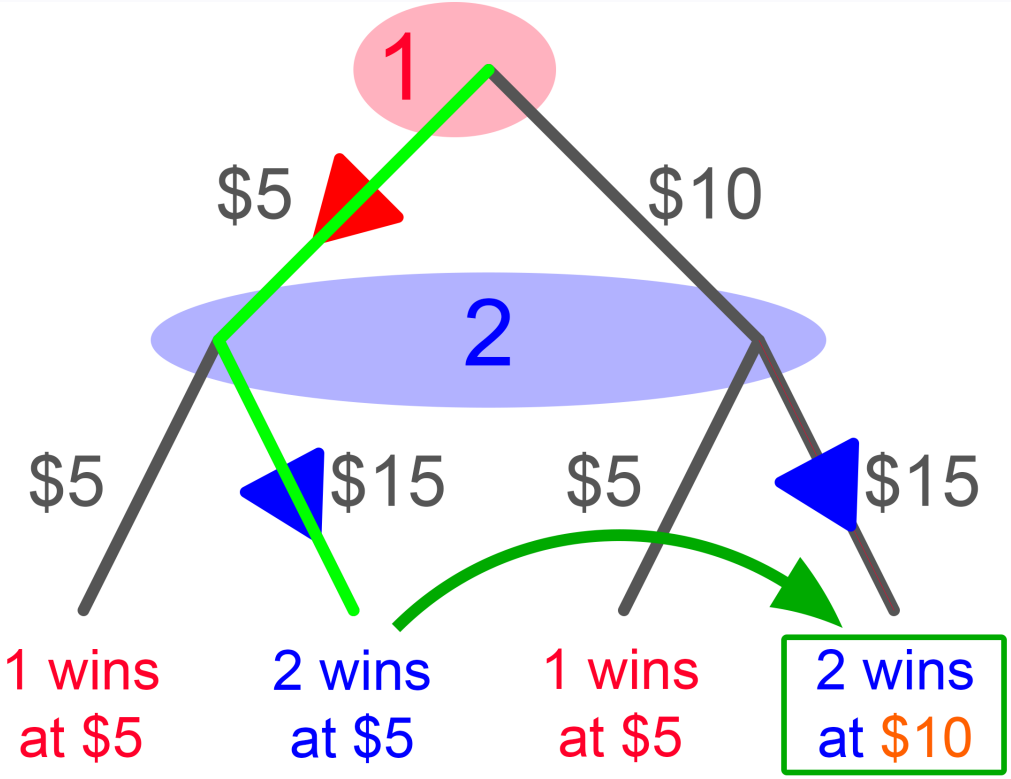
1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



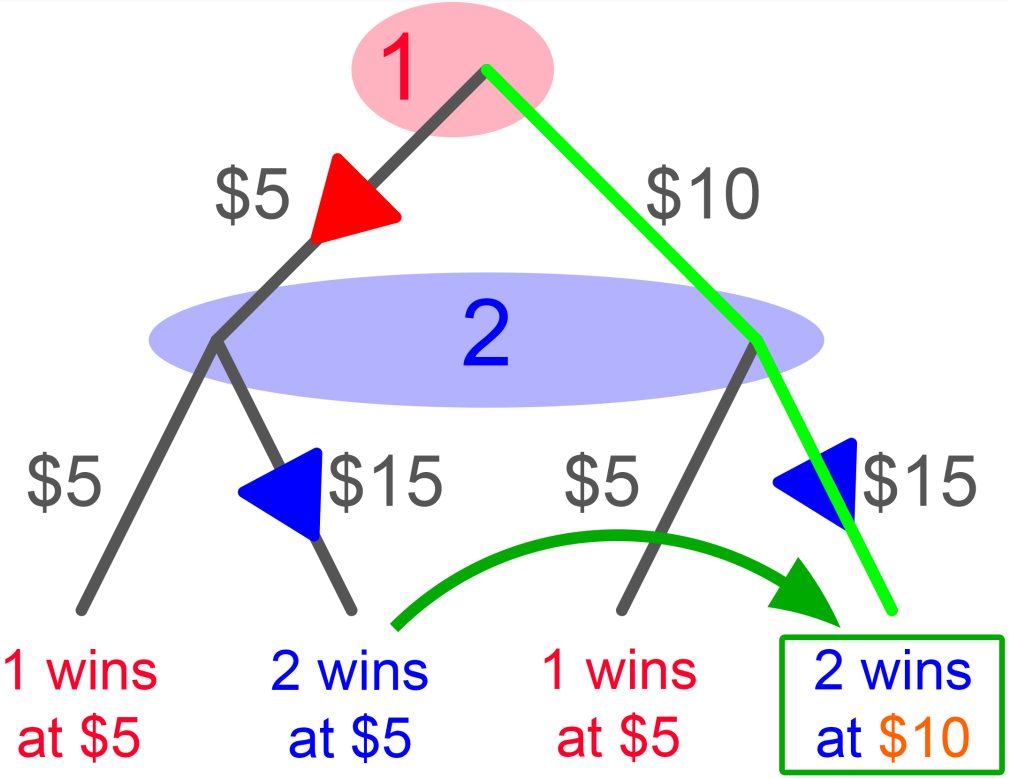
1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



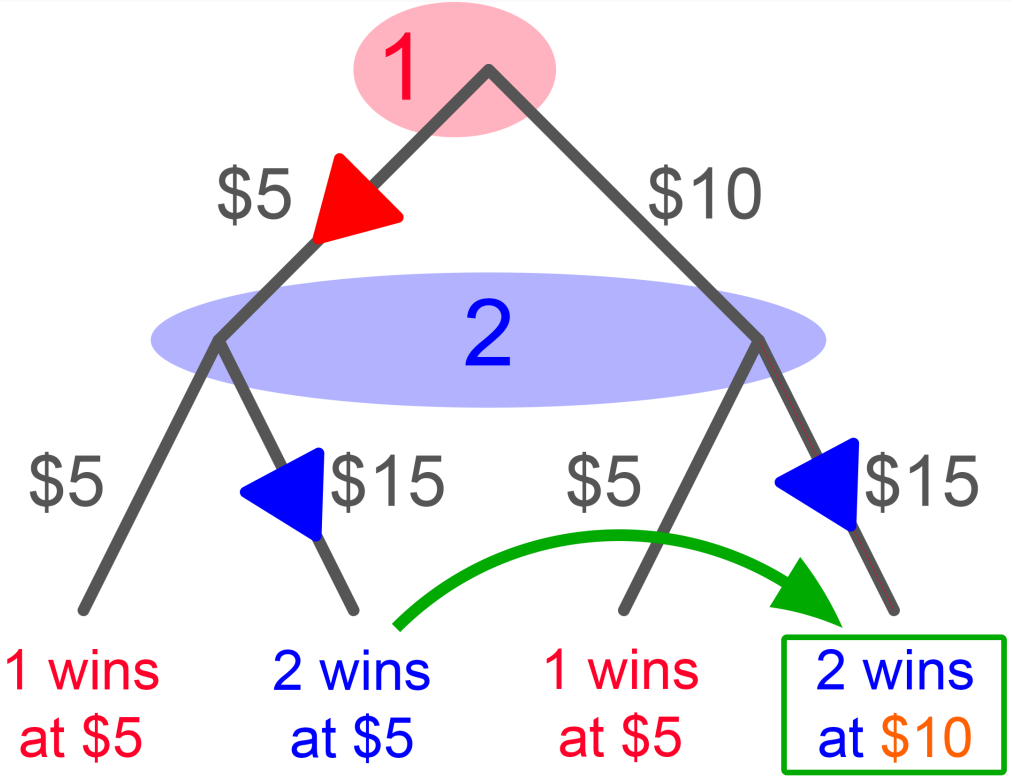
1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }

A Messaging Game

1. Auctioneer can:
   1.1 Either: Choose an outcome and end the game.
   1.2 Or: Go to Step 2.
2. Auctioneer chooses some agent i, sends message $m \in \{i\text{'s infosets in G}\}$
3. i privately observes m, chooses reply $r \in \{\text{actions available at } m\}$.
4. Auctioneer privately observes r. Go to Step 1.

A Messaging Game

1. Auctioneer can:
   1.1 Either: Choose an outcome and end the game.
   1.2 Or: Go to Step 2.
2. Auctioneer chooses some agent i, sends message $m \in \{i\text{'s infosets in G}\}$
3. i privately observes m, chooses reply $r \in \{\text{actions available at } m\}$.
4. Auctioneer privately observes r. Go to Step 1.

Full commitment: To 'run' G, auctioneer commits to S_0^G.

A Messaging Game

1. Auctioneer can:
   1.1 Either: Choose an outcome and end the game.
   1.2 Or: Go to Step 2.
2. Auctioneer chooses some agent i, sends message $m \in \{i\text{'s infosets in G}\}$
3. i privately observes m, chooses reply $r \in \{\text{actions available at } m\}$.
4. Auctioneer privately observes r. Go to Step 1.

Full commitment: To 'run' G, auctioneer commits to S_0^G.

Partial commitment: Auctioneer can deviate to any S_0 that an individual agent cannot distinguish from S_0^G.

A Messaging Game

1. Auctioneer can:
   1.1 Either: Choose an outcome and end the game.
   1.2 Or: Go to Step 2.
2. Auctioneer chooses some agent i, sends message $m \in \{i\text{'s infosets in G}\}$
3. i privately observes m, chooses reply $r \in \{\text{actions available at } m\}$.
4. Auctioneer privately observes r. Go to Step 1.

Full commitment: To 'run' G, auctioneer commits to S_0^G.

Partial commitment: Auctioneer can deviate to any S_0 that an individual agent cannot distinguish from S_0^G.

How the Auctioneer Can Deviate

Consider protocol (G, S_N), and S_0^G that 'runs' G.

How the Auctioneer Can Deviate

Consider protocol (G, S_N), and S_0^G that 'runs' G.

o_i observation for i =
communication sequence $(m_i^t, r_i^t)_{t=1}^T$
& whether i wins, how much i pays

How the Auctioneer Can Deviate

Consider protocol (G, S_N), and S_0^G that 'runs' G.

o_i observation for i =
communication sequence $(m_i^t, r_i^t)_{t=1}^T$
& whether i wins, how much i pays

resulting observation denoted $o_i(S_0, S_N, \theta_N)$

How the Auctioneer Can Deviate

Consider protocol (G, S_N), and S_0^G that 'runs' G.

o_i observation for i =
communication sequence $(m_i^t, r_i^t)_{t=1}^T$
& whether i wins, how much i pays

resulting observation denoted $o_i(S_0, S_N, \theta_N)$

$o_i(S_0, S_N, \theta_N)$ has an **innocent explanation** if:

$$\exists \theta'_{-i} : o_i(S_0, S_N, \theta_N) = o_i(S_0^G, S_N, (\theta_i, \theta'_{-i}))$$

How the Auctioneer Can Deviate

Consider protocol (G, S_N), and S_0^G that 'runs' G.

o_i observation for i =
communication sequence $(m_i^t, r_i^t)_{t=1}^T$
& whether i wins, how much i pays

resulting observation denoted $o_i(S_0, S_N, \theta_N)$

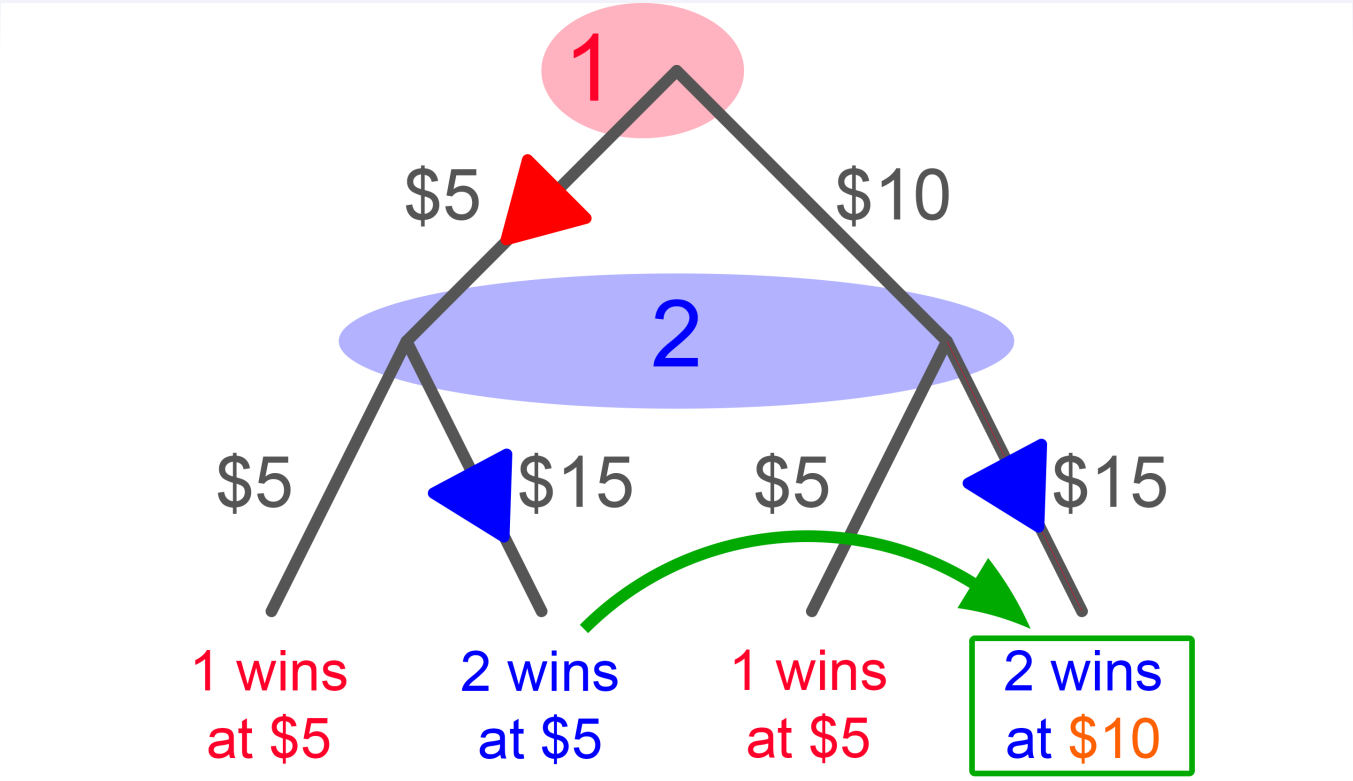
$o_i(S_0, S_N, \theta_N)$ has an **innocent explanation** if:

$$\exists \theta'_{-i} : o_i(S_0, S_N, \theta_N) = o_i(S_0^G, S_N, (\theta_i, \theta'_{-i}))$$

S_0 is **safe** if $\forall i : \forall \theta_N : o_i(S_0, S_N, \theta_N)$ has an innocent explanation.

Definition

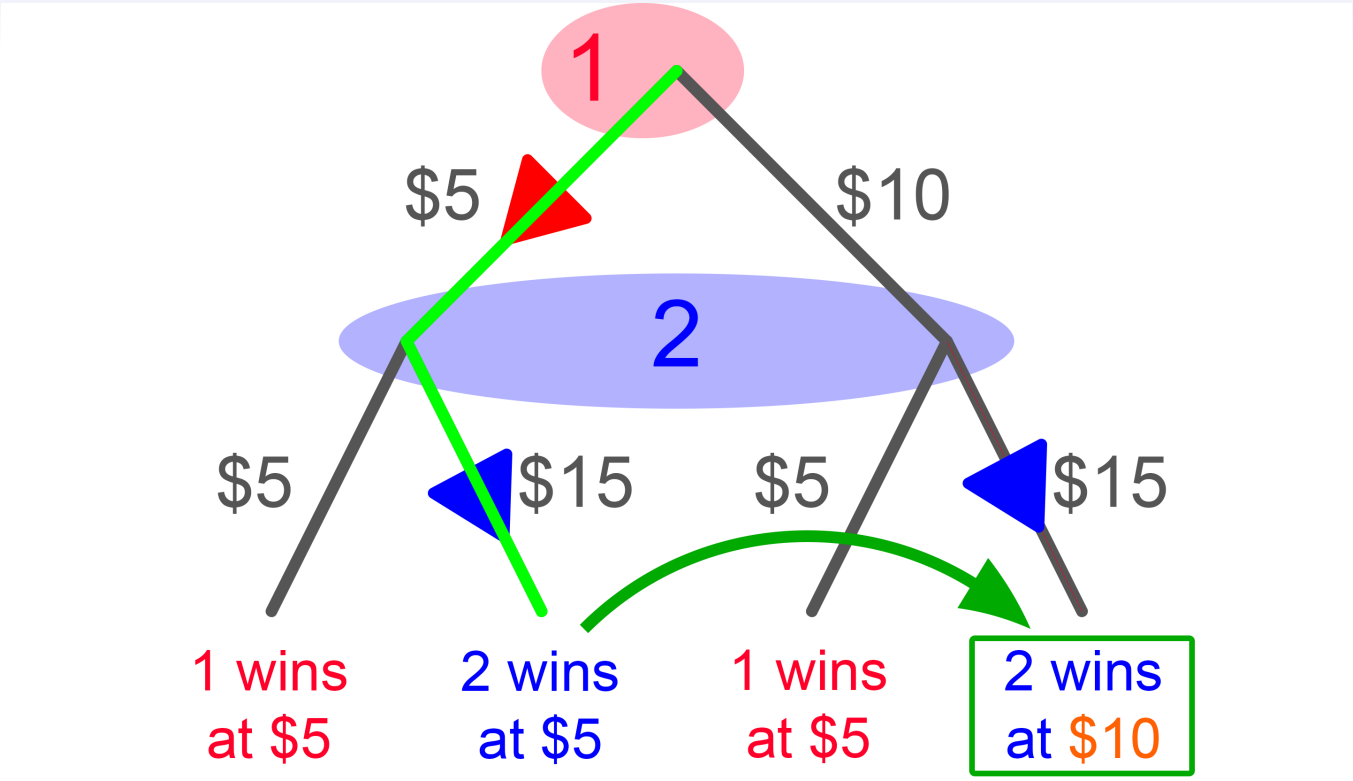
(G, S_N) *is* **credible** *if* S_0^G *is a best-response to* S_N.
(yields at least as much expected utility as any safe deviation)



Auctioneer's deviation

1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

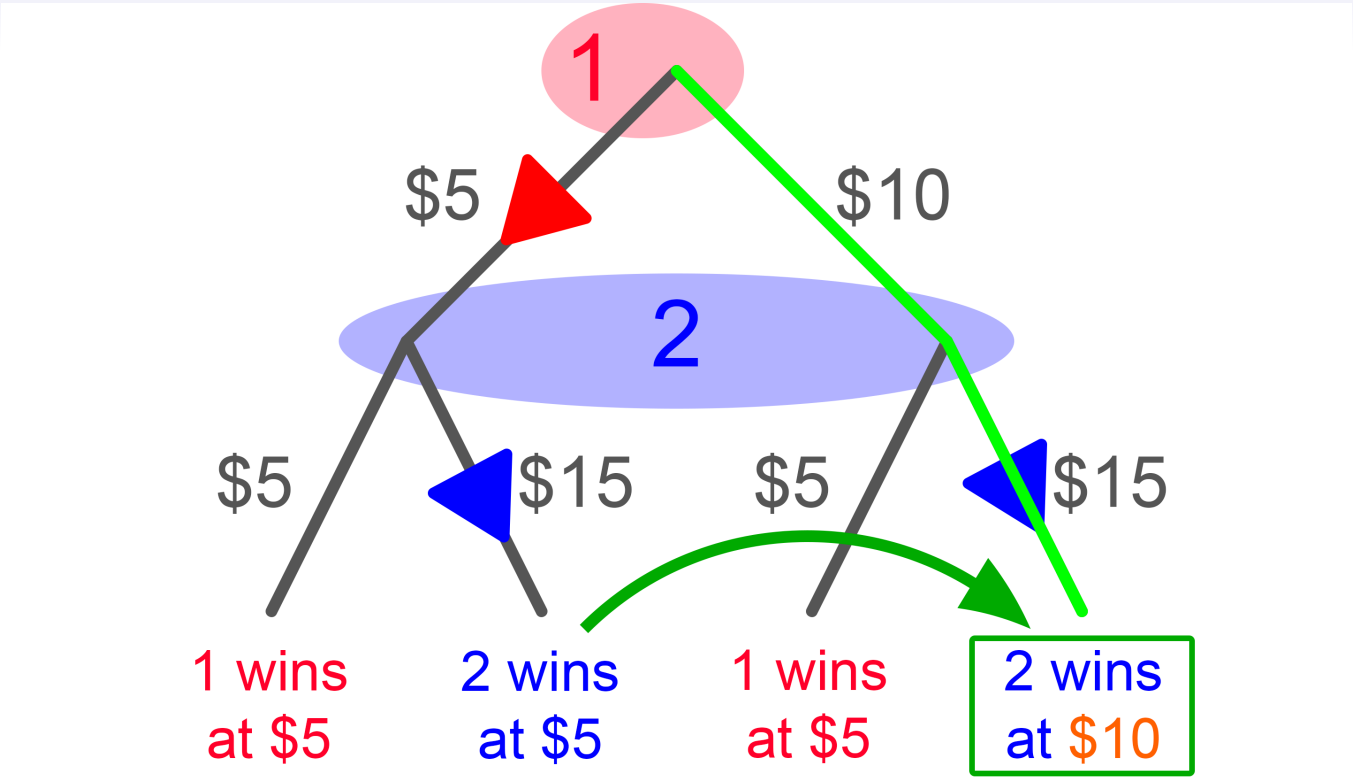
2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



Innocent explanation for 1's observation

1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }



Innocent explanation for 2's observation

1 observes: { 1 wins at $5 } { 2 wins at $5 , 2 wins at $10 }

2 observes: { 1 wins at $5 } { 2 wins at $5 } { 2 wins at $10 }

Safe deviations can involve 'midway' deception.

The auctioneer can send different messages during the mechanism.

She may adopt any communication strategy that is safe.

(For every type profile and every agent i, i's observation has an innocent explanation.)

Related literature

As above, but restricted to direct mechanisms

Dequiedt & Martimort 2015

This talk: Extensive forms.

Related literature

As above, but restricted to direct mechanisms

Dequiedt & Martimort 2015

This talk: Extensive forms.

Commit to today's auction, not tomorrow's auction

Milgrom 1987, McAfee & Vincent 1997, Skreta 2015, Liu *et al* 2017, Doval and Skreta 2019

This talk: Not a repeated game.

Related literature

As above, but restricted to direct mechanisms

Dequiedt & Martimort 2015

This talk: Extensive forms.

Commit to today's auction, not tomorrow's auction

Milgrom 1987, McAfee & Vincent 1997, Skreta 2015, Liu *et al* 2017, Doval and Skreta 2019

This talk: Not a repeated game.

Auctions as bargaining games

McAdams & Schwarz 2007, Vartiainen 2013, Lobel & Paes Leme 2017

This talk: No 'red-handed' rule-breaking.

credible static optimal auctions

Definition

(G, S_N) *is* **optimal** *if it maximizes revenue subject to BIC and participation constraints.*
(G, S_N) *is* **static** *if every agent has exactly one infoset and is always called to play.*

Definition

(G, S_N) *is a* **first-price auction** *if it is static, and each i either chooses a bid in some feasible set $B_i \subset \mathbb{R}$ or declines. The player with the highest bid wins, and pays his bid. (break ties arbitrarily)*

credible static optimal auctions

Definition

(G, S_N) *is* **optimal** *if it maximizes revenue subject to BIC and participation constraints.*
(G, S_N) *is* **static** *if every agent has exactly one infoset and is always called to play.*

Definition

(G, S_N) *is a* **first-price auction** *if it is static, and each i either chooses a bid in some feasible set $B_i \subset \mathbb{R}$ or declines. The player with the highest bid wins, and pays his bid. (break ties arbitrarily)*

We represent a reserve price by restricting B_i.

Notice: Bids defined by consequences, not by labels.

credible static optimal auctions

Definition

(G, S_N) *is* **optimal** *if it maximizes revenue subject to BIC and participation constraints.*
(G, S_N) *is* **static** *if every agent has exactly one infoset and is always called to play.*

Definition

(G, S_N) *is a* **first-price auction** *if it is static, and each i either chooses a bid in some feasible set $B_i \subset \mathbb{R}$ or declines. The player with the highest bid wins, and pays his bid. (break ties arbitrarily)*

Theorem 1

Assume (G, S_N) is optimal. If (G, S_N) is a first-price auction, then it is credible and static. If (G, S_N) is credible and static, then it is almost-everywhere a first-price auction.

credible static optimal auctions

Definition

(G, S_N) *is* **optimal** *if it maximizes revenue subject to BIC and participation constraints.*
(G, S_N) *is* **static** *if every agent has exactly one infoset and is always called to play.*

Definition

(G, S_N) *is a* **first-price auction** *if it is static, and each i either chooses a bid in some feasible set $B_i \subset \mathbb{R}$ or declines. The player with the highest bid wins, and pays his bid. (break ties arbitrarily)*

Theorem 1

Assume (G, S_N) is optimal. If (G, S_N) is a first-price auction, then it is credible and static. If (G, S_N) is credible and static, then it is almost-everywhere a first-price auction.

Proof Idea

first-price $\rightarrow$ credible and static

By inspection.

Proof Idea

first-price $\rightarrow$ credible and static

By inspection.

credible and static $\rightarrow$ almost-everywhere first-price

Suppose after i plays a, there are two prices that i might pay. Deviate to charge the higher price.

(Real proof must ensure deviation is measurable.)

Proof Idea

first-price $\rightarrow$ credible and static

By inspection.

credible and static $\rightarrow$ almost-everywhere first-price

Suppose after i plays a, there are two prices that i might pay. Deviate to charge the higher price.

(Real proof must ensure deviation is measurable.)

Lemma: Almost everywhere, for each action i takes, there is a unique price he pays if he wins.

Proof Idea

first-price $\rightarrow$ credible and static

By inspection.

credible and static $\rightarrow$ almost-everywhere first-price

Suppose after i plays a, there are two prices that i might pay. Deviate to charge the higher price.

(Real proof must ensure deviation is measurable.)

Lemma: Almost everywhere, for each action i takes, there is a unique price he pays if he wins.

Optimal allocation + envelope theorem pins down payments, so (almost everywhere) the winning bidder has highest bid. QED.

Dominant-strategy or credible?

Big Changes Coming To Auctions, As Exchanges Roll The Dice On First-Price

by Sarah Sluis // Tuesday, September 5th, 2017 – 8:00 am

Share:

The second-price auction is crumbling.

Buyers, publishers, and ad tech companies who advocate a switch to first-price auctions say it's because fair second-price auctions don't exist any more. [Online auctioneers] have polluted them with hidden fees and manipulative auction dynamics.

Dominant-strategy or credible?

Big Changes Coming To Auctions, As Exchanges Roll The Dice On First-Price

by Sarah Sluis // Tuesday, September 5th, 2017 – 8:00 am

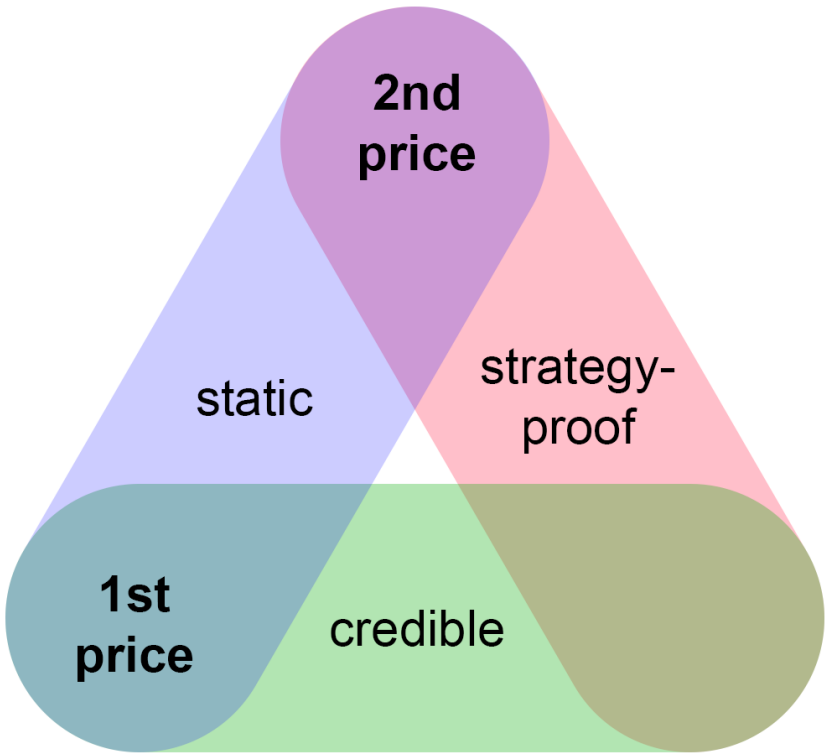
Share:

The second-price auction is crumbling.

Buyers, publishers, and ad tech companies who advocate a switch to first-price auctions say it's because fair second-price auctions don't exist any more. [Online auctioneers] have polluted them with hidden fees and manipulative auction dynamics.

March 2019: Google Ad Manager announces switch to first-price.

The story so far



Strategy-proof

Definition

(G, S_N) *is* **strategy-proof** *if* $\forall i : \forall S'_{N \setminus i} : S_i$ *best responds to* $S'_{N \setminus i}$.

Strategy-proof

Definition

(G, S_N) *is* **strategy-proof** *if* $\forall i : \forall S'_{N\setminus i} : S_i$ *best responds to* $S'_{N\setminus i}$.

Goal: Characterize the set of optimal extensive game forms credible $\cap$ strategy-proof.

Strategy-proof

Definition

(G, S_N) *is* **strategy-proof** *if* $\forall i : \forall S'_{N \setminus i} : S_i$ *best responds to* $S'_{N \setminus i}$.

Goal: Characterize the set of optimal extensive game forms credible $\cap$ strategy-proof.

No revelation principle.

1. Auctioneer could make any queries in any order.
2. Agents may receive information when called to play.

Strategy-proof

Definition

(G, S_N) *is* **strategy-proof** *if* $\forall i : \forall S'_{N \setminus i} : S_i$ *best responds to* $S'_{N \setminus i}$.

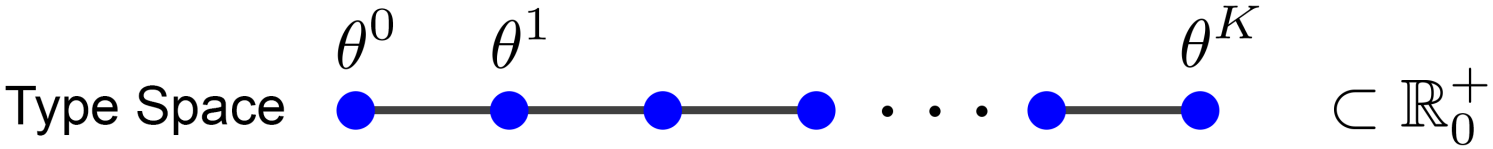
Goal: Characterize the set of optimal extensive game forms credible $\cap$ strategy-proof.

No revelation principle.

1. Auctioneer could make any queries in any order.
2. Agents may receive information when called to play.

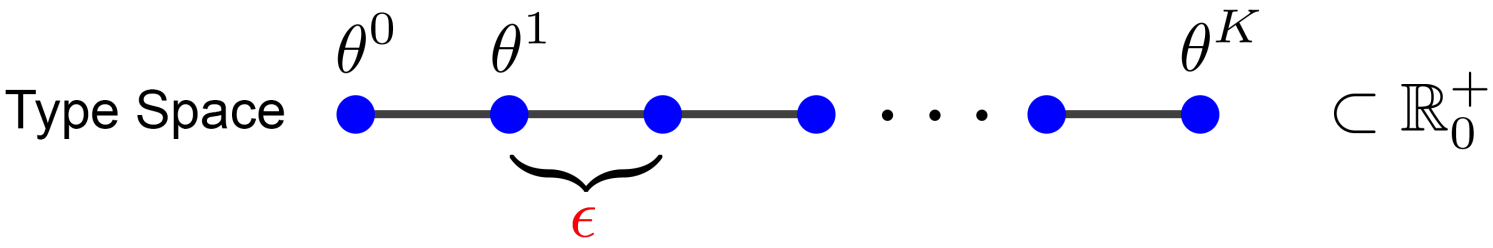
Problem: Extensive-form ascending auctions move in discrete steps, can't be exactly optimal.
Solution: Discretize the distribution.

Discretize the distribution



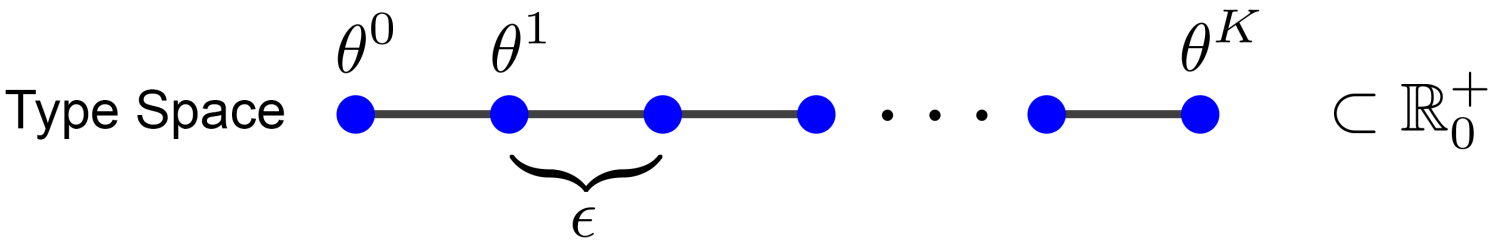
i.i.d. probability mass function $p : \Theta_i \to (0, 1]$

Discretize the distribution



i.i.d. probability mass function $p : \Theta_i \to (0, 1]$

Discretize the distribution

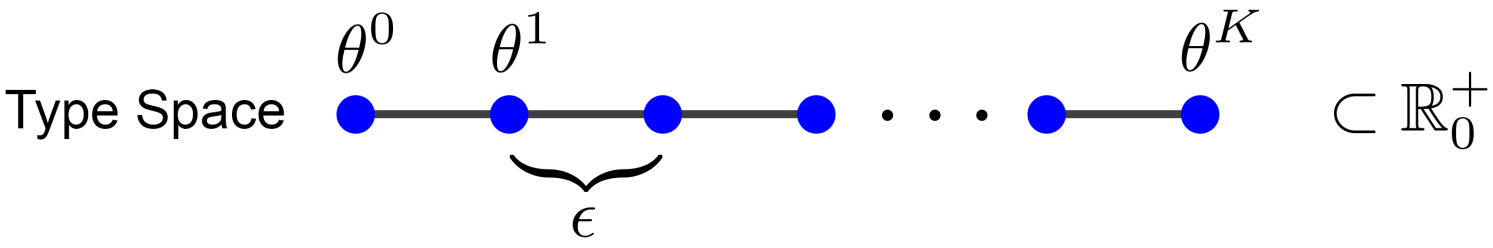


i.i.d. probability mass function $p : \Theta_i \to (0, 1]$

$$\text{pseudo-pdf} \quad f(\theta^k) \equiv \frac{p(\theta^k)}{\epsilon}$$

$$\text{cdf} \quad F(\theta^k) \equiv \sum_{j=1}^{k} p(\theta^j)$$

Discretize the distribution



i.i.d. probability mass function $p : \Theta_i \to (0, 1]$

$$\text{pseudo-pdf} \quad f(\theta^k) \equiv \frac{p(\theta^k)}{\epsilon}$$

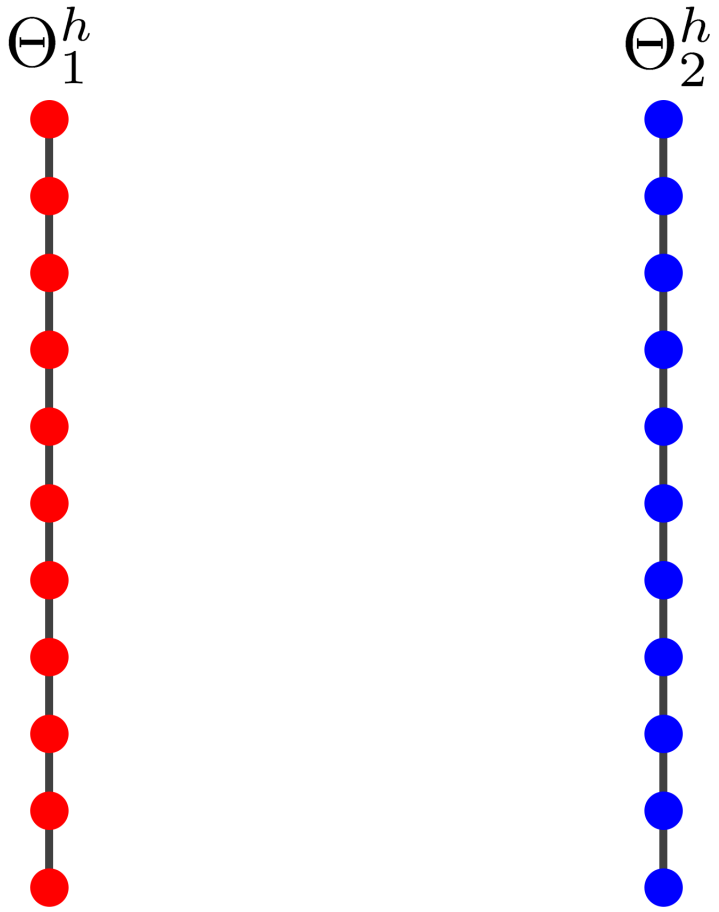
$$\text{cdf} \quad F(\theta^k) \equiv \sum_{j=1}^{k} p(\theta^j)$$

$$\text{virtual value} \quad \eta(\theta_i) \equiv \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}$$

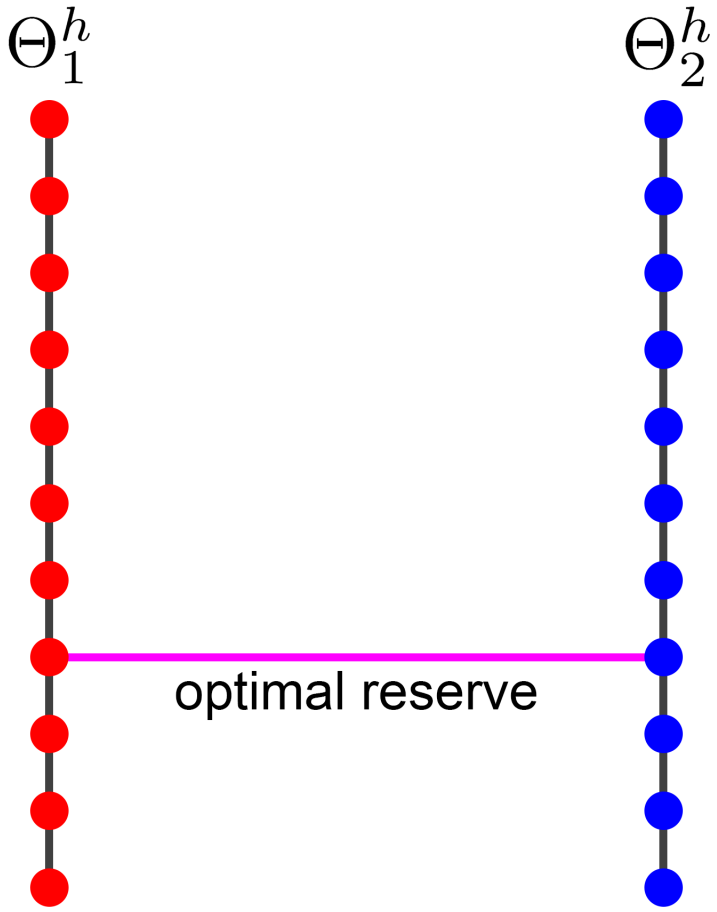
Assumption. F is regular, *i.e.* $\eta(\cdot)$ is strictly increasing.

A credible strategy-proof auction

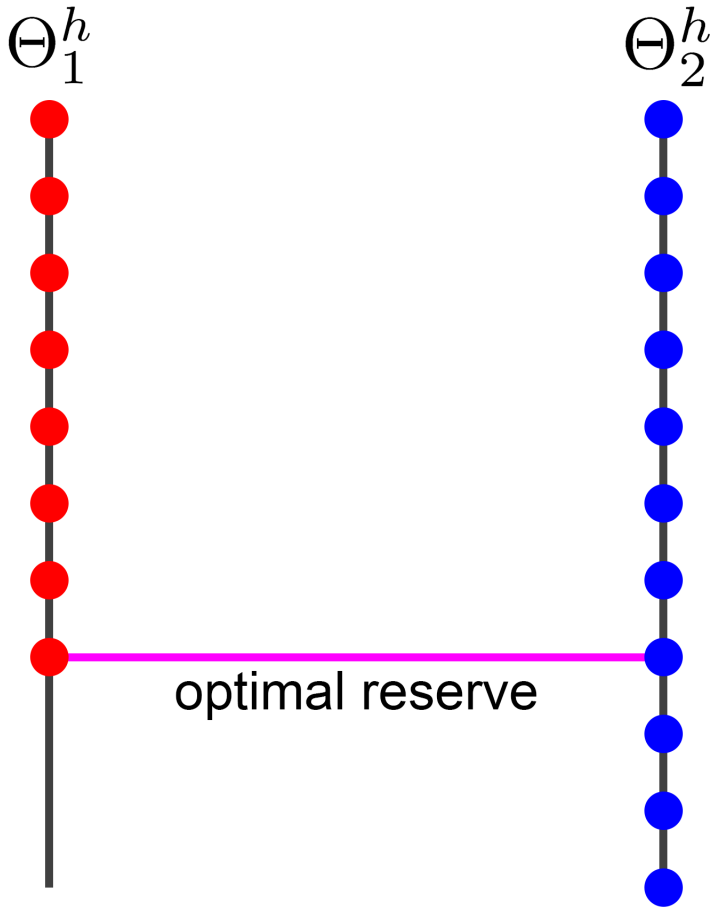
A credible strategy-proof auction



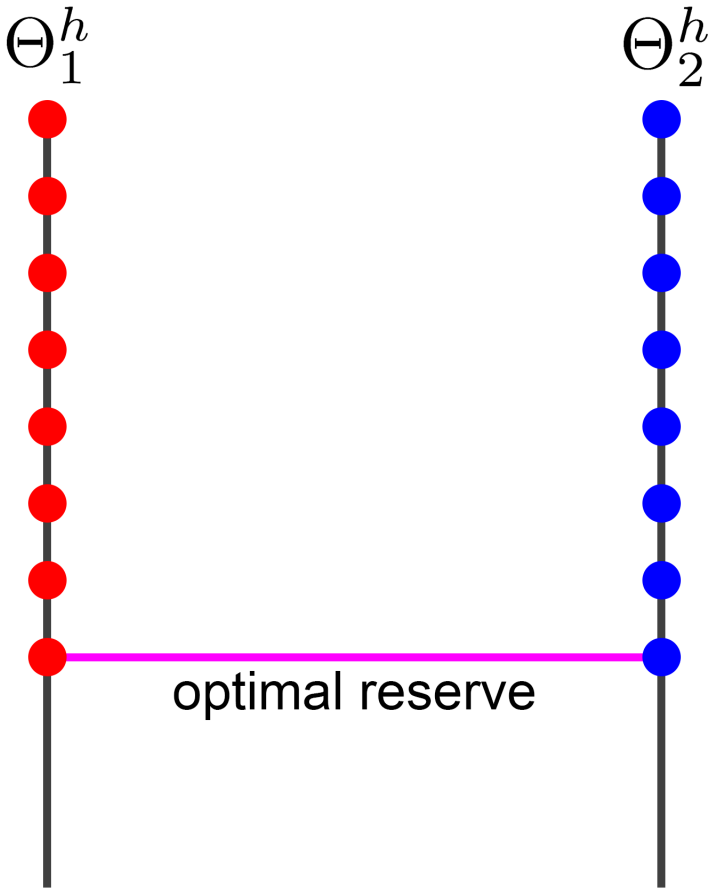
A credible strategy-proof auction



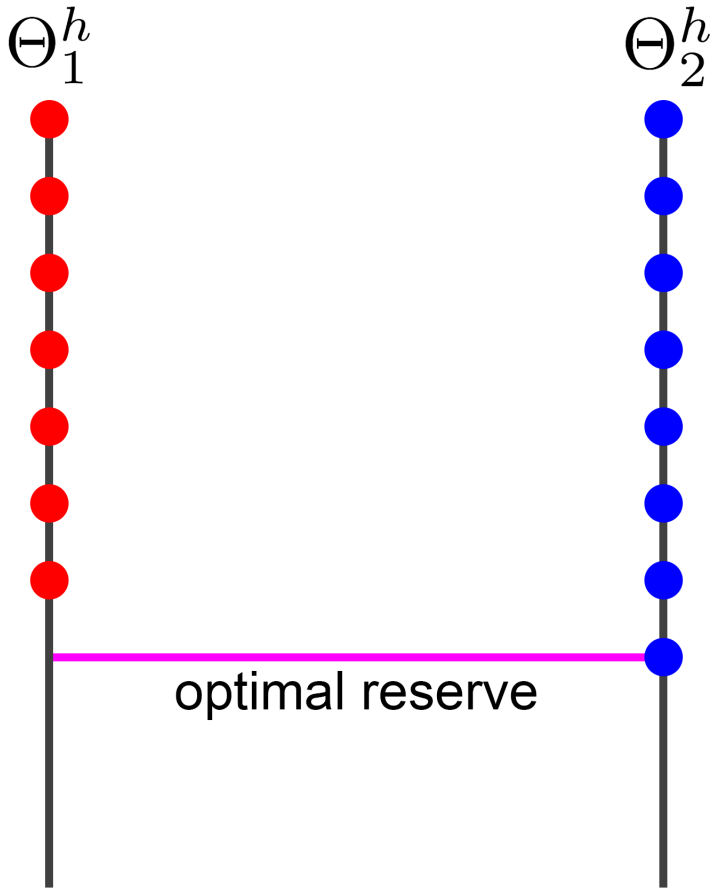
A credible strategy-proof auction



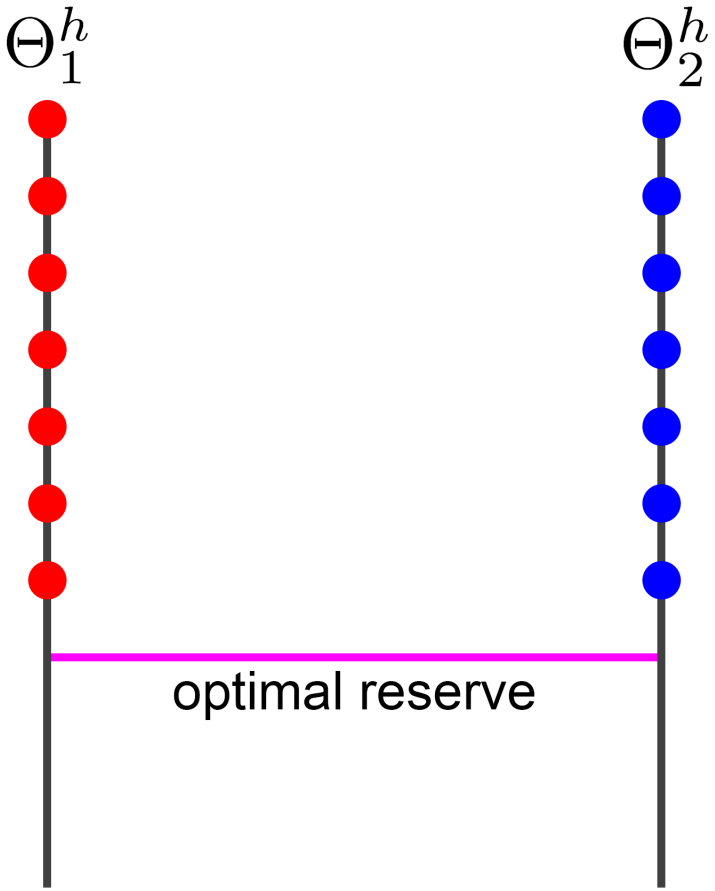
A credible strategy-proof auction



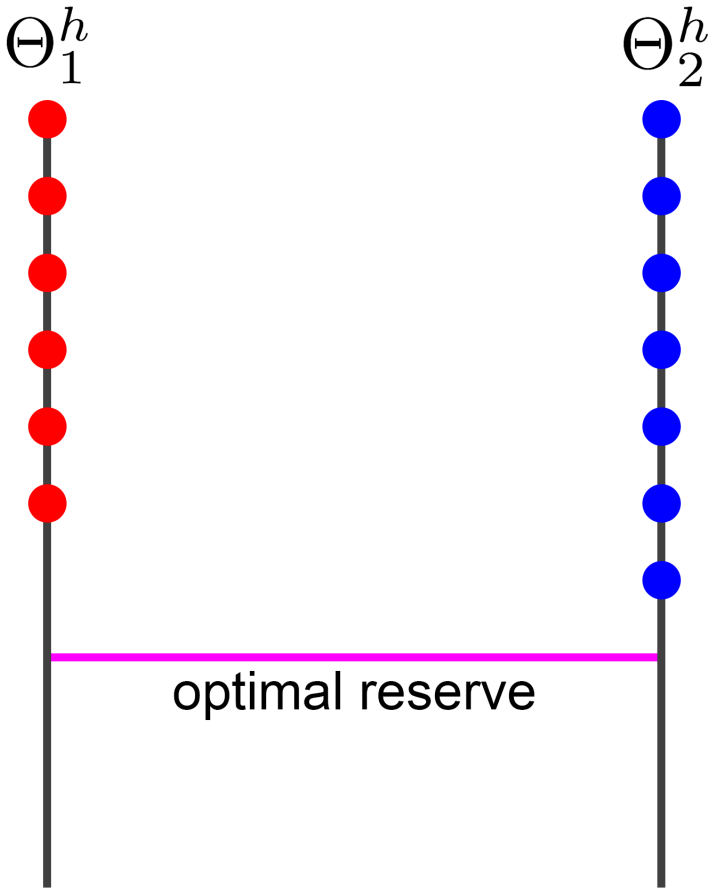
A credible strategy-proof auction



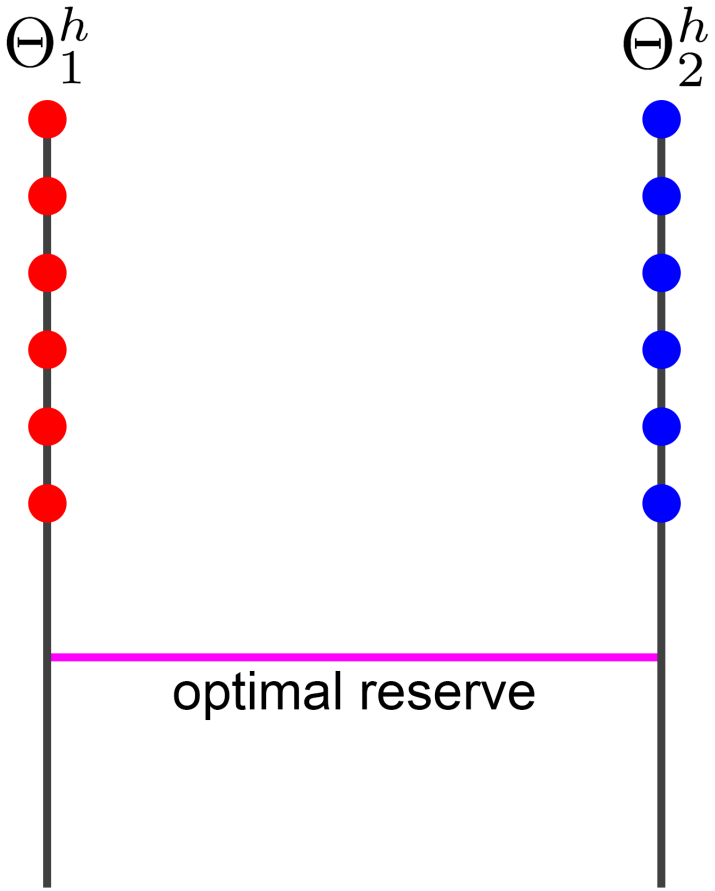
A credible strategy-proof auction



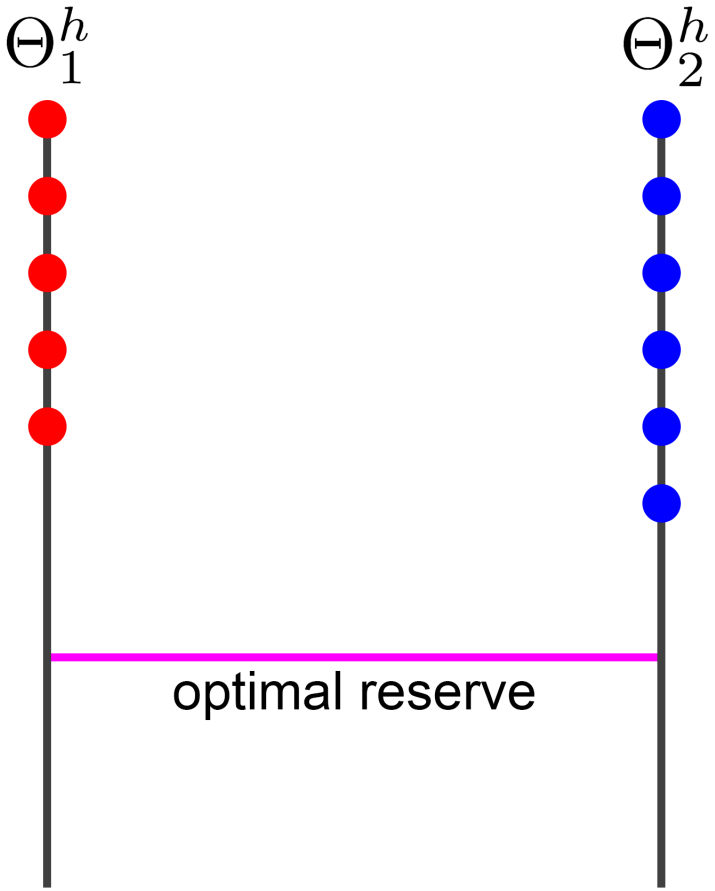
A credible strategy-proof auction



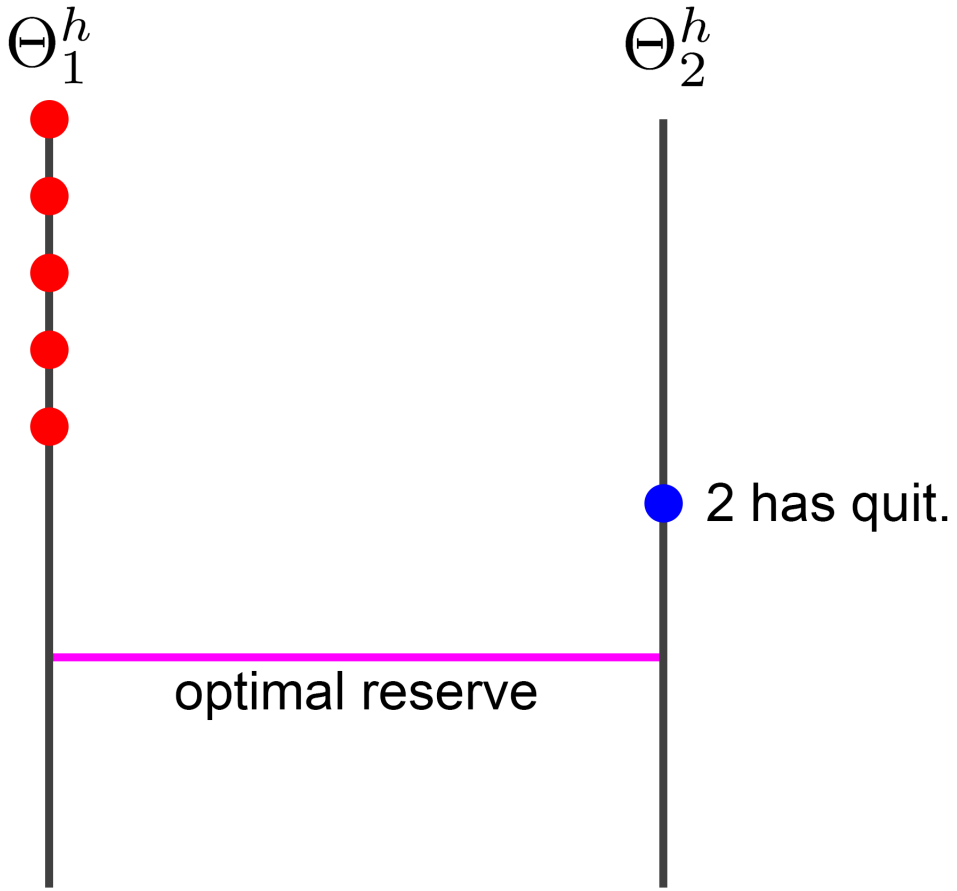
A credible strategy-proof auction



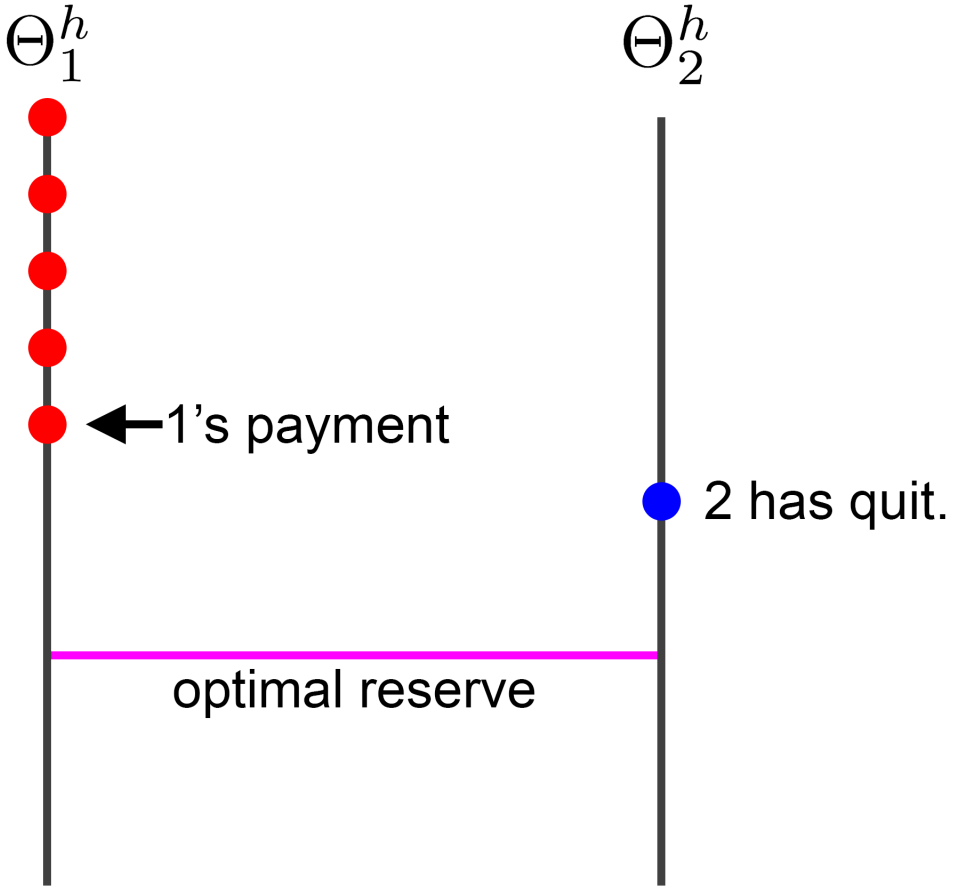
A credible strategy-proof auction



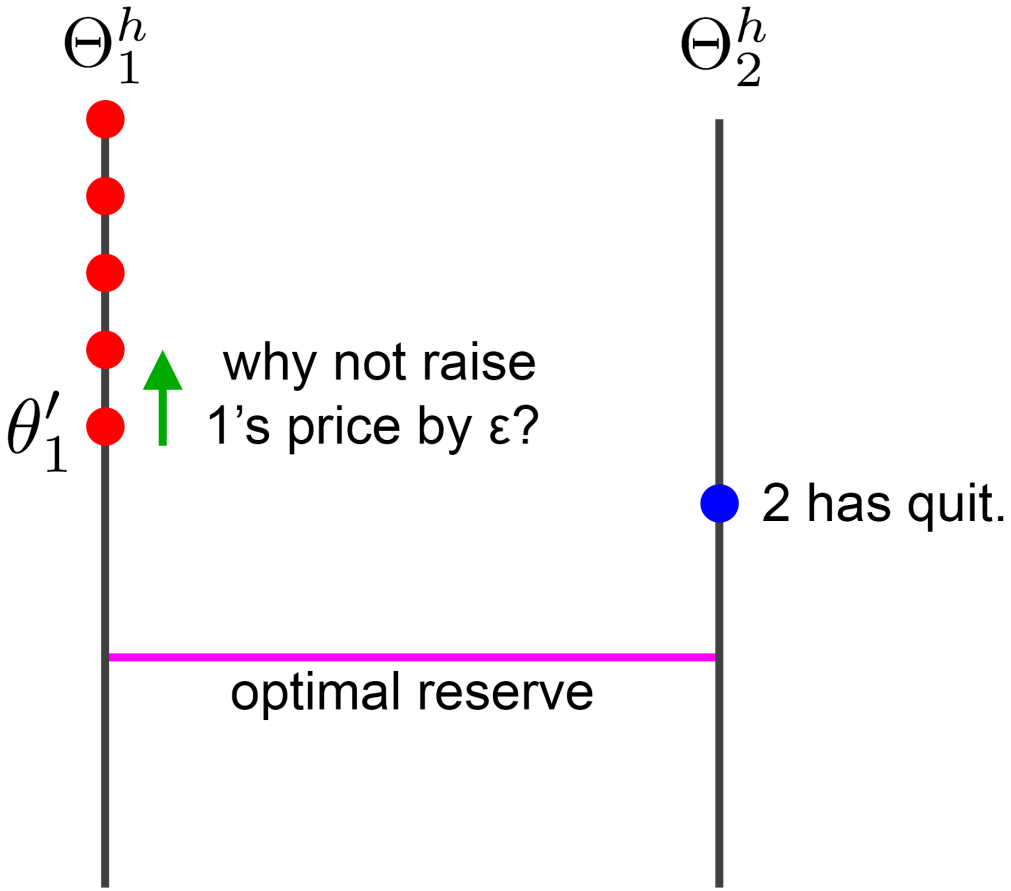
A credible strategy-proof auction



A credible strategy-proof auction



A credible strategy-proof auction



Q: Why not raise 1's price to $b_1 + \epsilon$, even after bidder 2 has quit?
A: 1's virtual value is positive.

Q: Why not raise 1's price to $b_1 + \epsilon$, even after bidder 2 has quit?
A: 1's virtual value is positive.

'The book' requires that 1 pay b_1.

$$\underbrace{-\epsilon f(b_1) b_1}_{\text{expected loss from 1 quitting}} \quad + \quad \underbrace{(1 - F(b_1))\epsilon}_{\text{expected gain from raising price}}$$

Q: Why not raise 1's price to $b_1 + \epsilon$, even after bidder 2 has quit?
A: 1's virtual value is positive.

'The book' requires that 1 pay b_1.

$$\underbrace{-\epsilon f(b_1) b_1}_{\text{expected loss from 1 quitting}} \quad + \quad \underbrace{(1 - F(b_1))\epsilon}_{\text{expected gain from raising price}}$$

Q: Why not raise 1's price to $b_1 + \epsilon$, even after bidder 2 has quit?
A: 1's virtual value is positive.

'The book' requires that 1 pay b_1.

$$\underbrace{-\epsilon f(b_1) b_1}_{\text{expected loss from 1 quitting}} \quad + \quad \underbrace{(1 - F(b_1))\epsilon}_{\text{expected gain from raising price}}$$

Q: Why not raise 1's price to $b_1 + \epsilon$, even after bidder 2 has quit?
A: 1's virtual value is positive.

'The book' requires that 1 pay b_1.

$$\underbrace{-\epsilon f(b_1) b_1}_{\text{expected loss from 1 quitting}} \quad + \quad \underbrace{(1 - F(b_1))\epsilon}_{\text{expected gain from raising price}}$$

Q: Why not raise 1's price to $b_1 + \epsilon$, even after bidder 2 has quit?
A: 1's virtual value is positive.

'The book' requires that 1 pay b_1.

$$\underbrace{-\epsilon f(b_1) b_1}_{\text{expected loss from 1 quitting}} \quad + \quad \underbrace{(1 - F(b_1))\epsilon}_{\text{expected gain from raising price}}$$

divide through by $\epsilon f(b_1)$

$$-\underbrace{[b_1 - \frac{1 - F(b_1)}{f(b_1)}]}_{\text{virtual value}} < 0$$

Q: Why not raise 1's price to $b_1 + \epsilon$, even after bidder 2 has quit?
A: 1's virtual value is positive.

'The book' requires that 1 pay b_1.

$$\underbrace{-\epsilon f(b_1) b_1}_{\text{expected loss from 1 quitting}} \quad + \quad \underbrace{(1 - F(b_1))\epsilon}_{\text{expected gain from raising price}}$$

divide through by $\epsilon f(b_1)$

$$-\underbrace{[b_1 - \frac{1 - F(b_1)}{f(b_1)}]}_{\text{virtual value}} < 0$$

Ceci n'est pas une proof.

How to deal with ties?

For technical convenience:

Definition

(G, S_N) *is* **orderly** *if, for some reserve* $\rho \leq \theta^K$*, and some strict order* $\rhd$ *on* N*, bidder* i *wins the object iff:*

1. $\theta_i \geq \rho$*, and*
2. *For all* $j \neq i$*,* θ_i *is more than* θ_j*, breaking ties with* $\rhd$*.*

How to deal with 'long-winded' auctioneers?

'You're the only bidder left, so you'll win if you bid the reserve. But first, tell me is your type a prime number? Is it a Mersenne prime?'

For technical convenience:

Definition

i **faces a posted price** *at history h if*

1. *i is called to play at h.*
2. *$\exists$ price τ_h such that $\forall$ successors of h, if i wins then i pays τ_h.*

(G, S_N) is **concise** *if, at any history h at which i faces a posted price, then h is the last time i is called to play, and the infoset containing h is singleton.*

Definition

(G, S_N) *is an* **ascending auction** *if:*

1. *The induced allocation rule is orderly.*
2. *The induced payment rule has threshold pricing.*

Definition

(G, S_N) *is an* **ascending auction** *if:*

1. *The induced allocation rule is orderly.*
2. *The induced payment rule has threshold pricing.*
3. *Each bidder has an initial bid equal to* 0.
4. *At each non-terminal history:*
   - 4.1 *Some bidder i is called to play, and offered some price $\geq$ his current bid.*
   - 4.2 *The available actions either accept the price or quit.*
   - 4.3 *If i quits, then i is not called to play again, does not win, and pays* 0.
   - 4.4 *If i accepts price p, then his bid is updated to p.*

Definition

(G, S_N) *is an* **ascending auction** *if:*

1. *The induced allocation rule is orderly.*
2. *The induced payment rule has threshold pricing.*
3. *Each bidder has an initial bid equal to* 0.
4. *At each non-terminal history:*
   - 4.1 *Some bidder i is called to play, and offered some price $\geq$ his current bid.*
   - 4.2 *The available actions either accept the price or quit.*
   - 4.3 *If i quits, then i is not called to play again, does not win, and pays* 0.
   - 4.4 *If i accepts price p, then his bid is updated to p.*
5. *At each infoset, either there is a unique accepting action, or any accepting action guarantees that i wins at current price.*
6. *At each terminal history, the winner pays his latest bid.*

credible, strategy-proof $\leftrightarrow$ ascending

Theorem 2

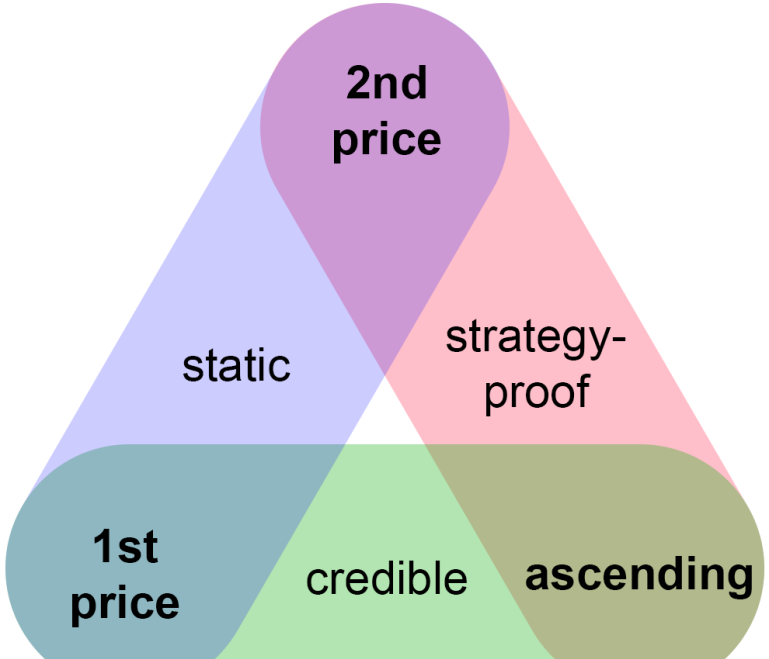
Assume (G, S_N) is optimal. If (G, S_N) is an ascending auction then it is credible and strategy-proof.
Assume additionally that (G, S_N) is orderly and concise. If (G, S_N) is credible and strategy-proof, then (G, S_N) is an ascending auction.

credible, strategy-proof $\leftrightarrow$ ascending

Theorem 2

Assume (G, S_N) is optimal. If (G, S_N) is an ascending auction then it is credible and strategy-proof.
Assume additionally that (G, S_N) is orderly and concise. If (G, S_N) is credible and strategy-proof, then (G, S_N) is an ascending auction.

Green-Laffont-Holmström, Theorem 1, and Theorem 2 $\rightarrow$



Proof: ascending $\rightarrow$ credible

$\pi(G, S_N)$

1. Ascending (G, S_N) is optimal.

Proof: ascending $\rightarrow$ credible

$$\pi(G, S_N) = \pi(S_0^G, S_N)$$

1. Ascending (G, S_N) is optimal.
2. Consider S_0^G that runs G.

Proof: ascending $\rightarrow$ credible

$$\pi(G, S_N) = \pi(S_0^G, S_N) < \pi(S_0', S_N)$$

1. Ascending (G, S_N) is optimal.
2. Consider S_0^G that runs G.
3. Suppose S_0' is a profitable safe deviation.

Proof: ascending $\rightarrow$ credible

$$\pi(G, S_N) = \pi(S_0^G, S_N) < \pi(S_0', S_N)$$

1. Ascending (G, S_N) is optimal.
2. Consider S_0^G that runs G.
3. Suppose S_0' is a profitable safe deviation.
4. For all i, S_i remains a best response to $(S_0', S_{N \setminus i})$.

Proof: ascending $\rightarrow$ credible

$$\pi(G, S_N) = \pi(S_0^G, S_N) < \pi(S_0', S_N) = \pi(G', S_N)$$

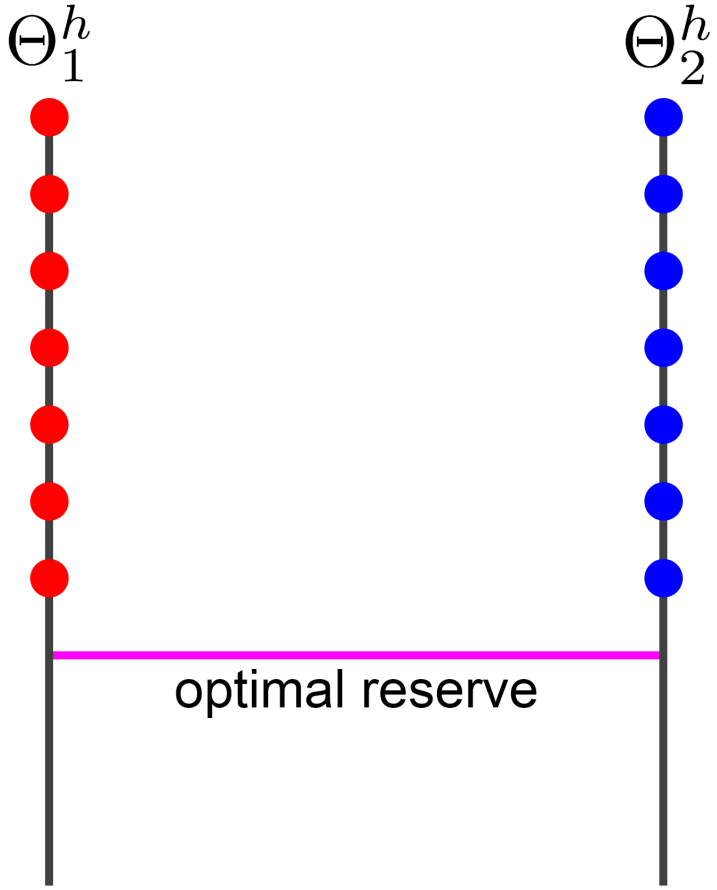
1. Ascending (G, S_N) is optimal.
2. Consider S_0^G that runs G.
3. Suppose S_0' is a profitable safe deviation.
4. For all i, S_i remains a best response to $(S_0', S_{N \setminus i})$.
5. (G', S_N) is also BIC, yields more revenue than (G, S_N). Contradiction, QED.

Proof sketch: credible, SP $\rightarrow$ ascending

A key feature of ascending auctions:

All the types who might still win **pool on the same action**.

Suppose (G, S_N) SP. not pooling $\rightarrow$ not credible.

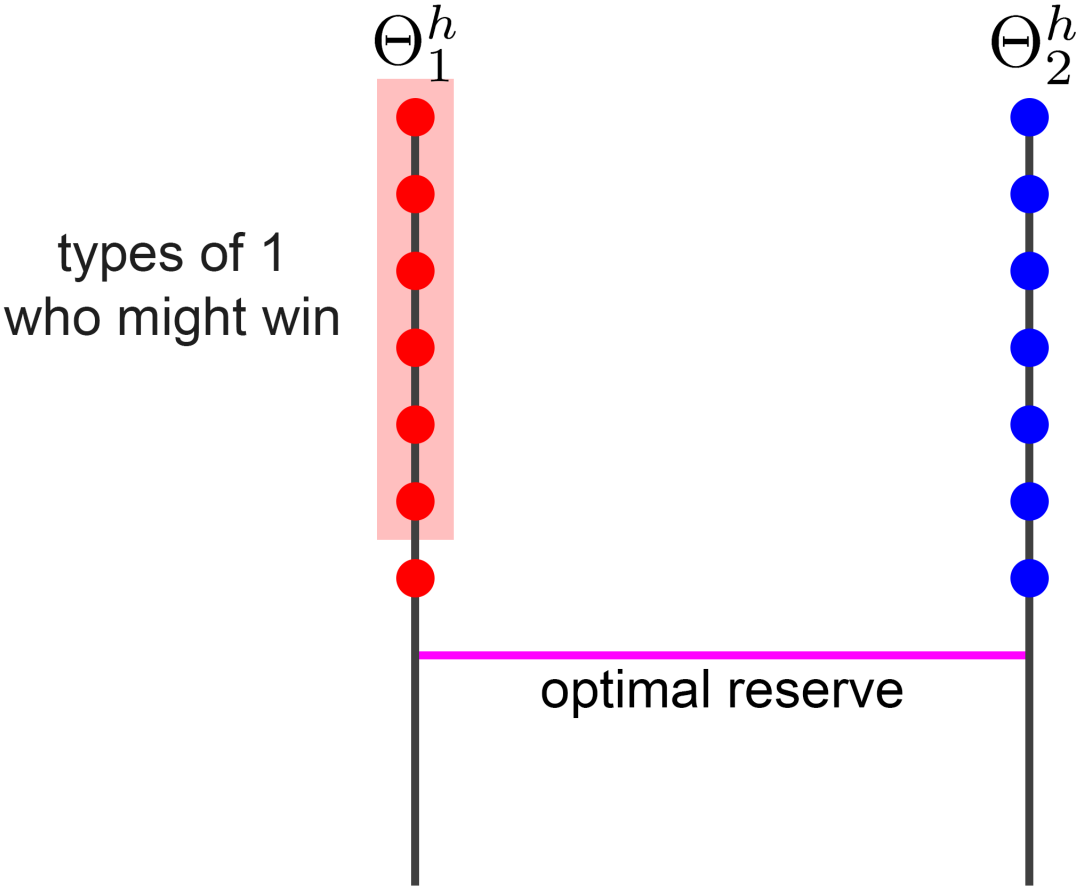


Proof sketch: credible, SP $\rightarrow$ ascending

A key feature of ascending auctions:

All the types who might still win **pool on the same action**.

Suppose (G, S_N) SP. not pooling $\rightarrow$ not credible.

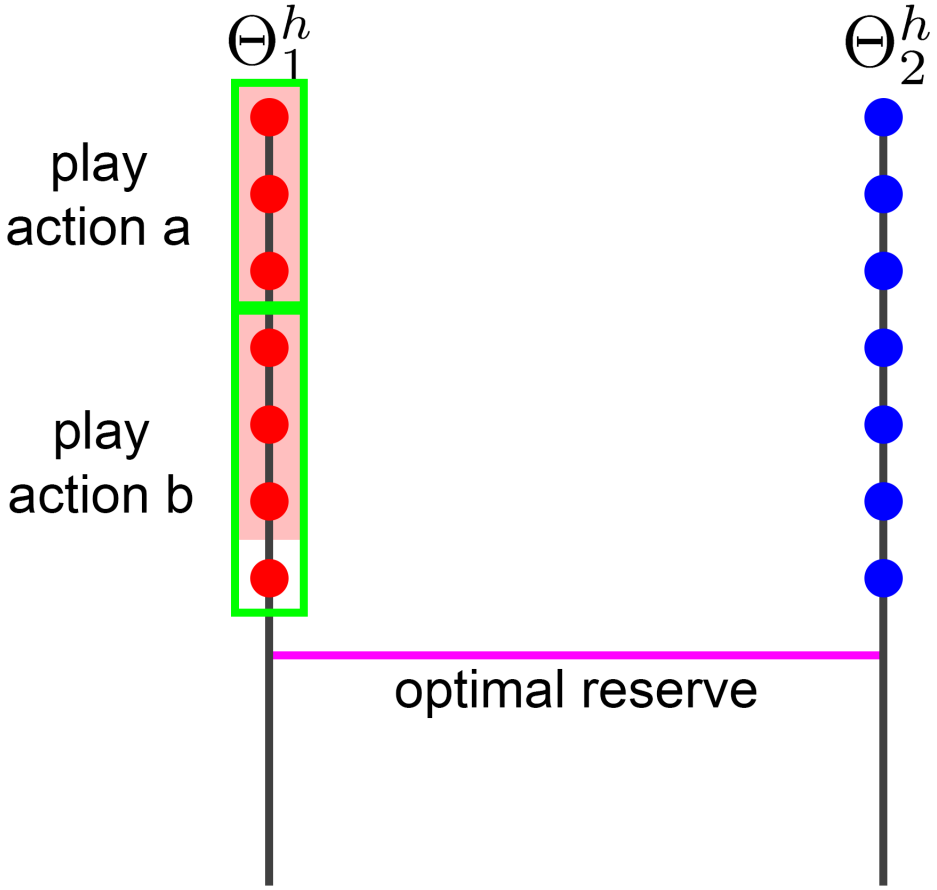


Proof sketch: credible, SP $\rightarrow$ ascending

A key feature of ascending auctions:

All the types who might still win **pool on the same action**.

Suppose (G, S_N) SP. not pooling $\rightarrow$ not credible.

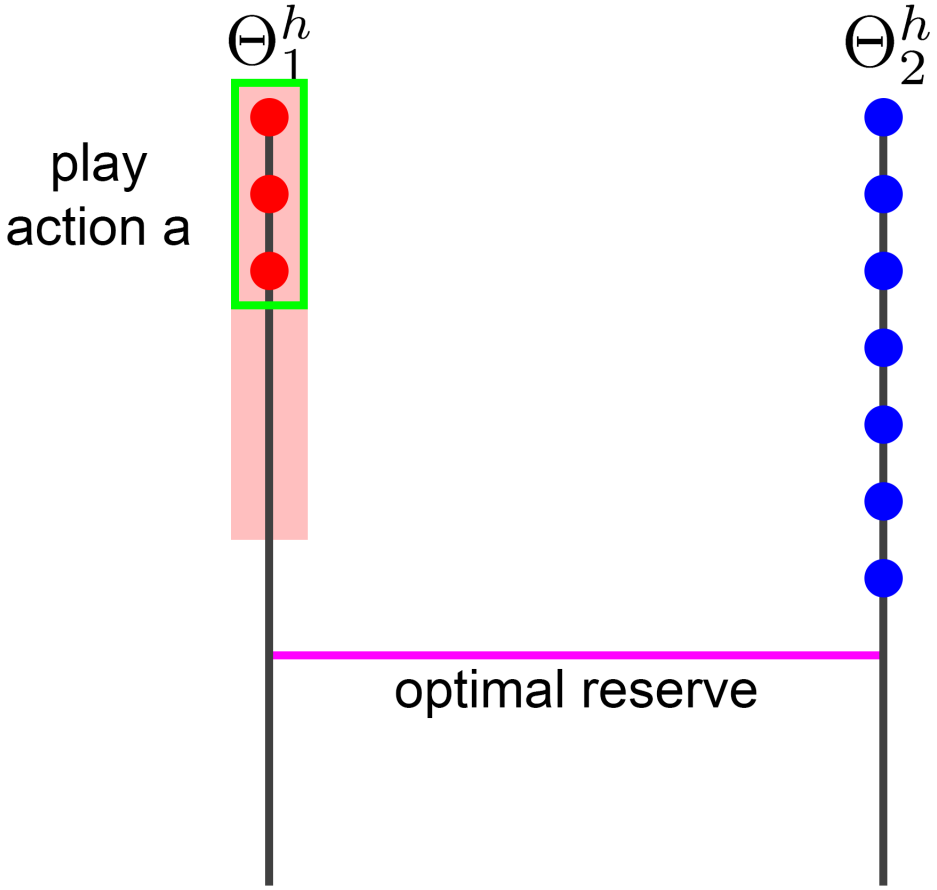


Proof sketch: credible, SP $\rightarrow$ ascending

A key feature of ascending auctions:

All the types who might still win **pool on the same action**.

Suppose (G, S_N) SP. not pooling $\rightarrow$ not credible.

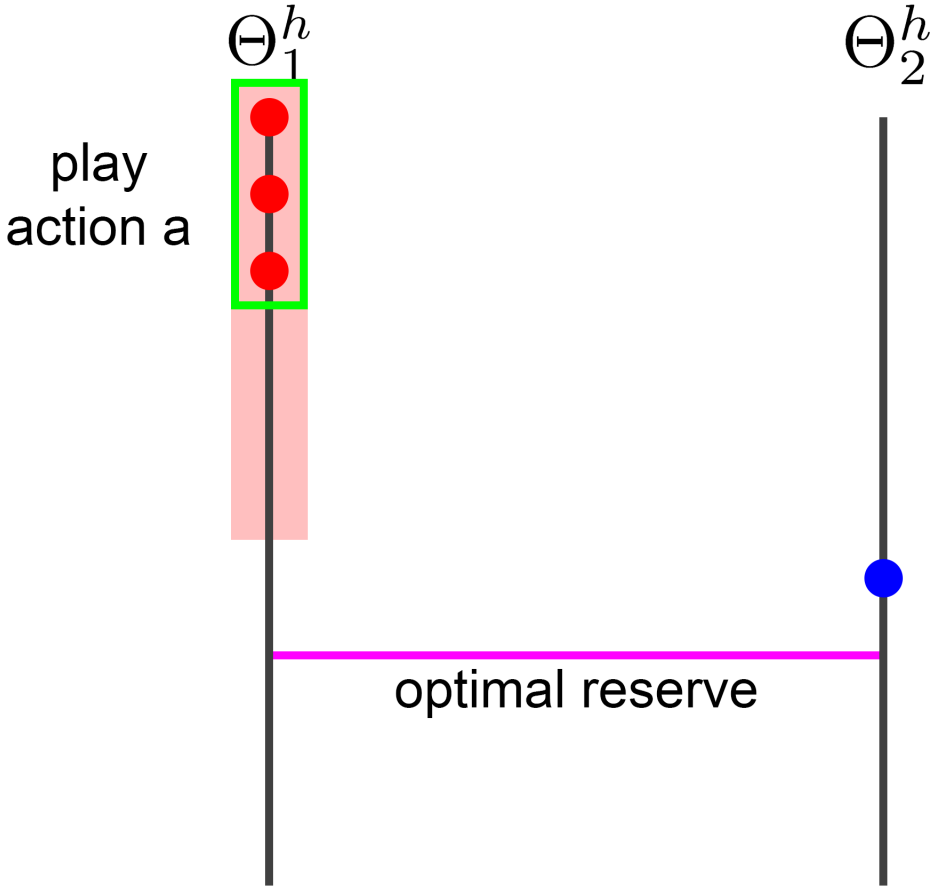


Proof sketch: credible, SP $\rightarrow$ ascending

A key feature of ascending auctions:

All the types who might still win **pool on the same action**.

Suppose (G, S_N) SP. not pooling $\rightarrow$ not credible.

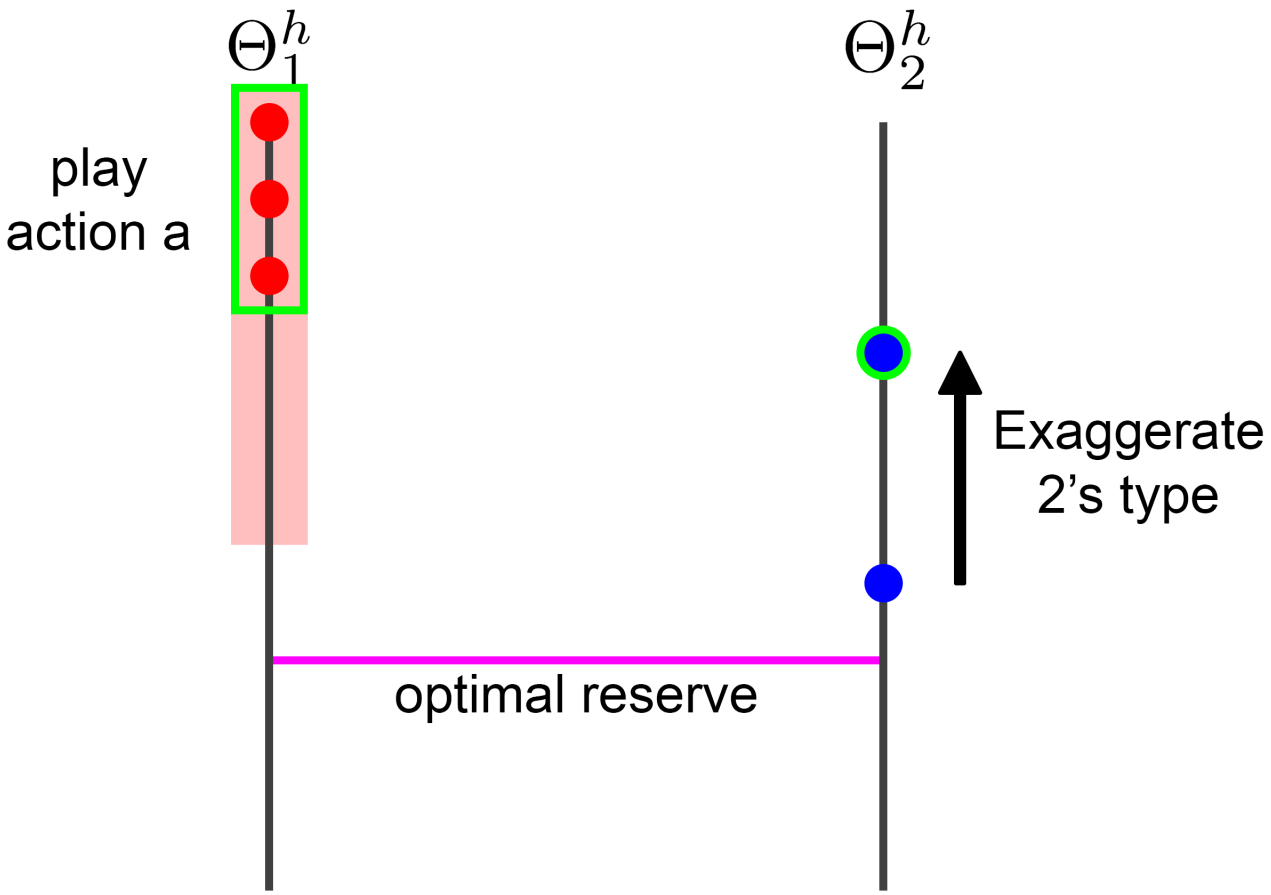


Proof sketch: credible, SP $\rightarrow$ ascending

A key feature of ascending auctions:

All the types who might still win **pool on the same action**.

Suppose (G, S_N) SP. not pooling $\rightarrow$ not credible.

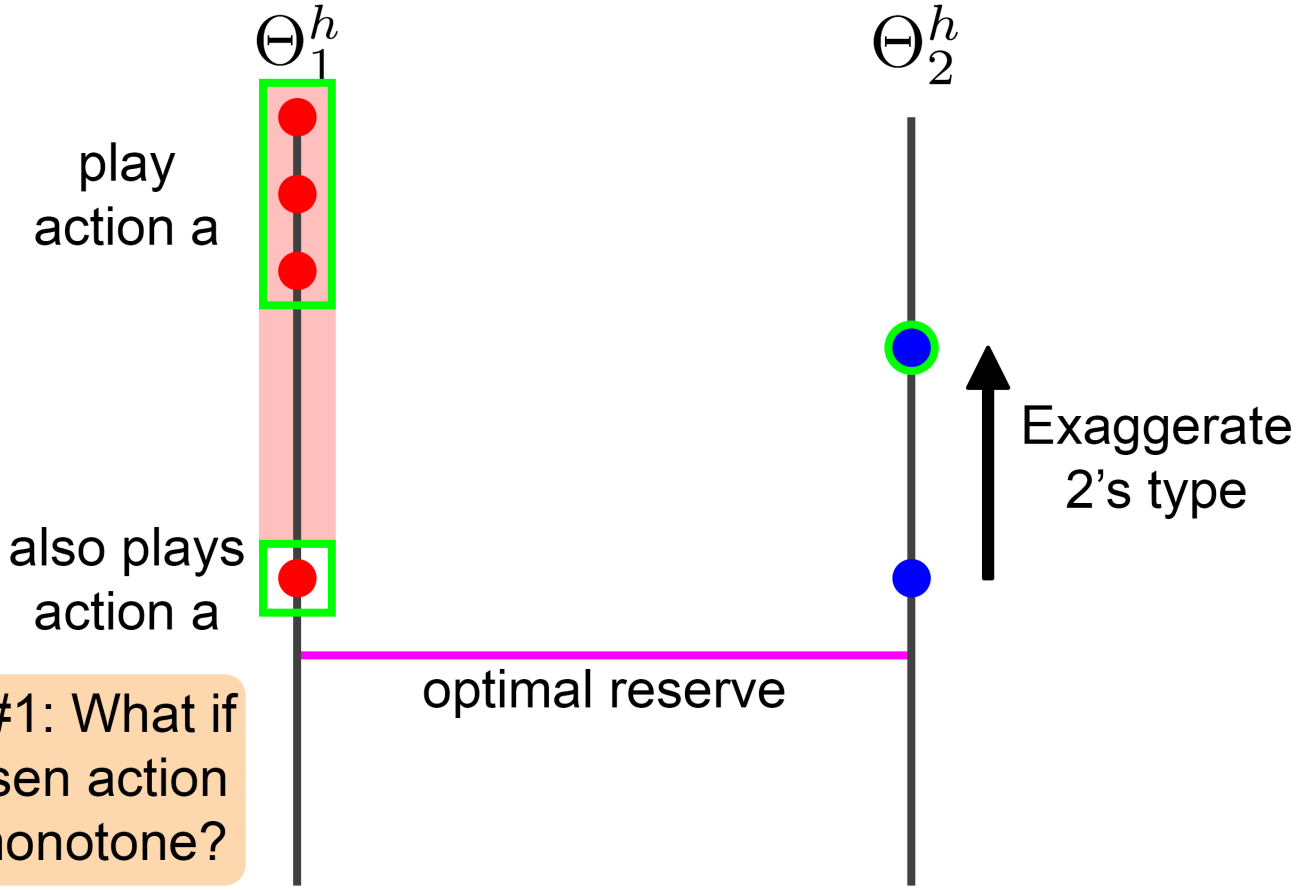


Proof sketch: credible, SP → ascending

A key feature of ascending auctions:

All the types who might still win **pool on the same action**.

Suppose (G, S_N) SP. not pooling → not credible.



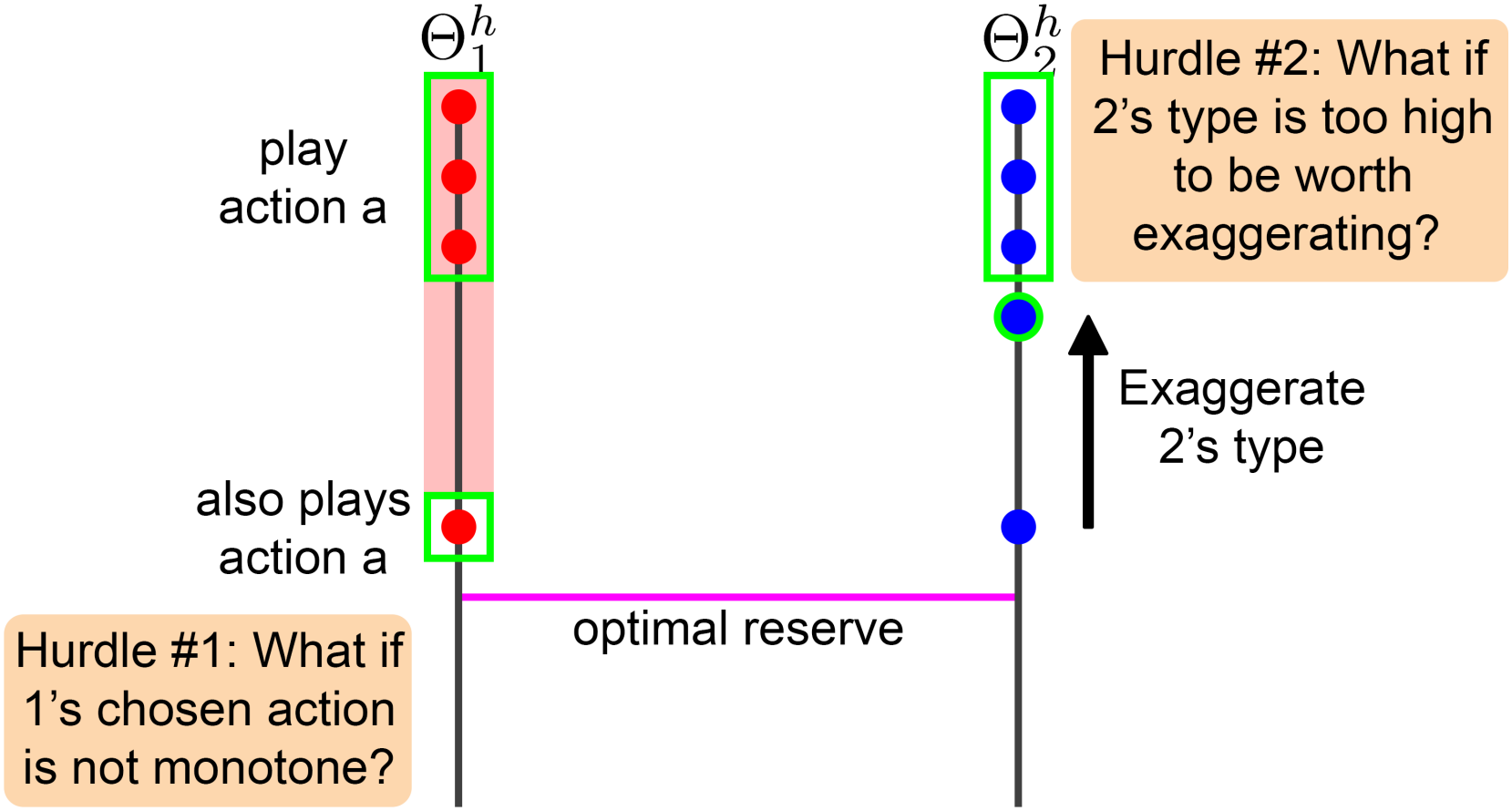
Hurdle #1: What if 1's chosen action is not monotone?

Proof sketch: credible, SP → ascending

A key feature of ascending auctions:

All the types who might still win **pool on the same action**.

Suppose (G, S_N) SP. not pooling → not credible.



A deviating algorithm

(High-level description, omits fine details.)

Given an arbitrary extensive form, take some history where bidder 1's winning types don't pool:

A deviating algorithm

(High-level description, omits fine details.)

Given an arbitrary extensive form, take some history where bidder 1's winning types don't pool:

1. Check if 1's type is high enough to exploit.
   - If not, sell to bidder 2.

A deviating algorithm

(High-level description, omits fine details.)

Given an arbitrary extensive form, take some history where bidder 1's winning types don't pool:

1. Check if 1's type is high enough to exploit.
   - If not, sell to bidder 2.
2. Check if 2's type is low enough to be worth exaggerating.
   - If not, sell the object 'by the book'.

A deviating algorithm

(High-level description, omits fine details.)

Given an arbitrary extensive form, take some history where bidder 1's winning types don't pool:

1. Check if 1's type is high enough to exploit.
   - If not, sell to bidder 2.
2. Check if 2's type is low enough to be worth exaggerating.
   - If not, sell the object 'by the book'.
3. Exaggerate 2's type, sell to bidder 1.

A deviating algorithm

(High-level description, omits fine details.)

Given an arbitrary extensive form, take some history where bidder 1's winning types don't pool:

1. Check if 1's type is high enough to exploit.
   - If not, sell to bidder 2.
2. Check if 2's type is low enough to be worth exaggerating.
   - If not, sell the object 'by the book'.
3. Exaggerate 2's type, sell to bidder 1.
4. Don't get caught.

A deviating algorithm

(High-level description, omits fine details.)

Given an arbitrary extensive form, take some history where bidder 1's winning types don't pool:

1. Check if 1's type is high enough to exploit.
   - If not, sell to bidder 2.
2. Check if 2's type is low enough to be worth exaggerating.
   - If not, sell the object 'by the book'.
3. Exaggerate 2's type, sell to bidder 1.
4. Don't get caught.

strategy-proof, not pooling $\rightarrow$ profitable safe deviation

A deviating algorithm

(High-level description, omits fine details.)

Given an arbitrary extensive form, take some history where bidder 1's winning types don't pool:

1. Check if 1's type is high enough to exploit.
   - If not, sell to bidder 2.
2. Check if 2's type is low enough to be worth exaggerating.
   - If not, sell the object 'by the book'.
3. Exaggerate 2's type, sell to bidder 1.
4. Don't get caught.

strategy-proof, not pooling → profitable safe deviation

credible, strategy-proof → pooling → ascending auction

bonus

What if we relax optimality and winner-paying?

Definition

(G, S_N) is **contestable** if, almost everywhere, if i wins at (θ_i, θ_{-i}), then there exists θ'_{-i} such that i loses at (θ_i, θ'_{-i}).

Definition

(G, S_N) is a **twin-bid auction** if each bidder i chooses bids in $B_i \subset \mathbb{R} \times \mathbb{R}$, such that:

1. i pays b_i^1 if he wins and b_i^2 if he loses.
2. If $\max_i b_i^1 - b_i^2 > 0$, then some bidder wins.
3. If bidder i wins, then $b_i^1 - b_i^2 \geq 0$ and $\forall j : b_i^1 - b_i^2 \geq b_j^1 - b_j^2$.

What if we relax optimality and winner-paying?

Definition

(G, S_N) is **contestable** if, almost everywhere, if i wins at (θ_i, θ_{-i}), then there exists θ'_{-i} such that i loses at (θ_i, θ'_{-i}).

Definition

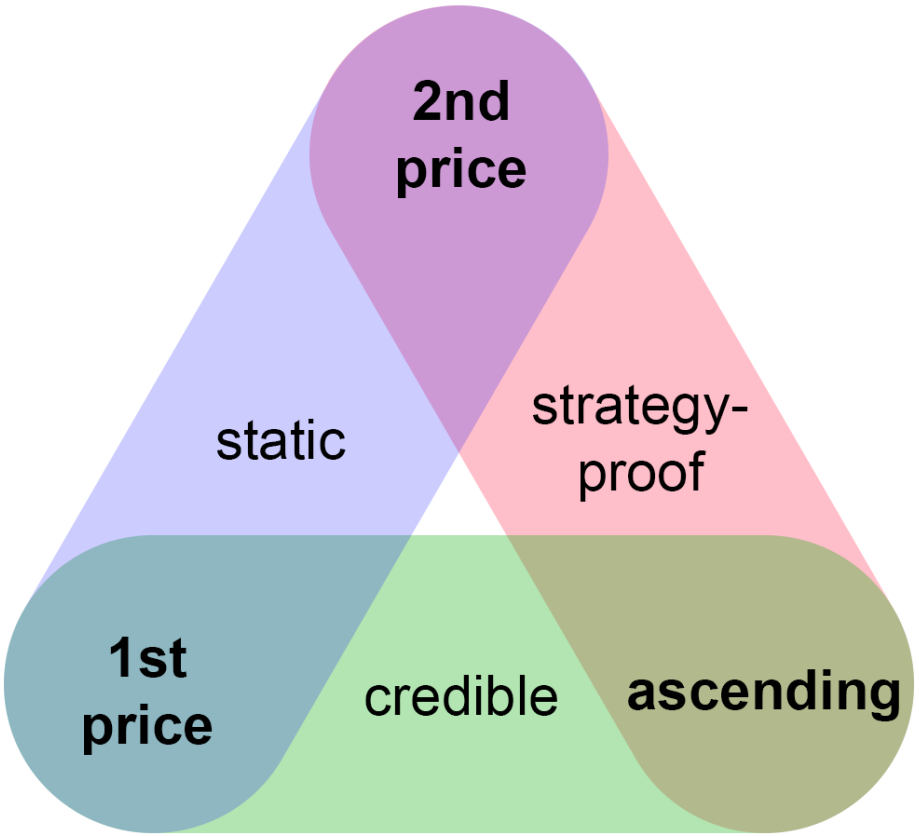
(G, S_N) is a **twin-bid auction** if each bidder i chooses bids in $B_i \subset \mathbb{R} \times \mathbb{R}$, such that:

1. i pays b_i^1 if he wins and b_i^2 if he loses.
2. If $\max_i b_i^1 - b_i^2 > 0$, then some bidder wins.
3. If bidder i wins, then $b_i^1 - b_i^2 \geq 0$ and $\forall j : b_i^1 - b_i^2 \geq b_j^1 - b_j^2$.

Theorem

Assume (G, S_N) is contestable. (G, S_N) is credible and static if and only if it is a twin-bid auction almost everywhere.

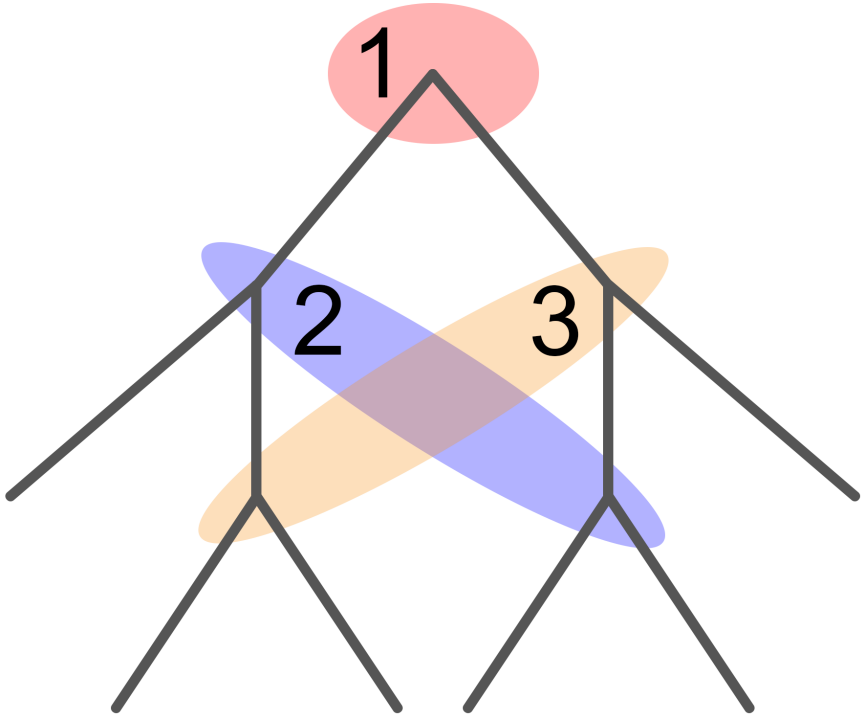
An Auction Trilemma



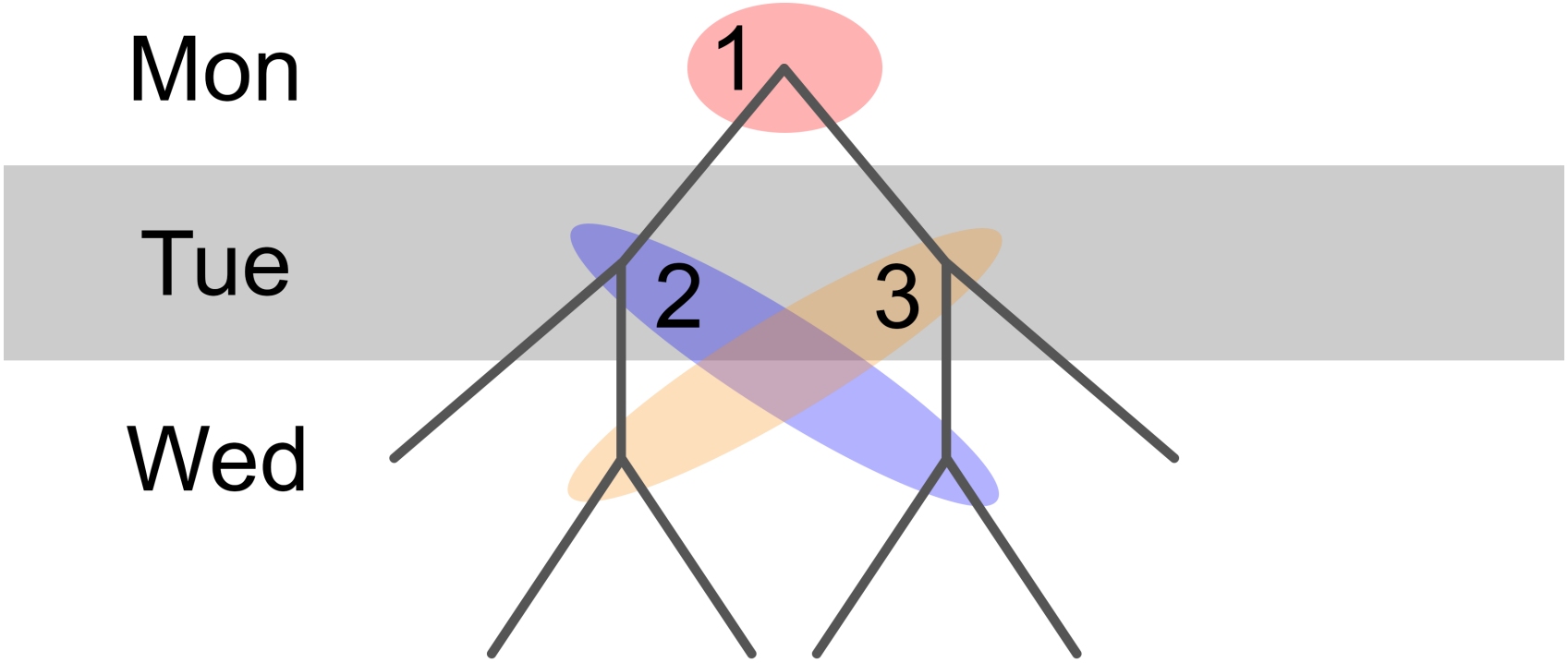
Modeling auctioneer IC yields a simple explanation of real-world auction formats.

Of course, these are not the only desiderata.

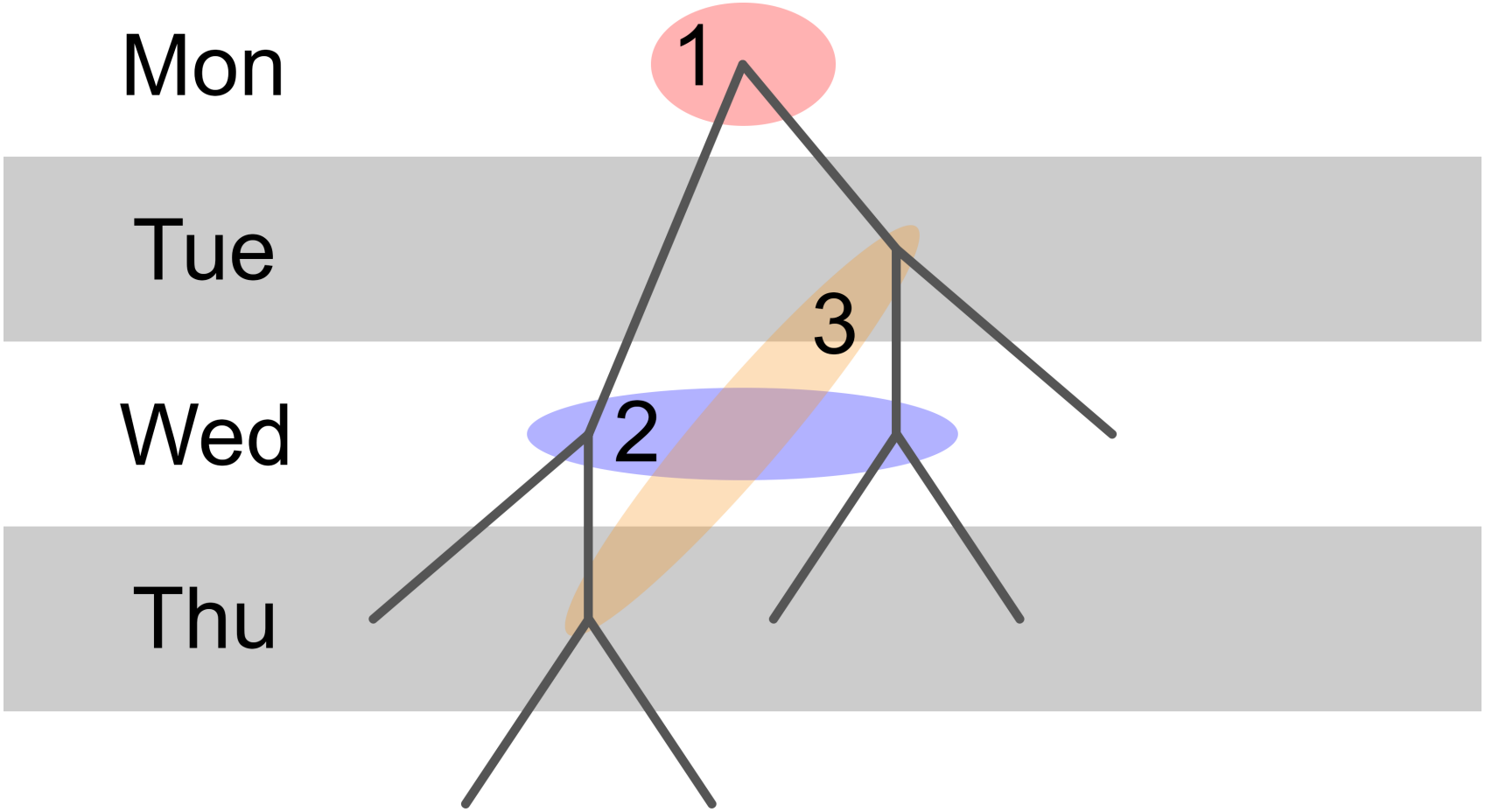
Calendar time isn't 'built into' extensive forms



Calendar time isn't 'built into' extensive forms



Calendar time isn't 'built into' extensive forms



What about asymmetric distributions?

First-price auction (static, credible)

'Robustly' credible. May not be optimal.
Sometimes impossible to restore optimality.

return

What about asymmetric distributions?

First-price auction (static, credible)

'Robustly' credible. May not be optimal.
Sometimes impossible to restore optimality.

Proposition

There exist asymmetric distributions such that no credible static (G, S_N) *is* ϵ*-optimal.*

return

What about asymmetric distributions?

First-price auction (static, credible)

'Robustly' credible. May not be optimal.
Sometimes impossible to restore optimality.

Proposition

There exist asymmetric distributions such that no credible static (G, S_N) *is* ϵ*-optimal.*

Ascending auction (strategy-proof, credible)

May not be credible or optimal.
Easy to restore both.
The **virtual values** ascending auction.

return

Bidders seldom display types on placards.

> *In the English system bids are . . . usually transmitted by signal. Such signals may be in the form of a wink, a nod, scratching an ear, lifting a pencil, tugging at the coat of the auctioneer or even staring into the auctioneer's eyes – all of them perfectly legal.*

Cassady 1967

Public communication affects aftermarkets and thus incentives.
Ausubel & Cramton 2004, Carroll & Segal 2016, Dworczak 2017.
(Outside the model today.)

A Menagerie

Table: ϵ-optimal auctions

	1P	2P	Asc
Strategy-proof		X	X
Static	X	X	
Credible	X		X
Ex Post IR	X	X	X
Non-winner 0 transfer	X	X	X

A Menagerie

Table: ϵ-optimal auctions

	1P	2P	Asc	Dutch
Strategy-proof		X	X	
Static	X	X		
Credible	X		X	X
Ex Post IR	X	X	X	X
Non-winner 0 transfer	X	X	X	X

A Menagerie

Table: ϵ-optimal auctions

	1P	2P	Asc	Dutch	All-Pay
Strategy-proof		X	X		
Static	X	X			X
Credible	X		X	X	X
Ex Post IR	X	X	X	X	
Non-winner 0 transfer	X	X	X	X	

A Menagerie

Table: ϵ-optimal auctions

	1P	2P	Asc	Dutch	All-Pay	Consol
Strategy-proof		X	X			
Static	X	X			X	X
Credible	X		X	X	X	X
Ex Post IR	X	X	X	X		X
Non-winner 0 transfer	X	X	X	X		

optimal $\cap$ first-price $= \emptyset$

$N = \{1, 2\}$
$\Theta_i = \{4, 5, 6\}$
Tie-breaking order: $1 \lhd 2$

Optimal reserve $= 4$.

Optimality requires:
$b_1(5) = 5$
$b_2(5) = 4.5$

When type profile is $(5, 5)$, tie-breaking rule requires to sell to bidder 2, even though he bid less. Not first-price auction!

return