

# Thickness and Information in Dynamic Matching Markets<sup>‡</sup>

Mohammad Akbarpour<sup>‡</sup>    Shengwu Li<sup>§</sup>    Shayan Oveis Gharan<sup>¶</sup>

First Draft: February 2014    This Draft: December 2018

## Abstract

We introduce a simple model of dynamic matching in networked markets, where agents arrive and depart stochastically, and the composition of the trade network depends endogenously on the matching algorithm. Varying the timing properties of matching algorithms can substantially affect their performance, and this depends crucially on the information structure. More precisely, if the planner can identify agents who are about to depart, then waiting to thicken the market substantially reduces the fraction of unmatched agents. If the planner cannot identify such agents, then matching agents greedily is close-to-optimal. We specify conditions under which local algorithms that choose the right time to match agents, but do not exploit the global network structure, are close-to-optimal. Finally, we consider a setting where agents have private information about their departure times, and design a continuous-time dynamic mechanism to elicit this information.

Keywords: Market Design, Matching, Networks, Continuous-time Markov Chains, Mechanism Design

JEL Classification Numbers: D47, C78, C60

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\*This paper has been previously distributed with the title “Dynamic Matching Market Design.”

<sup>†</sup>We thank Paul Milgrom, Alvin Roth, and several anonymous referees for valuable guidance. We also thank Eric Budish, Ben Golub, Scott Kominers, Alex Wolitzky, and multiple anonymous referees for their insights and advice. We are also grateful to Daron Acemoglu, Nikhil Agarwal, Nick Arnosti, Itai Ashlagi, Peter Biro, Timothy Bresnahan, Jeremy Bulow, Gabriel Carroll, Sylvain Chassang, Constantinos Daskalakis, Darrell Duffie, Alex Frankel, Matthew Jackson, Fuhito Kojima, Soohyung Lee, Jacob Leshno, Malwina Luczak, Joshua Mollner, Markus Mobius, Roger Myerson, Stephen Nei, Muriel Niederle, Afshin Nikzad, Michael Ostrovsky, Parag Pathak, Robert Shimer, Takuo Sugaya, Bob Wilson, and several anonymous referees for their valuable comments. All errors remain our own.

<sup>‡</sup>Graduate School of Business, Stanford University. Email: [mohamwad@stanford.edu](mailto:mohamwad@stanford.edu)

<sup>§</sup>Department of Economics, Harvard University. Email: [shengwu\\_li@fas.harvard.edu](mailto:shengwu_li@fas.harvard.edu)

<sup>¶</sup>Department of Computer Science and Engineering, University of Washington. Supported by a Miller fellowship. Email: [shayan@cs.washington.edu](mailto:shayan@cs.washington.edu)

# 1 Introduction

We study the problem of matching in a dynamic market with network constraints. In many markets, only some pairs of agents can be feasibly matched. For instance, in carpooling platforms, two riders can share a ride if they have nearby locations and destinations. In paired kidney exchange, patient-donor pairs must be biologically compatible before a swap can be made. Because of these frictions, any matching decision is constrained by a network, comprised of agents (nodes) and compatible pairs (links).

In many such markets, the set of compatible agents is not fixed. Instead, agents arrive and depart over time, a social planner continually observes the network of compatible agents, and chooses how to match them. Matched agents leave the market, and unmatched agents either persist or depart. Consequently, the planner’s decision today affects the sets of agents and options tomorrow. For instance, in carpooling platforms, the compatibility network evolves when new ride requests arrive, when two riders are matched, or when unmatched riders leave the market.

In such environments, the planner must decide not only which agents to match, but also *when* to match them. The planner could match agents frequently, or wait to thicken the market. If the planner waits, agents may perish. However, waiting has benefits. For example, in the left scenario in [Figure 1](#), where each node represents an agent and each link represents a compatible pair, if the planner matches agent 1 to agent 2 at time  $t$ , then the planner will be unable to match agents 3 and 4 at time  $t + 1$ . By contrast, if the planner waits until  $t + 1$ , he can match all four agents by matching 1 to 4 and 2 to 3. Moreover, waiting might bring information about which agents will soon depart, enabling the planner to give priority to those agents. For example, in the right scenario in [Figure 1](#), the planner learns at  $t + 1$  that agent 3 will imminently leave the market if not matched. If the planner matches agent 1 to agent 2 at time  $t$ , then he will be unable to react to this information at  $t + 1$ .

The optimal timing policy in a dynamic matching problem is not obvious *a priori*. Ride-sharing platforms, for instance, have been extensively experimenting with their timing policies to analyze the trade-off between matching frequency and market thickness<sup>1</sup>. Many paired kidney exchanges enact static matching algorithms (‘match-runs’) at fixed intervals. Even

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<sup>1</sup>For instance, Uber’s website writes: “In the early days, a rider was immediately matched with the closest available driver... But if we wait just a few seconds after a request, it can make a big difference. It’s enough time for a batch of potential rider-driver matches to accumulate.” (<https://marketplace.uber.com/matching>, accessed 2018/12/02) Also, see [Korolko et al. \(2018\)](#) for an introduction to *dynamic waiting* algorithm, a recent innovation in the timing policy in Uber.

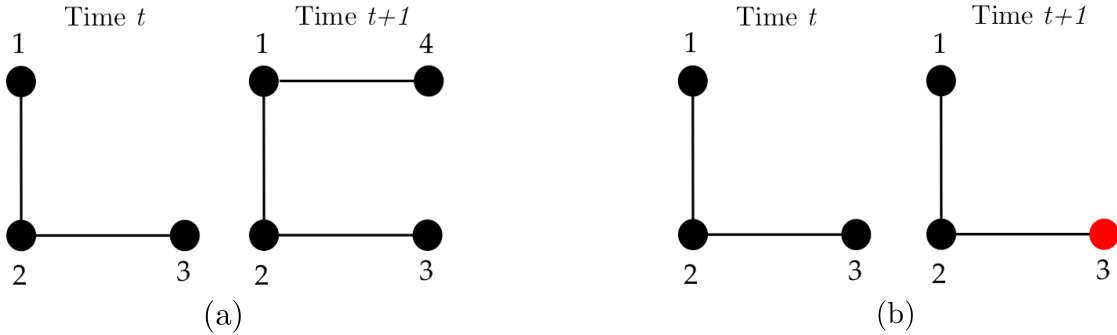


Figure 1: Waiting expands information about the set of options and departure times. Here, each node represents an agent and each link represents a compatible pair. In (a), the planner observes the set of new agents and options at time  $t + 1$ . If he matches agent 1 to agent 2 at time  $t$ , then the planner will be unable to match agents 3 and 4 at time  $t + 1$ . In (b), the planner gets the information that agent 3 is about to depart at time  $t + 1$ . If he matches 1 and 2 at time  $t$ , then he will be unable to react to the information about urgency of agent 3 at time  $t + 1$ .

then, matching intervals differ substantially between exchanges.<sup>2</sup> Under what conditions is it valuable to wait to thicken the market?

We now introduce our model. In the classic Erdős-Rényi random graph setting, there are  $m$  agents, and any two agents are compatible with probability  $\frac{d}{m}$ , where  $d$  scales the expected degree of each agent (Erdős and Rényi, 1960). A planner observes the network and chooses a matching, seeking to minimize the number of unmatched agents. We create a natural dynamic analogue: Agents arrive at Poisson rate  $m$ , any two agents are compatible with probability  $\frac{d}{m}$ , and each agent departs (*perishes*) at a Poisson rate, normalized to 1. Links persist over time. The planner observes the current network and chooses a matching; matched agents leave the market. The planner also observes which agents are *critical*, in the sense that he knows which agents will perish imminently if not matched. As in the static case, the planner seeks to minimize the proportion of unmatched agents (the *loss*).

In essence, our model is about matching agents to carry out some valuable activity in pairs. Agents gradually arrive and depart, and only certain pairs of agents can cooperate. The model can be interpreted as the problem faced by a ride-sharing firm that offers a car-pooling service. Holding the pricing scheme fixed, customers arrive and depart over time

<sup>2</sup>The Alliance for Paired Kidney Donation conducts a match-run every weekday (APKD, 2017), the United Network for Organ Sharing conducts a match-run twice a week (UNOS, 2015), the South Korean kidney exchange conducts a match-run once a month, and the Dutch kidney exchange conducts a match-run once a quarter (Akkina et al., 2011).

and two riders are compatible if they have nearby locations and destinations. It can also be interpreted as a stylized representation of barter exchange; a pair is compatible if they can gainfully trade with each other. Kidney exchange is one example of such a trade, although the abstractions in the model are particularly aggressive for this application.<sup>3</sup>

What are the key features of the optimal dynamic matching algorithm? Since we explicitly model the network of potential matches, the resulting Markov Decision Problem is combinatorially complex. Thus, it is not feasible to compute the optimal solution with standard dynamic programming techniques. To address this issue, we employ a different approach: We formulate simple algorithms with different timing properties, which are analytically tractable because they naïvely ignore the network structure. By comparing these algorithms, we show that the choice of *when* to match agents has large effects on performance. Then, we produce theoretical bounds on the performance of optimal algorithms that additionally exploit the network structure. We show that, when  $d$  is not too small, our simple algorithms come close to these bounds on optimum performance. This suggests that timing is a first-order concern in our environment.

The simple algorithms are as follows: The *Greedy* algorithm attempts to match agents as soon as possible; it treats each instant as a static matching problem without regard for the future. The *Patient* algorithm attempts to match only critical agents (potentially to a non-critical partner). Both these algorithms are *local*, in the sense that they look only at the immediate neighbors of the agent they attempt to match, rather than at the global network structure.

It is intuitive that the Patient algorithm will achieve a lower loss than the Greedy algorithm, but is the difference substantial? Our first result answers this question: The Greedy algorithm’s loss is at least  $\frac{1}{2d+1}$ , whereas the Patient algorithm’s loss is at most  $\frac{e^{-d/2}}{2}$ . To place these results in context, the static model provides a useful benchmark. Given a maximum matching on an Erdős-Rényi random graph, the expected fraction of unmatched agents is exponentially small in  $d$ , so the loss falls rapidly as  $d$  rises (Zhou and Ou-Yang, 2003). In the case with arrivals and departures, our result shows that running a statically-optimal matching at every instant does not yield exponentially small loss. However, waiting to match agents suffices to achieve exponentially small loss, and thus the Patient algorithm substantially outperforms the Greedy. For instance, in a market where  $d = 8$ , the loss of the Patient algorithm is no more than 16% of the loss of the Greedy algorithm.

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<sup>3</sup>In particular, while some agents in the model have more links than others, the Erdős-Rényi structure rules out the possibility of persistently under-demanded and over-demanded *types*, an important feature of kidney exchange (Roth et al., 2005).

The intuition behind this result is as follows: The composition of the network as well as the number of agents in the market depends endogenously on the matching algorithm. As  $d$  rises, the Greedy algorithm matches agents more rapidly, reducing the equilibrium stock of available agents. This effect cancels out the exponential improvements that would accrue from raising  $d$  in a static model. In addition, under the Greedy algorithm, there are no compatible agents among the set of agents in the market (the market is *thin*) and so all critical agents perish. On the contrary, under the Patient algorithm, an increase in  $d$  will not rapidly reduce the equilibrium stock of available agents, so the market is always thick. This market thickness enables the planner to react to critical cases.

Our second result states that the loss of the Patient algorithm is “close to” the loss of the optimum algorithm; the optimum algorithm’s loss is at least  $\frac{e^{-\frac{d}{2}(1+\epsilon)}}{d+1}$  where  $\epsilon \leq e^{-d/2}$ . Recall that the Patient algorithm is local; it looks only at the immediate neighborhood of the agents it seeks to match. By contrast, the optimum algorithm chooses the optimal time to match agents, as well as the optimal agents to match, by exploiting the entire network structure. When we compare the performance of the Greedy algorithm to the optimum algorithm, we find that most of the gain is achieved merely by thickening the market, rather than optimizing over the network structure.

Essentially, our first two results delineate conditions under which the timing of matching matters much more than optimizing over the network. They suggest that it may be unwise to abstract away from timing considerations in order to focus on the network structure.

Next, we drop the assumption that the planner can identify critical agents and ask: What if the planner has more or less information about departure times? Our next results show that departure information and market thickness are complements, in the following sense. Any algorithm that cannot identify critical agents has a loss of at least  $\frac{1}{2d+1}$ , no matter how long it waits. Suppose on the other hand that the planner is constrained to match agents as soon as possible, but knows agents’ departure times far in advance. Any algorithm that does not wait has a loss of at least  $\frac{1}{2d+1}$ , no matter how much information it has about departure times.

Recall that the Patient algorithm requires only short-horizon information about agent departures. What if the planner has even more information? For instance, the planner may be able to forecast departures long in advance, or foresee how many new agents will arrive, or know that certain agents are more likely than others to have new links. We prove that *no* expansion of the planner’s information allows him to achieve a loss smaller than  $\frac{e^{-d}}{d+1}$ . Taken together, these results suggest that short-horizon information about departure times

is especially valuable to the planner. Lacking this information leads to large losses, and having more than this information does not yield large gains.

In some settings, however, agents know that they are critical, but the planner does not. For instance, riders know whether they are in urgent need for a car or they can wait with low cost, but the carpooling platform does not. Our final result concerns the incentive-compatible implementation of the Patient algorithm. Suppose that the planner observes the network, but does not know when agents are critical. Suppose that agents know when they are critical, but do not observe the network (*i.e.* they do not know when they have a compatible partner).<sup>4</sup> When agents have waiting costs, they may have incentives to misreport their urgency so as to hasten their match or to increase their probability of getting matched. We show that if agents are not too impatient, a dynamic mechanism without transfers can elicit such information. The mechanism treats agents who report that they are critical, but persist, as though they had left the market. This means that as an agent, I trade off the possibility of a swifter match (by declaring that I am critical now) with the option value of being matched to another agent before I truly become critical. We prove that it is arbitrarily close to optimal for agents to report the truth in large markets.

We close by stating a caveat: Our results build on a model with several stylized assumptions. Whether these are plausible depends on the context. For instance, consider the assumption of exponential departure times. In kidney exchanges, it seems implausible to assume that a patient’s departure time is memoryless, but this assumption is consistent with empirical evidence from ride-sharing (Liu et al., 2018). Hence, our results do not imply that network optimization is never important. Rather, they specify assumptions under which timing considerations are more important than network optimization. If an analyst believes (or finds out) that employing a complicated algorithm that accounts for the network structure will yield large gains, then their environment must depart materially from the tractable benchmark studied here.

## 1.1 Related Work

There have been several studies on dynamic matching in economics, computer science, and operations research. To the best of our knowledge, no prior work has examined dynamic matching on a general graph, where agents stochastically depart.

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<sup>4</sup>One of the reasons that agents enter centralized matching markets is that they are unable to find partners by themselves.

Kurino (2009) and Bloch and Houy (2012) study an overlapping generations model of the housing market. In their models, agents have deterministic arrivals and departures and the housing side of the market is infinitely durable and static. In the same context, Leshno (2012) studies a one-sided dynamic housing allocation problem in which there are two types of houses that arrive stochastically over time. In subsequent papers, Baccara et al. (2015) and Loertscher et al. (2016) study the problem of optimal dynamic matching and thickness in two-sided models with two types on each side.

In the context of kidney exchanges, a problem first studied by Roth et al. (2004) and Roth et al. (2005), Ünver (2010) is the first paper that considers dynamics in a model with multiple types of agents. In his model, agents never perish. Thus, one insight of his model is that waiting to thicken the market is not helpful when only bilateral exchanges are allowed. We show that this result changes when agents depart stochastically. Some other aspects of dynamic kidney exchange have been studied in Zenios (2002); Su and Zenios (2005); Awasthi and Sandholm (2009); Dickerson et al. (2012); Sönmez and Ünver (2015). Ashlagi et al. (2013) construct a finite-horizon model of kidney exchange with agents who never depart. They show that (with 2-way exchanges) waiting yields large gains only if the planner waits for a constant fraction of total agents to arrive. Since our model is infinite-horizon and agents depart, it is not possible to wait for a constant fraction of the total agents to arrive. Nevertheless, the Patient algorithm ensures that the size of the market is linear in  $m$ , which makes thickness valuable. Finally, a recent paper builds on our framework to study the competition of two platforms with Greedy and Patient algorithms (Das et al., 2015).

In concurrent work, Anderson et al. (2014) analyze a model in which the main objective is to minimize the average waiting time, and agents never perish. They show that with two-way exchanges, the Greedy algorithm is optimal in the class of ‘periodic Markov policies’, which is similar to Theorem 4 in this paper. Our paper shows that when agents’ departure times are observable, then Greedy performs weakly, and the option value of waiting can be large. In other related work, Arnosti et al. (2014) model a two-sided dynamic matching market to analyze congestion in decentralized markets. Some recent papers study the problem of stability in dynamic matching markets (Du and Livne, 2014; Kadam et al., 2014; Doval, 2014). The objective of matching in such environments is stability, whereas we are concerned with total welfare.

In recent years, research on ride-sharing platforms has grown rapidly. While the problem of pricing is widely studied, the literature on the problem of dynamic matching in ride-sharing is relatively sparse. Subsequent to our paper, Ozkan and Ward (2017) and Ashlagi et al.

(2018) study this problem from a theoretical perspective. More recently, Liu et al. (2018) develop a two-sided generalization of our model and estimate it using data from DiDi, the world’s largest ride-sharing platform. Their comparison of the two algorithms developed here (Patient and Greedy) shows that if drivers are not too heterogeneous, Patient outperforms Greedy.

The literature on online advertising is also related to our work. In this setting, advertisements are static, but queries arrive adversarially or stochastically over time. Unlike our model, queries persist in the market for exactly one period. Karp et al. (1990) introduced the problem and designed a randomized matching algorithm. Subsequently, the problem has been considered under several arrival models with pre-specified budgets for the advertisers, (Mehta et al., 2007; Goel and Mehta, 2008; Feldman et al., 2009; Manshadi et al., 2012; Blum et al., 2015).

The problem of dynamic matching has been extensively studied in the literature on labor market search. Shimer and Smith (2001) study a decentralized search market and discuss efficiency issues. This paper and its descendants are different from ours in at least two ways: First, rather than modeling market thickness via a fixed match-function, we explicitly account for the network structure that affects the planner’s options, endogenously determining market thickness. In addition, in Shimer and Smith (2001), the benefit of waiting is in increasing the *match quality*, whereas in our model we show that even if you cannot increase match quality, waiting can still be beneficial because it increases the *number* of agents who get matched. Ebrahimy and Shimer (2010) study a decentralized version of the Greedy algorithm from a labor-search perspective<sup>5</sup>.

## 2 Model

In this section, we introduce the pieces of our continuous-time model for a matching market on stochastic networks that runs in the interval  $[0, T]$ .

**Arrivals and Departures.** Agents arrive at the market at Poisson rate  $m$ . Hence, in any interval  $[t, t + 1]$ ,  $m$  new agents enter the market in expectation. Throughout the paper we

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<sup>5</sup>A less directly related literature is dynamic auctions and dynamic mechanism design. Budish et al. (2015) study the problem of timing and frequent batch auctions in the high frequency setting. Parkes and Singh (2003) generalize the VCG mechanism to a dynamic setting. Athey and Segal (2007) construct efficient and incentive-compatible dynamic mechanisms for private information settings. We refer interested readers to Parkes (2007) for a review of the dynamic mechanism design literature.



assume  $m \geq 1$ . Let  $A_t$  be the set of the agents in our market at time  $t$ , and let  $Z_t := |A_t|$ . We refer to  $A_t$  as the *pool* of the market and to  $Z_t$  as the *pool size*. We start by describing the evolution of  $A_t$  as a function of  $t \in [0, T]$ . Since we are interested in the limit behavior of  $A_t$ , we assume  $A_0 = \emptyset$ . We use  $A_t^n$  to denote<sup>6</sup> the set of agents who enter the market at time  $t$ . Note that with probability 1,  $|A_t^n| \leq 1$ . Also, let  $A_{t_0, t_1}^n$  denote the set of agents who enter the market in time interval  $[t_0, t_1]$ . Each agent becomes *critical* according to an independent Poisson process with rate  $\lambda$ , which, without loss of generality, we normalize to 1. This implies that, if an agent  $a$  enters the market at time  $t_0$ , then she becomes critical at some time  $t_0 + X$  where  $X$  is an exponential random variable with mean 1. Any critical agent leaves the market immediately; so the last point in time that an agent can get matched is the time that she gets critical. We say an agent  $a$  *perishes* if  $a$  leaves the market unmatched.

We assume that an agent  $a \in A_t$  leaves the market at time  $t$  if either  $a$  is not critical but is matched with another agent  $b \in A_t$ , or if  $a$  becomes critical and gets matched to another agent, or if  $a$  becomes critical and leaves the market unmatched and so perishes. Consequently, for any matching algorithm,  $a$  leaves the pool at some time  $t_1$  where  $t_0 \leq t_1 \leq t_0 + X$ . The *sojourn* of  $a$  is the length of the interval that  $a$  is in the pool, i.e.,  $s(a) := t_1 - t_0$ . We use  $A_t^c$  to denote the set of agents that are critical at time  $t$ .<sup>7</sup> Also, note that for any  $t \geq 0$ , with probability 1,  $|A_t^c| \leq 1$ .

It is essential to note that the arrival of the criticality event with some Poisson rate is *not* equivalent to discounting with the same rate, because the criticality event might be observed by the planner and the planner can react to that information.

**The Compatibility Network.** For any pair of agents, they are compatible with probability  $p$ , where  $0 \leq p \leq 1$ , and these probabilities are independent across pairs. Let  $d = m \cdot p$  be the density parameter of the model. In the paper, we use this definition and replace  $p$  with  $d/m$ .

For any  $t \geq 0$ , let  $E_t \subseteq A_t \times A_t$  be the set of compatible pairs of agents in the market (the set of *edges*) at time  $t$ , and let  $G_t = (A_t, E_t)$  be the network at time  $t$ . Compatible pairs persist over time; i.e.,  $a, b \in A_t$  and  $a, b \in A_{t'}$ , then  $(a, b) \in E_t$  if and only if  $(a, b) \in E_{t'}$ . For an agent  $a \in A_t$  we use  $N_t(a) \subseteq A_t$  to denote the set of neighbors of  $a$  in  $G_t$ . It follows

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<sup>6</sup>As a notational guidance, we use subscripts to refer to a point in time or a time interval, while superscripts  $n, c$  refer to *new* agents and *critical* agents, respectively.

<sup>7</sup>In our proofs, we use the fact that  $A_t^c \subseteq \cup_{0 \leq \tau \leq t} A_\tau$ . In the example of the text, we have  $a \in A_{t_0+X}^c$ . Note that even if agent  $a$  is matched before getting critical (i.e.,  $t_1 < t_0 + X$ ), we still have that  $a \in A_{t_0+X}^c$ . Hence,  $A_t^c$  is not necessarily a subset of  $A_t$  since it may have agents who are already matched and left the pool. This generalized definition of  $A_t^c$  is helpful in our proofs.

that, if the planner does not match any agents, then for any fixed  $t \geq 0$ ,  $G_t$  is distributed as an Erdős-Rényi graph with parameter  $d/m$  and in the long-run,  $d$  is the average degree<sup>8</sup> of agents (Erdős and Rényi, 1960).

Let  $A = \cup_{t \leq T} A_t^n$ , let  $E \subseteq A \times A$  be the set of acceptable transactions between agents in  $A$ , and let  $G = (A, E)$ <sup>9</sup>. Observe that any realization of the above stochastic process is uniquely defined given  $A_t^n, A_t^c$  for all  $t \geq 0$  and the set of compatible pairs,  $E$ . A vector  $(m, d)$  represents a *dynamic matching market*.

**Matching Algorithms.** A set of edges  $M_t \subseteq E_t$  is a *matching* if no two edges share the same endpoints. A *matching algorithm*, at any time  $t \geq 0$ , selects a (possibly empty) matching,  $M_t$ , in the current graph  $G_t$ , and the endpoints of the edges in  $M_t$  leave the market immediately. We assume that any matching algorithm at any time  $t_0$  only knows the current graph  $G_t$  for  $t \leq t_0$  and does not know anything about  $G_{t'}$  for  $t' > t_0$ . In the benchmark case that we consider, the matching algorithm can depend on the set of critical agents at time  $t$ . Nonetheless, we will extend several of our theorems to the case where the algorithm knows more than this, or less than this.

We emphasize that the random sets  $A_t$  (the set of agents in the pool at time  $t$ ),  $E_t$  (the set of compatible pairs of agents at time  $t$ ),  $N_t(a)$  (the set of an agent  $a$ 's neighbors), as well as the random variable  $Z_t$  (pool size at time  $t$ ) are all functions of the underlying matching algorithm. We abuse notation and do not include the name of the algorithm when we analyze these variables.

**The Goal.** Let  $\text{ALG}(T)$  be the set of matched agents by time  $T$ ,

$$\text{ALG}(T) := \{a \in A : a \text{ is matched by ALG by time } T\}.$$

We may drop the  $T$  in the notation  $\text{ALG}(T)$  if it is clear from context.

The goal of the planner is to match the maximum number of agents, or, equivalently, to minimize the number of perished agents. The *loss* of a matching algorithm ALG is defined as the ratio of the expected number of perished agents to the expected number of agents,

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<sup>8</sup>In an undirected graph, *degree* of a vertex is equal to the total number of edges connected to that vertex.

<sup>9</sup>Note that  $E \supseteq \cup_{t \leq T} E_t$ , and the two sets are not typically equal, since two agents may find it acceptable to transact, even though they are not in the pool at the same time because one of them was matched earlier.

which is, by definition, a number in  $[0, 1]$ .

$$\mathbf{L}(\text{ALG}) := \frac{\mathbb{E}[|A - \text{ALG}(T) - A_T|]}{\mathbb{E}[|A|]} = \frac{\mathbb{E}[|A - \text{ALG}(T) - A_T|]}{mT}.$$

As in the static case, the planner seeks a maximum matching among all agents in a random graph. Unlike the static case, he faces two additional constraints: First, not all agents are present at the same time, and second, he is uncertain about future arrivals and departures.

Minimizing loss is equivalent to maximizing social welfare, for the case where the cost of waiting is negligible compared to the cost of leaving the market unmatched. We study the case where agents do not discount the future.<sup>10</sup>

Our problem can be modeled as a Markov Decision Problem (MDP) that is defined as follows. The state space is the set of pairs  $(H, B)$  where  $H$  is any undirected graph of any size, and if the algorithm knows the set of critical agents,  $B$  is a set of at most one vertex of  $H$  representing the corresponding critical agent. The action space for a given state is the set of matchings on the graph  $H$ . Under this view, an algorithm designer wants to minimize the loss over a time period  $T$ .

**Optimum Solutions.** In many parts of this paper we compare the performance of a matching algorithm to the performance of an optimal *Omniscient* algorithm. Unlike any matching algorithm, the Omniscient algorithm has full information about the future, i.e., it knows the full realization of the graph  $G$  of agents who will be in the system in future and the edges between them. Therefore, it can return the (static) maximum matching in this graph as its output, and thus minimize the fraction of perished agents. Let  $\text{OMN}(T)$  be the set of matched agents in the maximum matching of  $G$ . The loss function under the omniscient algorithm at time  $T$  is

$$\mathbf{L}(\text{OMN}) := \frac{\mathbb{E}[|A - \text{OMN}(T) - A_T|]}{mT}$$

Observe that for any matching algorithm  $\text{ALG}$ , and any realization of the probability space, we have  $|\text{ALG}(T)| \leq |\text{OMN}(T)|$ , because the Omniscient algorithm knows the realization of the stochastic process, and it could select the same matching as any matching algorithm.

The optimum matching algorithm, *i.e.*, the solution to the above MDP, is the algorithm

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<sup>10</sup>The case where discount rate is not zero is extensively studied in a subsequent working paper (Akbarpour et al., 2017).

that minimizes loss. We first consider  $\text{OPT}^c$ , the algorithm that knows the set of critical agents at time  $t$ . We then relax this assumption and consider  $\text{OPT}$ , the algorithm that does not know these sets.

Let  $\text{ALG}^c$  be any matching algorithm that knows the set of critical agents at time  $t$ . It follows that

$$\mathbf{L}(\text{ALG}^c) \geq \mathbf{L}(\text{OPT}^c) \geq \mathbf{L}(\text{OMN}).$$

Similarly, let  $\text{ALG}$  be any matching algorithm that does not know the set of critical agents at time  $t$ . It follows that

$$\mathbf{L}(\text{ALG}) \geq \mathbf{L}(\text{OPT}) \geq \mathbf{L}(\text{OPT}^c) \geq \mathbf{L}(\text{OMN}).$$

Note that unlike the loss function that considers the expected number of matched agents,  $|\text{ALG}|$  and  $|\text{OPT}|$  (the number of matched agents under  $\text{ALG}$  and  $\text{OPT}$ ) are generally incomparable, and depending on the realization of  $G$  we may even have  $|\text{ALG}| > |\text{OPT}|$ .

### 3 Simple Matching Algorithms

In our model, solving for the optimal matching algorithm is computationally complex. This is because there are at least  $2^{\binom{m}{2}}/m!$  distinct graphs of size  $m$ , so for even moderately large markets, we cannot apply standard dynamic programming techniques to find the optimum online matching algorithm.<sup>11</sup>

Nevertheless, we are not fully agnostic about the optimal algorithm. In particular, we know that  $\text{OPT}^c$  has at least two properties:

- i) A pair of agents  $a, b$  gets matched in  $\text{OPT}^c$  only if one of them is critical, because if  $a, b$  can be matched and neither of them is critical, then we are weakly better off if we wait and match them later.
- ii) If an agent  $a$  is critical at time  $t$  and  $N_t(a) \neq \emptyset$  then  $\text{OPT}^c$  matches  $a$ . This is because allowing a critical agent to perish now decreases the number of future perished agents by at most one.

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<sup>11</sup>This lower bound is derived as follows: When there are  $m$  agents, there are  $\binom{m}{2}$  possible edges, each of which may be present or absent. Some of these graphs may have the same structure but different agent indices. A conservative lower bound is to divide by all possible re-labellings of the agents ( $m!$ ). For instance, for  $m = 30$ , there are more than  $10^{98}$  states in the approximated MDP.

$\text{OPT}^c$  waits until some agent gets critical and if an agent is critical and has some compatible partner, then  $\text{OPT}^c$  matches that agent. But the choice of match partner depends on the entire network structure, which is what makes the problem combinatorially complex. Our goal here is to separate these two effects: How much is achieved merely by being patient? And how much more is achieved by optimizing over the network structure?

To do this, we start by designing a matching algorithm (the Greedy algorithm), which mimics ‘match-as-you-go’ algorithms used in many real marketplaces. It delivers maximal matchings at any point in time, without regard for the future.

**Definition 1** (Greedy Algorithm). *If any new agent  $a$  enters the market at time  $t$ , then match her with an arbitrary agent in  $N_t(a)$  whenever  $N_t(a) \neq \emptyset$ .*

Since no two agents arrive at the same time almost surely, we do not need to consider the case where more than one agent enters the market. Moreover, the graph  $G_t$  in the Greedy algorithm is almost always an empty graph. Hence, the Greedy algorithm cannot use any information about the set of critical agents.

To separate the value of waiting from the value of optimizing over the network structure, we design a second algorithm which chooses the optimal *time* to match agents, but ignores the network structure.

**Definition 2** (Patient Algorithm). *If an agent  $a$  becomes critical at time  $t$ , then match her uniformly at random with an agent in  $N_t(a)$  whenever  $N_t(a) \neq \emptyset$ .*

To run the Patient algorithm, we need access to the set of critical agents at time  $t$ . Note that the Patient algorithm exploits *only* short-horizon information about critical agents, as compared to the Omniscient algorithm which has full information about the future. Of course, the knowledge of the exact departure times is an abstraction from reality and we do not intend the timing assumptions about critical agents to be interpreted literally. An agent’s point of perishing represents the point at which it ceases to be socially valuable to match that agent and the agent will incur a high waiting cost if not matched. Letting the planner observe the set of critical agents is a modeling convention that represents high-accuracy short-horizon information about agents’ departures.

We now state results for the case of large markets with sparse graphs, in the steady state:  $m \rightarrow \infty$ ,  $d$  is held constant, and  $T \rightarrow \infty$ . Clearly, this implies that  $\frac{d}{m} = p \rightarrow 0$ , which should not be taken literally. This method eliminates nuisance terms and is a standard way to state results for large but sparse graphs (Erdős and Rényi, 1960). [Appendix B](#) and [Appendix C](#) study the performance of each algorithm as a function of  $m$ ,  $T$ , and  $d$ , without taking

limits. Simulations in [Appendix H](#) indicate that the key comparisons hold for small values of  $m$ . Moreover, the algorithms we examine converge rapidly to the stationary distribution ([Theorem 7](#)). The readers interested in technical non-limit results can see [Theorem 9](#) and [Theorem 10](#) for the dependence of our results on  $m$  and  $T$ .

## 4 Timing, Thickness, and Network Optimization

Does timing substantially affect the performance of dynamic matching algorithms? Our first result establishes that varying the timing properties of simple algorithms has large effects on their performance.

**Theorem 1.** *For (constant)  $d \geq 2$ , as  $T, m \rightarrow \infty$ ,*

$$\begin{aligned} \mathbf{L}(\text{Greedy}) &\geq \frac{1}{2d+1} \\ \mathbf{L}(\text{Patient}) &\leq \frac{1}{2} \cdot e^{-d/2} \end{aligned}$$

We already knew that the Patient algorithm outperforms the Greedy algorithm. What this theorem shows is that Patient algorithm achieves exponentially small loss, but the Greedy algorithm does not. [Theorem 1](#) provides an *upper bound* for the value of waiting: We shut down the channels by which waiting can be costly (negligible waiting cost, while the planner observes critical agents) and show that in this world, the *option value* of waiting is large.

Why does this happen? The Greedy algorithm attempts to match agents upon arrival and the Patient algorithm attempts to match them upon departure. Now, suppose a new agent enters the market under the Greedy algorithm. If there are  $z$  other agents in the market, the probability that the agent has no feasible partner is  $(1 - \frac{d}{m})^z$ , which falls exponentially as  $d$  rises. Equally, suppose an agent becomes critical, and the Patient algorithm attempts to match him. If there are  $z$  other agents in the market, the probability that the agent has no feasible partner is again  $(1 - \frac{d}{m})^z$ . What, then, explains the difference in performance?

The key is to see that the composition and number of agents in the market depends endogenously on the matching algorithm. As  $d$  rises, the Greedy algorithm matches agents more rapidly, depleting the stock of available agents, and reducing the equilibrium  $z$ . This effect cancels out the exponential improvements that would accrue from raising  $d$  in a static model. By contrast, because the Patient algorithm waits, the market is thick. We prove that,



Figure 2: If  $a_2$  gets critical in the above graph, it is strictly better to match him to  $a_1$  as opposed to  $a_3$ . The Patient algorithm, however, chooses either of  $a_1$  or  $a_3$  with equal probability.

under the Patient algorithm, equilibrium  $z$  is always above  $\frac{m}{2}$ , which entails that  $(1 - \frac{d}{m})^z$  falls exponentially as  $d$  rises.

The next question is, are the gains from patience large compared to the total gains from optimizing over the network structure and choosing the right agents to match? First we show by example that the Patient algorithm is not optimal, because it ignores the global network structure.

**Example 1.** Let  $G_t$  be the graph shown in [Figure 2](#), and let  $a_2 \in A_t^c$ , i.e.,  $a_2$  is critical at time  $t$ . Observe that it is strictly better to match  $a_2$  to  $a_1$  as opposed to  $a_3$ . Nevertheless, since the Patient algorithm makes decisions that depend only on the immediate neighbors of the agent it is trying to match, it cannot differentiate between  $a_1$  and  $a_3$  and will choose either of them with equal probability.

The next theorem provides a bound for gains from network optimizations by proving a lower bound for the optimum.

**Theorem 2.** Consider any algorithm ALG that observes the set of critical agents. Then, for (constant)  $d \geq 2$ , as  $T, m \rightarrow \infty$ ,

$$\mathbf{L}(\text{OPT}^c) \geq \frac{e^{-\frac{d}{2}(1+\mathbf{L}(\text{ALG}))}}{d+1}.$$

Recall that  $\mathbf{L}(\text{Patient}) \leq \frac{1}{2} \cdot e^{-d/2}$ . Substituting for  $\mathbf{L}(\text{ALG})$  implies that<sup>12</sup>  $\mathbf{L}(\text{OPT}^c) \geq \frac{e^{-\frac{d}{2}(1+e^{-d/2})}}{d+1}$ . This exponential term in  $\mathbf{L}(\text{OPT}^c)$  is close to that of the  $\mathbf{L}(\text{Patient})$  for even moderate values of  $d$ . The preceding results show that the gains from the right timing decision (moving from the Greedy algorithm to the Patient algorithm) are larger than the remaining gains from optimizing over the entire network (moving from the Patient algorithm to the optimum algorithm). In many settings, optimal solutions may be computationally demanding and difficult to implement. Thus, this result suggests that, in settings that

<sup>12</sup>Paradoxically, this argument exploits the fact that Patient has a small loss to show that  $\text{OPT}^c$  has a large loss. The bound in [Theorem 2](#) is decreasing  $\mathbf{L}(\text{ALG})$ , so new algorithms that improve on Patient automatically result in a tighter lower bound on  $\mathbf{L}(\text{OPT}^c)$ .

approximately satisfy our key assumptions, it will often be more worthwhile for policymakers to find ways to thicken the market, rather than to seek complicated optimal policies.

It is worth emphasizing that this result (as well as [Theorem 4](#)) proves that “local” algorithms are close-to-optimal. Since in our model agents are *ex ante* homogeneous, this shows that “whom to match” is not as important as “when to match”. In settings where agents have multiple types, however, the decision of “whom to match” can be an important one even when it is local. We defer the discussion of conditions under which “whom to match” is important to [Section 6](#).

We now sketch the proof and offer intuition for [Theorem 1](#). The proof of [Theorem 2](#) has similar ideas as the proof of [Theorem 4](#), so we discuss them together in the next section.

*Proof Overview.* The key idea in proving [Theorem 1](#) is to carefully study the structure of the graph induced by these algorithms and the distribution of the pool size,  $Z_t$ . In particular, we show that  $Z_t$  is Markov chain with a unique stationary distribution that mixes rapidly, and it is a sufficient statistic for the structure of the graph under the Greedy and Patient algorithms.

**Greedy algorithm.** Under the Greedy algorithm, conditional on  $Z_t$ , the pool is almost always an empty graph; *i.e.*, a graph with no edges. Now note that the rate that some agent in the pool becomes critical is  $Z_t$ . Because the graph is empty, critical agents perish with probability one. Therefore, in steady state,  $\mathbf{L}(\text{Greedy}) \approx \mathbb{E}[Z_t]/m$ .

Next, we show that for the Greedy algorithm,  $\mathbb{E}[Z_t] \geq \frac{m}{2d+1}$ . Take any pool size  $z$ . At rate  $m$ , a new agent arrives. With probability  $(1 - d/m)^z$  the new agent has no compatible matches, which increases the pool size by 1. With probability  $1 - (1 - d/m)^z$ , the new agent has a compatible match and the pool size falls by 1. At rate  $z$ , an agent perishes, in which case the pool size falls by 1. Let  $z^*$  be the point where these forces balance; *i.e.*, the solution to:

$$m(1 - d/m)^z(+1) + z(-1) + m(1 - (1 - d/m)^z)(-1) = 0.$$

By algebraic manipulation,  $z^* \geq \frac{m}{2d+1}$ . We show that under the stationary distribution,  $Z_t$  is highly concentrated around  $z^*$ , which then implies that  $\mathbb{E}[Z_t]$  is close to  $z^*$ . This produces the lower bound for  $\mathbf{L}(\text{Greedy})$ .

**Patient algorithm.** Under the Patient algorithm, conditional on  $Z_t$ , the pool is an Erdős-Rényi random graph with parameter  $d/m$ . To see why, suppose an agent gets critical. The Patient algorithm’s choice of a match partner for that agent depends only on the immediate neighbors of that agent. Consequently, after the critical agent leaves, the rest of



the graph is still distributed as an Erdős-Rényi random graph. The rate that some agent becomes critical is  $Z_t$ . Because the graph is a random graph, critical agents perish with probability  $(1 - d/m)^{Z_t}$ . Therefore, in steady state,  $\mathbf{L}(\text{Patient}) \approx \mathbb{E} [Z_t(1 - d/m)^{Z_t}] / m$ .

The next step is to show that  $Z_t$  is highly concentrated around  $\mathbb{E}[Z_t]$ , so  $\mathbf{L}(\text{Patient}) \approx \mathbb{E}[Z_t](1 - d/m)^{\mathbb{E}[Z_t]} / m$ . This step involves long arguments. But once this step is established, it remains to prove that  $\mathbb{E}[Z_t] \geq \frac{m}{2}$ . The exact proof for this is involved, but a simple thought experiment gives good intuition. Suppose the Patient algorithm is not able to match any agents. Then  $\mathbb{E}[Z_t] = m$ . On the other hand, suppose the Patient algorithm can match *all* agents. Then agents arrive at rate  $m$  and get matched at rate  $2\mathbb{E}[Z_t]$  because for each critical agent, two agents get matched. This implies that  $\mathbb{E}[Z_t] = \frac{m}{2}$ . In fact, the Patient algorithm can match some but not all agents, so,  $\frac{m}{2} \leq \mathbb{E}[Z_t] \leq m$ . This produces the upper bound for  $\mathbf{L}(\text{Patient})$ .  $\square$

What if the planner has more than short-horizon information about agents' departure times? Suppose the planner knows the exact departure times of all agents who are in the pool. Is it still the case that waiting is highly valuable? To answer this question, we design a new class of algorithms which are constrained to match agents as soon as they can, but have access to exact departure times of all agents in the market. We refer to this class of algorithms as *Departure-Aware Greedy* (DAG) algorithms.

**Definition 3** (Departure-Aware Greedy Algorithms). *If any new agent  $a$  enters the market at time  $t$ , then match her with an agent in  $N_t(a)$  whenever  $N_t(a) \neq \emptyset$ , where the choice of match partner can depend on the profile of departure times for agents in the pool.*

For instance, if a newly arrived agent has multiple matches, a DAG algorithm can break the tie in favor of the partner who departs soonest. If this algorithm can perform 'close to' the Patient algorithm, then it suggests that waiting is not valuable if the planner has access to sufficiently rich information. Our next theorem, however, shows that even this long-horizon information cannot substantially help the planner.

**Theorem 3.** *For any DAG algorithm, as  $T \rightarrow \infty$ ,*

$$\mathbf{L}(\text{DAG}) \geq \frac{1}{2d + 1}$$

*Proof.* Suppose we run a DAG algorithm from time 0 to time  $T$ . Pick an agent  $i$  uniformly at random from the set of all agents who arrive between 0 and  $T$ , that is, uniformly at random from the set  $A = \cup_{t \leq T} A_t^n$ . Let  $\phi_i^n$  denote the probability that  $i$  is not matched upon arrival

and let  $\phi_i$  denote the probability that  $i$  is not matched at all. Our goal is to provide a lower bound for  $\phi_i$ .

For any realization of the stochastic process under a DAG algorithm, the number of agents who are matched is twice the number of agents who are matched upon arrival. Thus, since  $i$  was drawn uniformly at random,

$$1 - \phi_i = 2(1 - \phi_i^n) \quad (4.1)$$

DAG algorithms cannot condition matching decisions on information about agents who have not yet arrived. For all other agent's  $j$ ,  $i$ 's sojourn length is drawn independently from the probability that  $i$  is compatible with  $j$ . Thus, conditional on  $i$  not being matched on arrival,  $i$ 's sojourn length is still distributed exponentially with parameter 1. New agents compatible with  $i$  arrive at Poisson rate  $m \times d/m = d$ . Thus, the probability that no compatible partner arrives during  $i$ 's sojourn is  $\frac{1}{d+1}$ , so  $\phi_i \geq \frac{\phi_i^n}{d+1}$ .<sup>13</sup> Substituting for  $\phi_i^n$  with Equation 4.1 yields  $\phi_i \geq \frac{1}{2d+1}$ .

By definition of the loss,

$$\mathbf{L}(DAG) = \frac{\mathbb{E}[|A - DAG(T)|]}{mT} - \frac{\mathbb{E}[|A_T|]}{mT} \geq \frac{mT \frac{1}{2d+1}}{mT} - \frac{m}{mT} = \frac{1}{2d+1} - \frac{1}{T} \quad (4.2)$$

$\mathbb{E}[|A_T|] \leq m$  holds since any algorithm has a smaller expected pool size than the inactive algorithm. Taking the limit as  $T \rightarrow \infty$  completes the proof.  $\square$

The theorem shows that any matching algorithm that does not wait, even with access to long-horizon information about departure times of agents who are in the market, cannot perform close to the Patient algorithm. Therefore, the Patient algorithm strongly outperforms the Greedy algorithm because it *waits* long enough to create a thick market.

## 5 Value of Information and Incentive-Compatibility

Up to this point, we have assumed that the planner knows the set of critical agents; i.e. he has accurate short-horizon information about departures. We now relax this assumption in both directions.

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<sup>13</sup>For two independent Poisson arrival processes with rates 1 and  $d$ , the probability that the first process arrives before the second is  $\frac{1}{d+1}$ . Notably, this inequality holds regardless of the state of the system at the instant when  $i$  arrives, so the argument does not require that the system has converged to a stationary distribution (if it exists).

First, we consider the case that the planner does not know the set of critical agents. That is, the planner’s policy may depend on the graph  $G_t$ , but not the set of critical agents  $A_t^c$ . Recall that OPT is the optimum algorithm subject to these constraints. Second, we consider OMN, the case under which the planner knows *everything* about the future realization of the market. Our main result in this section is stated below:

**Theorem 4.** *For (constant)  $d \geq 2$ , as  $T, m \rightarrow \infty$ ,*

$$\frac{1}{2d+1} \leq \mathbf{L}(\text{OPT}) \leq \mathbf{L}(\text{Greedy}) \leq \frac{\log(2)}{d}$$

$$\frac{e^{-d}}{d+1} \leq \mathbf{L}(\text{OMN}) \leq \mathbf{L}(\text{Patient}) \leq \frac{1}{2} \cdot e^{-d/2}.$$

This shows that the loss of OPT and Greedy are relatively close, which indicates that waiting and criticality information are complements: Waiting to thicken the market is substantially valuable only when the planner can identify critical agents. Observe that OPT could in principle wait to thicken the market, but this result proves that the gains from doing so (compared to running the Greedy algorithm) are not large.

What if the planner knows *more* than just the set of critical agents? For instance, the planner may have long-horizon forecasts of agent departure times, or the planner may know that certain agents are more likely to have matches in future than other agents<sup>14</sup>. However, **Theorem 4** shows that *no* expansion of the planner’s information set yields a better-than-exponential loss.

Under these new information assumptions, we once more find that local algorithms can perform close to computationally intensive global optima: Greedy is close to OPT under without access to criticality information, and Patient is close to OPT<sup>c</sup>.

We now sketch the proof ideas of **Theorem 4** and **Theorem 2**.

*Proof Overview.* We now sketch the proof of bounds that we provided for our optimum benchmarks. The full proofs can be found in **Appendix D**.

**OPT algorithm.** We show how we bound  $\mathbf{L}(\text{OPT})$ , without knowing anything about the way OPT works. The idea is to provide lower bounds on the performance of *any* matching algorithm as a function of its expected pool size. Let  $\zeta$  be the expected size of the pool at time  $t$ , where  $t$  is picked uniformly at random from  $[0, T]$ . The rate that some agent gets

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<sup>14</sup>In our model, the number of acceptable transactions that a given agent will have with the next  $N$  agents to arrive is Bernoulli distributed. If the planner knows beforehand whether a given agent’s realization is above or below the 50th percentile of this distribution, it is as though agents have different ‘types’.

critical is  $\zeta$ . When the planner does not observe critical agents, all critical agents perish. Hence, in steady state,  $\mathbf{L}(\text{OPT}) \simeq \zeta/m$ . Note that this is an increasing function of  $\zeta$ , so from this perspective the planner prefers to decrease the pool size as much as possible.

Next, we count the fraction of agents who do not form any edges upon arrival and during their sojourn. No matching algorithm can match these agents and so the fraction of those agents is a lower bound on the performance of *any* matching algorithm, including OPT. The probability of having no edges upon arrival is at least  $(1 - d/m)^\zeta$ , while (as  $T \rightarrow \infty$ ) the probability of not forming any edges during a sojourn is  $\int_{t=0}^{\infty} e^{-t} \cdot (1 - d/m)^{mt} dt$ , because an agent who becomes critical  $t$  periods after arrival meets  $mt$  new agents in expectation. Simple algebra (see [Subsection D.1](#)) shows that for any algorithm ALG,  $\mathbf{L}(\text{ALG}) \geq \frac{e^{-\zeta(1+d/m)d/m}}{1+d+d^2/m} \geq \frac{1-\zeta(d/m+d^2/m^2)}{1+2d+d/m^2}$ . From this perspective, the planner prefers to increase the pool size as much as possible. One can then easily show that if  $\zeta \leq 1/(2d + 1)$ , this lower bound guarantees that the fraction of agents with no matches is at least  $1/(2d + 1)$ , and if  $\zeta > 1/(2d + 1)$  our previous bound guarantees that loss is at least  $1/(2d + 1)$ . So  $\mathbf{L}(\text{OPT}) \geq 1/(2d + 1)$ .

**OMN algorithm.** We use a similar trick to provide a lower bound for  $\mathbf{L}(\text{OMN})$ . We have already established a lower bound on the fraction of agents with no matches, as a function of expected pool size. But we know that the expected pool size can never be more than  $m$ , because that is the expected pool size when the planner does not match any agents. Hence, the fraction of agents with no matches when the expected pool size is  $m$  is a lower bound on the loss of the OMN. (See [Subsection D.2](#) for the details.)

**OPT<sup>c</sup> algorithm.** Now we sketch the proof of [Theorem 2](#) and bound OPT<sup>c</sup>. The key idea behind this proof is the following: OPT<sup>c</sup> matches agents (to non-critical partners) if and only if they are critical. Suppose we run OPT<sup>c</sup> from time 0 to  $T$ . Take any realization of the stochastic process: All critical agents depart. A non-critical agent departs if and only if he is matched. Thus, the number of non-critical agents who are matched is equal to the number of departures (denoted  $\#depart$ ) minus the number of agents who become critical (denoted  $\#critical$ ). Each matched pair consists of one critical agent and one non-critical agent, so the total number of matched agents is equal to  $2(\#depart - \#critical)$ . The following approximations hold as  $T \rightarrow \infty$ :  $\mathbf{L}(\text{OPT}^c) \approx \frac{\mathbb{E}[\#arrive - 2(\#departures - \#critical)]}{mT}$ , where  $\#arrive$  is the number of agents who arrive between 0 and  $T$ .  $\mathbb{E}[\#critical] = \zeta^c$ , where  $\zeta^c$  is the expected pool size at  $t$  for  $t$  drawn uniformly at random from  $[0, T]$ .  $\frac{\mathbb{E}[\#arrive - 2\#depart]}{mT} \approx -1$ . Thus,  $\mathbf{L}(\text{OPT}^c) \approx 2\frac{\zeta^c}{m} - 1$ .

Finally, for any ALG,  $\mathbf{L}(\text{ALG}) \geq \mathbf{L}(\text{OPT}^c) \approx 2\frac{\zeta^c}{m} - 1$ , which leads to  $\frac{m}{2}(1 + \mathbf{L}(\text{ALG})) \geq \zeta^c$ . From before,  $\mathbf{L}(\text{OPT}^c) \geq \frac{e^{-\zeta^c(1+d/m)d/m}}{1+d+d^2/m}$ , and substituting for  $\zeta^c$  finishes the proof of

Theorem 2. □

The idea behind the proof of bound for OMN sheds light on the fundamental dilemma that any algorithm with no access to criticality information confronts. On the one hand, because the planner cannot identify critical agents, they all perish. Hence, the planner wishes to make the pool size as small as possible to avoid perishings. On the other hand, the planner wishes to thicken the market so that agents have more matches. In balancing these two opposing forces, we prove that the planner cannot do much to outperform the Greedy algorithm.

These results suggest that criticality information is particularly valuable. This information is necessary to achieve exponentially small loss, and no expansion of information enables an algorithm to perform much better. However, in many settings, agents have privileged insight into their own criticality times. For instance, a rider in a carpooling platform knows the time-sensitivity of her request, but the platform does not know that. In such cases, agents may have incentives to misreport whether they are critical, in order to increase their chance of getting matched. The situation is more subtle if agents have waiting costs.

To study this problem, we first formally introduce discounting to our model: An agent receives zero utility if she leaves the market unmatched. If she is matched, she receives a utility of 1 discounted at rate  $r$ . More precisely, if  $s(a)$  is the sojourn of agent  $a$ , then we define the utility of agent  $a$  as follows:

$$u(a) := \begin{cases} e^{-rs(a)} & \text{if } a \text{ is matched} \\ 0 & \text{otherwise.} \end{cases}$$

We assume that agents are fully rational and know the underlying parameters, and that they believe that the pool is in the stationary distribution when they arrive, but they do not observe the actual realization of the stochastic process. That is, agents observe whether they are critical, but do not observe  $G_t$ , while the planner observes  $G_t$  but does not observe which agents are critical. Consequently, agents' strategies are independent of the realized sample path. Our results are sensitive to this assumption; for instance, if the agent knew that she had a neighbor, or knew that the pool at that moment was very large, she would have an incentive under our mechanism to falsely report that she was critical. This assumption is plausible in many settings; generally, centralized brokers know more about the current state of the market than individual traders. Indeed, frequently agents approach centralized brokers because they do not know who is available to trade with them.

We now exhibit a truthful mechanism *without* transfers that elicits such information from agents, and implements the Patient algorithm.

**Definition 4** (Patient-Mechanism). *Ask agents to report when they get critical. When an agent reports being critical, the market-maker attempts to match her to a random neighbor. If the agent has no neighbors, the market-maker treats her as if she has perished, i.e., she will never be matched again.*

Each agent  $a$  selects a mixed strategy by choosing a function  $c_a(\cdot)$ ; at the interval  $[t, t+dt]$  after her arrival, if she is not yet critical, she reports being critical with rate  $c_a(t)dt$ , and when she truly becomes critical she reports immediately. Our main result in this section asserts that if agents are not too impatient, then the Patient Mechanism is incentive-compatible in the sense that the truthful strategy profile is a strong  $\epsilon$ -Nash equilibrium.<sup>15</sup>

**Theorem 5.** *Suppose that the market is in the stationary distribution,  $d \geq 2$  and  $d = \text{polylog}(m)$ .<sup>16</sup> If  $0 \leq r \leq e^{-d/2}$ , then the truthful strategy profile is a strong  $\epsilon$ -Nash equilibrium for Patient-Mechanism, where  $\epsilon \rightarrow 0$  as  $m \rightarrow \infty$ .*

*Proof Overview.* An agent can be matched in one of two ways under Patient Mechanism: Either she becomes critical and has a neighbor, or one of her neighbors becomes critical and is matched to her. By symmetry, the chance of either happening is the same, because with probability 1 every matched pair consists of one critical agent and one non-critical agent. When an agent declares that she is critical, she is taking her chance that she has a neighbor in the pool right now. By contrast, if she waits, there is some probability that another agent will become critical and be matched to her before she takes her chance of getting matched by reporting to be critical. Consequently, for small  $r$ , agents will opt to wait.

There is a hidden obstacle here. Even if one assumes that the market is in a stationary distribution at the point an agent enters, the agent's beliefs about the graph structure and  $Z_t$  may change as time passes. In particular, an agent makes inferences about the current distribution of pool size, *conditional* on not having been matched yet, and this conditional distribution is different from the stationary distribution. This makes it difficult to compute the payoffs from deviations from truthful reporting. We tackle this problem by using the concentration bounds (see [Proposition 16](#)) which limit how much an agent's posterior can

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<sup>15</sup>Any strong  $\epsilon$ -Nash equilibrium is an  $\epsilon$ -Nash equilibrium. For a definition of strong  $\epsilon$ -Nash equilibrium, see [Definition 5](#).

<sup>16</sup> $\text{polylog}(m)$  denotes any polynomial function of  $\log(m)$ . In particular,  $d = \text{polylog}(m)$  if  $d$  is a constant independent of  $m$ .

be different from her prior. We also focus on strong  $\epsilon$ -Nash equilibrium, which allows small deviations from full optimality.  $\square$

The key insight of [Theorem 5](#) is that remaining in the pool has a “continuation value”: The agent, while not yet critical, may be matched to a critical agent. If agents are not too impatient, then the planner can induce truth-telling by using punishments that decrease this continuation value. The Patient Mechanism sets this continuation value to zero, but in practice softer punishments could achieve the same goal. For instance, if there are multiple potential matches for a critical agent, the planner could break ties in favor of agents who have never misreported. However, such mechanisms can undermine the Erdős-Rényi property that makes the analysis tractable.<sup>17</sup>

## 6 Discussion

In this paper, we studied the problem of dynamic matching in a market with networked frictions. The key insight of our analysis is that thickness and information are complements and can be highly valuable. Waiting to thicken the market can yield large gains if the planner can forecast departures accurately. Information about departures is highly valuable if it is feasible to wait. In our simple benchmark, local algorithms perform well relative to algorithms that account for the entire network structure.

One goal of this paper is to identify conditions under which timing concerns are more important than network optimization. To illustrate the limits of these results, we close by discussing two environments in which network optimization is potentially important.

**Sparse networks:** When evaluating algorithms, our results mainly distinguish between loss that is fractionally small in  $d$  and loss that is exponentially small in  $d$ . Such bounds have bite when graph is not extremely sparse ( $d < 4$ ). For instance, when  $d = 3$ , our bounds only imply that  $\mathbf{L}(DAG) \geq .143$  and  $\mathbf{L}(\text{Patient}) \leq .112$ , so waiting may not yield large improvements. Similarly, in extremely sparse graphs the upper bound for  $\mathbf{L}(\text{Patient})$  leaves room for improvement by considering the network structure. For instance, one could modify the Patient algorithm so that when a critical agent has multiple compatible partners, it

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<sup>17</sup>If an agent could be matched even if he misreported previously, then we need to keep track of the edges of that agent off the equilibrium path. However, the fact that the agent was not matched indicates that the agent did not have any edges at the point he misreported, which means that the Markov process off the path does not have a tractable Erdős-Rényi random graph representation.

breaks ties in favor of partners with lower degree. Simulations indicate that this modified algorithm yields appreciable improvements when  $d$  is small (Figure 8 in Appendix G).

**Heterogeneous types:** One limitation of the model is the assumption that agents are *ex ante* homogeneous. Some of our results crucially depend on this assumption. Suppose instead that agents have types that affect the probability that they are compatible with other agents, as in the following example:

**Example 2.** *There are two types of agents: Those who are hard-to-match ( $H$ ) and those who are easy-to-match ( $E$ ). The probability that two agents of types  $i, j \in \{H, E\}$  are compatible is  $p_{ij}$ , where  $p_{HH} < p_{HE} = p_{EH} < p_{EE}$ . Suppose a critical agent has two neighbors, one who is hard-to-match and one who is easy-to-match. Then, ceteris paribus, the optimal policy should match the critical agent to the hard-to-match neighbor.*

Clearly, an algorithm that ignored types could have large losses. In this case, a *local* algorithm would exploit information about neighbors’ types, but ignore the overall network structure. It is an open question whether local algorithms perform well for heterogeneous types, since our current proofs do not extend straightforwardly to this case.<sup>18</sup>

A recent empirical study of the world’s largest carpooling platform (DiDi) by Liu et al. (2018) sheds some light on the role of heterogeneity. Liu et al. (2018) study a two-sided version of our model. Their simulations indicate that for some moderate levels of match heterogeneity, Patient continues to outperform the Greedy<sup>19</sup>. Nonetheless, for substantial levels of heterogeneity in matches, Greedy can outperform Patient. These findings show that heterogeneity interacts with our results in a subtle way, and a detailed theoretical treatment is an interesting avenue for future research.

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<sup>18</sup>Even extending our model to two types raises several technical challenges, discussed in the Online Appendix for Nikzad et al. (2017).

<sup>19</sup>In particular, Liu et al. (2018) write: “In our context, drivers have heterogeneous preferences, but we still find that the patient algorithm achieves the highest level of market thickness, number of matches and average driver utility at our estimates.”



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## A Roadmap to the Appendices

In [Appendix B](#), we model our algorithms (i.e. Greedy and Patient) as a Markov Chain and prove the existence of a stationary distribution for them. We also provide bounds for their mixing times and show that they converge to the stationary distribution rapidly. Next, and in [Appendix C](#), we prove the *upper-bounds* of Greedy and Patient algorithms. These upper-bounds are used in [Theorem 1](#) and [Theorem 4](#). In [Appendix D](#), we prove the bounds of all optimum algorithms: OPT, OPT<sup>c</sup> and OMN. This will prove [Theorem 2](#), [Theorem 3](#) and the remaining piece of [Theorem 4](#). In [Appendix E](#), we fully analyze the Patient-Mechanism and prove [Theorem 5](#).

Our analysis of [Appendix C](#) employs a series of mathematical lemmas, all of which are proved in [Appendix F](#). In [Appendix G](#) we study an algorithm that partially exploits the network structure. [Appendix H](#) presents small market simulations. [Appendix I](#) covers some known inequalities that we have exploited in proofs.

## B Modeling a Matching Algorithm as a Markov Chain

In this section, we establish that under both of the Patient and Greedy algorithms the random processes  $Z_t$  are Markovian, have unique stationary distributions, and mix rapidly to the stationary distribution. To do so, this section contains a brief overview on continuous time Markov Chains. We refer interested readers to [Norris \(1998\)](#); [Levin et al. \(2006\)](#) for detailed discussions.

First, we argue that the pool size  $Z_t$  is a Markov process under the Patient and Greedy algorithms. This follows from the following simple observation.

**Proposition 6.** *Under either of Greedy or Patient algorithms, for any  $t \geq 0$ , conditioned on  $Z_t$ , the distribution of  $G_t$  is uniquely defined. So, given  $Z_t$ ,  $G_t$  is conditionally independent of  $Z_{t'}$  for  $t' < t$ .*

*Proof.* Under the Greedy algorithm, at any time  $t \geq 0$ ,  $|E_t| = 0$ . Therefore, conditioned on  $Z_t$ ,  $G_t$  is an empty graph with  $|Z_t|$  vertices.

For the Patient algorithm, note that the algorithm never looks at the edges between non-critical agents, so the algorithm is oblivious to these edges. It follows that under the Patient algorithm, for any  $t \geq 0$ , conditioned on  $Z_t$ ,  $G_t$  is an Erdős-Rényi random graph with  $|Z_t|$  vertices and parameter  $d/m$ .  $\square$

Now we will review some known results from the theory of Markov chains. Let  $Z_t$  be a continuous time Markov Chain on the non-negative integers ( $\mathbb{N}$ ) that starts from state 0. For any two states  $i, j \in \mathbb{N}$ , we assume that the rate of going from  $i$  to  $j$  is  $r_{i \rightarrow j} \geq 0$ . The rate matrix  $Q \in \mathbb{N} \times \mathbb{N}$  is defined as follows,

$$Q(i, j) := \begin{cases} r_{i \rightarrow j} & \text{if } i \neq j, \\ \sum_{k \neq i} -r_{i \rightarrow k} & \text{otherwise.} \end{cases}$$

Note that, by definition, the sum of the entries in each row of  $Q$  is zero. It turns out that

(see e.g., (Norris, 1998, Theorem 2.1.1)) the transition probability in  $t$  units of time is,

$$e^{tQ} = \sum_{i=0}^{\infty} \frac{t^i Q^i}{i!}.$$

Let  $P_t := e^{tQ}$  be the transition probability matrix of the Markov Chain in  $t$  time units. It follows that,

$$\frac{d}{dt} P_t = P_t Q. \quad (\text{B.1})$$

In particular, in any infinitesimal time step  $dt$ , the chain moves based on  $Q \cdot dt$ .

A Markov Chain is irreducible if for any pair of states  $i, j \in \mathbb{N}$ ,  $j$  is reachable from  $i$  with a non-zero probability. Fix a state  $i \geq 0$ , and suppose that  $Z_{t_0} = i$ , and let  $T_1$  be the first jump out of  $i$  (note that  $T_1$  is distributed as an exponential random variable). State  $i$  is *positive recurrent* iff

$$\mathbb{E}[\inf\{t \geq T_1 : Z_t = i\} | Z_{t_0} = i] < \infty \quad (\text{B.2})$$

The ergodic theorem (Norris, 1998, Theorem 3.5.3) entails that a continuous time Markov Chain has a unique stationary distribution if and only if it has a positive recurrent state.

Let  $\pi : \mathbb{N} \rightarrow \mathbb{R}_+$  be the stationary distribution of a Markov chain. It follows by the definition that for any  $t \geq 0$ ,  $\pi = \pi P_t$ . The *balance equations* of a Markov chain say that for any  $S \subseteq \mathbb{N}$ ,

$$\sum_{i \in S, j \notin S} \pi(i) r_{i \rightarrow j} = \sum_{i \in S, j \notin S} \pi(j) r_{j \rightarrow i}. \quad (\text{B.3})$$

Let  $z_t(\cdot)$  be the distribution of  $Z_t$  at time  $t \geq 0$ , i.e.,  $z_t(i) := \mathbb{P}[Z_t = i]$  for any integer  $i \geq 0$ . For any  $\epsilon > 0$ , we define the *mixing time* (in total variation distance) of this Markov Chain as follows,

$$\tau_{\text{mix}}(\epsilon) = \inf \left\{ t : \|z_t - \pi\|_{\text{TV}} := \sum_{k=0}^{\infty} |\pi(k) - z_t(k)| \leq \epsilon \right\}. \quad (\text{B.4})$$

The following is the main theorem of this section.

**Theorem 7.** *For the Patient and Greedy algorithms and any  $0 \leq t_0 < t_1$ ,*

$$\mathbb{P}[Z_{t_1} | Z_t \text{ for } 0 \leq t < t_1] = \mathbb{P}[Z_{t_1} | Z_t \text{ for } t_0 \leq t < t_1].$$

*The corresponding Markov Chains have unique stationary distributions and mix in time*

$O(\log(m) \log(1/\epsilon))$  in total variation distance:

$$\tau_{\text{mix}}(\epsilon) \leq O(\log(m) \log(1/\epsilon)).$$

This theorem is crucial in justifying our focus on long-run results in [Section 3](#), since these Markov chains converge very rapidly (in  $O(\log(m))$  time) to their stationary distributions.

## B.1 Proof of [Theorem 7](#)

### B.1.1 Stationary Distributions: Existence and Uniqueness

In this part we show that the Markov Chain on  $Z_t$  has a unique stationary distribution under each of the Greedy and Patient algorithms. By [Proposition 6](#),  $Z_t$  is a Markov chain on the non-negative integers ( $\mathbb{N}$ ) that starts from state zero.

First, we show that the Markov Chain is irreducible. First note that every state  $i > 0$  is reachable from state 0 with a non-zero probability. It is sufficient that  $i$  agents arrive at the market with no acceptable bilateral transactions. On the other hand, state 0 is reachable from any  $i > 0$  with a non-zero probability. It is sufficient that all of the  $i$  agents in the pool become critical and no new agents arrive at the market. So  $Z_t$  is an *irreducible* Markov Chain.

Therefore, by the ergodic theorem it has a unique stationary distribution if and only if it has a positive recurrent state ([Norris, 1998](#), Theorem 3.5.3). In the rest of the proof we show that state 0 is positive recurrent. It is useful to define the *inactive* algorithm, i.e., where the matching algorithm does nothing and no agents ever get matched. We adopt the notation  $\tilde{A}_t$  and  $\tilde{Z}_t$  to denote the agents in the pool and the pool size in this case. By [\(D.1\)](#)  $Z_t = 0$  if  $\tilde{Z}_t = 0$ . So, it is sufficient to show

$$\mathbb{E} \left[ \inf \{t \geq T_1 : \tilde{Z}_t = 0\} \mid \tilde{Z}_{t_0} = 0 \right] < \infty. \tag{B.5}$$

It follows that  $\tilde{Z}_t$  is just a continuous time birth-death process on  $\mathbb{N}$  with the following transition rates,

$$\tilde{r}_{k \rightarrow k+1} = m \text{ and } \tilde{r}_{k \rightarrow k-1} := k \tag{B.6}$$

It is well known (see e.g. ([Grimmett and Stirzaker, 1992](#), p. 249-250)) that  $\tilde{Z}_t$  has a stationary

distribution if and only if

$$\sum_{k=1}^{\infty} \frac{\tilde{r}_{0 \rightarrow 1} \tilde{r}_{1 \rightarrow 2} \cdots \tilde{r}_{k-1 \rightarrow k}}{\tilde{r}_{1 \rightarrow 0} \cdots \tilde{r}_{k \rightarrow k-1}} < \infty.$$

Using (B.6) we have

$$\sum_{k=1}^{\infty} \frac{\tilde{r}_{0 \rightarrow 1} \tilde{r}_{1 \rightarrow 2} \cdots \tilde{r}_{k-1 \rightarrow k}}{\tilde{r}_{1 \rightarrow 0} \cdots \tilde{r}_{k \rightarrow k-1}} = \sum_{k=1}^{\infty} \frac{m^k}{k!} = e^m - 1 < \infty$$

Therefore,  $\tilde{Z}_t$  has a stationary distribution. The ergodic theorem (Norris, 1998, Theorem 3.5.3) entails that every state in the support of the stationary distribution is positive recurrent. Thus, state 0 is positive recurrent under  $\tilde{Z}_t$ . This proves (B.5), so  $Z_t$  is an ergodic Markov Chain.

### B.1.2 Upper bounding the Mixing Times

In this part we complete the proof of Theorem 7 and provide an upper bound the mixing of Markov Chain  $Z_t$  for the Greedy and Patient algorithms. Let  $\pi(\cdot)$  be the stationary distribution of the Markov Chain.

**Mixing time of the Greedy Algorithm.** We use the *coupling* technique (see (Levin et al., 2006, Chapter 5)) to get an upper bound for the mixing time of the Greedy algorithm. Suppose we have two Markov Chains  $Y_t, Z_t$  (with different starting distributions) each running the Greedy algorithm. We define a joint Markov Chain  $(Y_t, Z_t)_{t=0}^{\infty}$  with the property that projecting on either of  $Y_t$  and  $Z_t$  we see the stochastic process of Greedy algorithm, and that they stay together at all times after their first simultaneous visit to a single state, i.e.,

$$\text{if } Y_{t_0} = Z_{t_0}, \text{ then } Y_t = Z_t \text{ for } t \geq t_0.$$

Next we define the joint chain. We define this chain such that for any  $t \geq t_0$ ,  $|Y_t - Z_t| \leq |Y_{t_0} - Z_{t_0}|$ . Assume that  $Y_{t_0} = y, Z_{t_0} = z$  at some time  $t_0 \geq 0$ , for  $y, z \in \mathbb{N}$ . Without loss of generality assume  $y < z$  (note that if  $y = z$  there is nothing to define). Consider any arbitrary labeling of the agents in the first pool with  $a_1, \dots, a_y$ , and in the second pool with  $b_1, \dots, b_z$ . Define  $z + 1$  independent exponential clocks such that the first  $z$  clocks have rate 1, and the last one has rate  $m$ . If the  $i$ -th clock ticks for  $1 \leq i \leq y$ , then both of  $a_i$  and  $b_i$  become critical (recall that in the Greedy algorithm the critical agent leaves the market right away). If  $y < i \leq z$ , then  $b_i$  becomes critical, and if  $i = z + 1$  new agents  $a_{y+1}, b_{z+1}$  arrive



to the markets. In the latter case we need to draw edges between the new agents and those currently in the pool. We use  $z$  independent coins each with parameter  $d/m$ . We use the first  $y$  coins to decide simultaneously on the potential transactions  $(a_i, a_{y+1})$  and  $(b_i, b_{z+1})$  for  $1 \leq i \leq y$ , and the last  $z - y$  coins for the rest. This implies that for any  $1 \leq i \leq y$ ,  $(a_i, a_{y+1})$  is an acceptable transaction iff  $(b_i, b_{z+1})$  is acceptable. Observe that if  $a_{y+1}$  has at least one acceptable transaction then so has  $b_{z+1}$  but the converse does not necessarily hold.

It follows from the above construction that  $|Y_t - Z_t|$  is a non-increasing function of  $t$ . Furthermore, this value decreases when either of the agents  $b_{y+1}, \dots, b_z$  become critical (we note that this value may also decrease when a new agent arrives but we do not exploit this situation here). Now suppose  $|Y_0 - Z_0| = k$ . It follows that the two chains arrive to the same state when all of the  $k$  agents that are not in common become critical. This has the same distribution as the maximum of  $k$  independent exponential random variables with rate 1. Let  $E_k$  be a random variable that is the maximum of  $k$  independent exponentials of rate 1. For any  $t \geq 0$ ,

$$\mathbb{P}[Z_t \neq Y_t] \leq \mathbb{P}[E_{|Y_0 - Z_0|} \geq t] = 1 - (1 - e^{-t})^{|Y_0 - Z_0|}. \quad (\text{B.7})$$

Now, we are ready to bound the mixing time of the Greedy algorithm. Let  $z_t(\cdot)$  be the distribution of the pool size at time  $t$  when there is no agent in the pool at time 0 and let  $\pi(\cdot)$  be the stationary distribution. Fix  $0 < \epsilon < 1/4$ , and let  $\beta \geq 0$  be a parameter that we fix later. Let  $(Y_t, Z_t)$  be the joint Markov chain that we constructed above where  $Y_t$  is started at the stationary distribution and  $Z_t$  is started at state zero. Then,

$$\begin{aligned} \|z_t - \pi\|_{\text{TV}} \leq \mathbb{P}[Y_t \neq Z_t] &= \sum_{i=0}^{\infty} \pi(i) \mathbb{P}[Y_t \neq Z_t | Y_0 = i] \\ &\leq \sum_{i=0}^{\infty} \pi(i) \mathbb{P}[E_i \geq t] \\ &\leq \sum_{i=0}^{\beta m/d} (1 - (1 - e^{-t})^{\beta m/d}) + \sum_{i=\beta m/d}^{\infty} \pi(i) \leq \frac{\beta^2 m^2}{d^2} e^{-t} + 2e^{-m(\beta-1)^2/2d} \end{aligned}$$

where the first inequality follows from [Equation B.7](#) (since  $Z_t$  starts at zero) and the last inequality follows by [equation \(I.4\)](#) and [Proposition 12](#). Letting  $\beta = 1 + \sqrt{2 \log(2/\epsilon)}$  and  $t = 2 \log(\beta m/d) \cdot \log(2/\epsilon)$  we get  $\|z_t - \pi\|_{\text{TV}} \leq \epsilon$ , which proves the theorem.

**Mixing time of the Patient Algorithm.** It remains to bound the mixing time of the Patient algorithm. The construction of the joint Markov Chain is very similar to the above construction except some caveats. Again, suppose  $Y_{t_0} = y$  and  $Z_{t_0} = z$  for  $y, z \in \mathbb{N}$  and  $t_0 \geq 0$  and that  $y < z$ . Let  $a_1, \dots, a_y$  and  $b_1, \dots, b_z$  be a labeling of the agents. We consider two cases.

Case 1)  $z > y + 1$ . In this case the construction is essentially the same as the Greedy algorithm. The only difference is that we toss random coins to decide on acceptable bilateral transactions at the time that an agent becomes critical (and not at the time of arrival). It follows that when new agents arrive the size of each of the pools increase by 1 (so the difference remains unchanged). If any of the agents  $b_{y+1}, \dots, b_z$  become critical then the size of second pool decrease by 1 or 2 and so is the difference of the pool sizes.

Case 2)  $z = y + 1$ . In this case we define a slightly different coupling. This is because, for some parameters and starting values, the Markov chains may not visit the same state for a long time for the coupling defined in Case 1. If  $z \gg m/d$ , then with a high probability any critical agent gets matched. Therefore, the magnitude of  $|Z_t - Y_t|$  does not quickly decrease<sup>20</sup>. Therefore, in this case we change the coupling. We use  $z + 2$  independent clocks where the first  $z$  are the same as before, i.e., they have rate 1 and when the  $i$ -th clock ticks  $b_i$  (and  $a_i$  if  $i \leq y$ ) become critical. The last two clocks have rate  $m$ , when the  $z + 1$ -st clock ticks a new agent arrives to the first pool and when  $z + 2$ -nd one ticks a new agent arrives to the second pool.

Let  $|Y_0 - Z_0| = k$ . By the above construction  $|Y_t - Z_t|$  is a decreasing function of  $t$  unless  $|Y_t - Z_t| = 1$ . In the latter case this difference goes to zero if a new agent arrives to the smaller pool and it increases if a new agent arrives to the bigger pool. Let  $\tau$  be the first time  $t$  where  $|Y_t - Z_t| = 1$ . Similar to the Greedy algorithm, the event  $|Y_t - Z_t| = 1$  occurs if the second to maximum of  $k$  independent exponential random variables with rate 1 is at most  $t$ . So, the probability that  $\tau > t$  is at most the probability that the maximum of  $k$  independent exponential random variables with rate 1 is more than  $t$ . Therefore,

$$\mathbb{P} \left[ \tau > t \mid |Y_0 - Z_0| = k \right] \leq \mathbb{P} [E_k > t] = 1 - (1 - e^{-t})^k$$

---

<sup>20</sup>For a concrete example, consider the case where  $d = m$ ,  $y = m/2$  and  $z = m/2 + 1$ : In this case whenever  $b_z$  gets critical it will be matched with high probability. So  $Z_t$  decreases by 2 and  $Y_t$  remains invariant. Therefore, the absolute value of the difference remains 1.

Now, suppose  $t \geq \tau$ ; we need to bound the time it takes to make the difference zero. First, note that after time  $\tau$  the difference is never more than 2. Let  $X_t$  be the (continuous time) Markov Chain illustrated in [Figure 3](#) and suppose  $X_0 = 1$ .

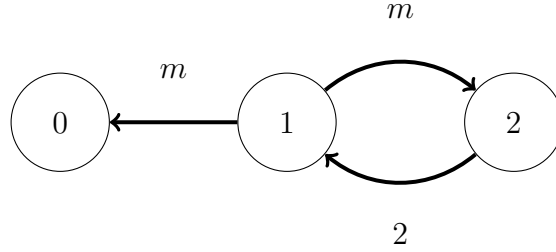


Figure 3: A three state Markov Chain used for analyzing the mixing time of the Patient algorithm.

We define a coupling between the  $X_t$  chain and the  $(Y_{t+\tau}, Z_{t+\tau})$  chain such for any  $t$ ,  $X_t \geq |Y_{t+\tau} - Z_{t+\tau}|$ . This, in particular, implies that when  $X_t = 0$ , then  $Y_{t+\tau} = Z_{t+\tau}$ . Now, let us define the coupling: When  $X_t = 1$ , we run two independent exponential clock each with rate  $m$ , if the first one ticks first we go to 0, otherwise we go to 2 (these clocks are exactly the last two clocks that we used in Case 2). When, we are at 2, we run two exponential clocks of rate 1 (corresponding to  $z - 1, z$  agents) when one of them ticks we move to 1. Note that there are other cases that we can transfer to  $Y_{t+\tau} = Z_{t+\tau}$  from a state where the difference is 1 or 2 but we do not consider those in the  $X_t$  chain. Hence,  $X_t = 0$  implies  $Y_{t+\tau} = Z_{t+\tau}$ .

**Claim 8.**  $\mathbb{P}[X_t \neq 0] \leq 1.3^{-t/2+1}$ .

*Proof.* Instead of analyzing the  $X_t$  chain we analyze a chain that goes to 0 slower but it is easier to analyze. Let  $U_t$  be the chain in [Figure 4](#). Note that starting from 1, this chain goes to 0 slower, because the number of times it goes out of state 1 is less than the  $X_t$  chain, and similar to the  $X_t$  chain whenever we go out of 1 we choose between 0, 2 uniformly at random. Therefore,  $\mathbb{P}[X_t \neq 0] \leq \mathbb{P}[U_t \neq 0]$ . To analyze the time to get 0 in the  $U_t$  chain we

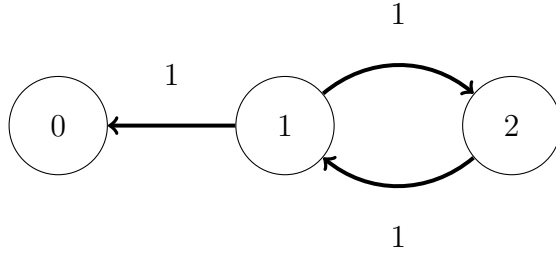


Figure 4: A simpler three state chain where all edges have rate 1

simply use the following formula (see (Levin et al., 2006, Chapter 20) for details)

$$\begin{aligned}
\mathbb{P}[U_t \neq 0] &= \sum_{k=0}^{\infty} \frac{e^{-t} t^k}{k!} 2^{-\lceil k/2 \rceil} \\
&\leq \sum_{k=0}^{t/2} \frac{e^{-t} t^k}{k!} + \sum_{k=t/2+1}^{\infty} \frac{e^{-t} t^k}{k!} 2^{-t/4} \\
&\leq \sum_{k=0}^{t/2} \frac{e^{-t} t^k}{k!} + 2^{-t/4} \\
&\leq \frac{e^{-t} (et)^{t/2}}{(t/2)^{t/2}} + 2^{-t/4} \\
&= e^{-t/2} 2^{t/2} + 2^{-t/4} \leq 1.3^{-t/2} + 2^{-t/4} \leq 1.3^{-t/2+1}. \tag{B.8}
\end{aligned}$$

The second inequality follows by the fact that  $\frac{e^{-t} t^k}{k!}$  is the density function of a Poisson random variable so it sums up to 1 for  $k = 0, 1, \dots, \infty$ . The third inequality follows by a concentration of Poisson random variables: For any Poisson random variable  $X$  with rate  $\lambda$  and  $x < \lambda$ ,  $\mathbb{P}[X \leq x] \leq \frac{e^{-\lambda} (\lambda)^x}{x^x}$  (see Mitzenmacher and Upfal (2005), page 97).  $\square$

Now, we are ready to upper-bound the mixing time of the Patient algorithm. Let  $z_t(\cdot)$  be the distribution of the pool size at time  $t$  where there is no agent at time 0, and let  $\pi(\cdot)$  be the stationary distribution. Fix  $\epsilon > 0$ , and let  $\beta \geq 2$  be a parameter that we fix later. Let  $(Y_t, Z_t)$  be the joint chain that we constructed above where  $Y_t$  is started at the stationary

distribution and  $Z_t$  is started at state zero.

$$\begin{aligned}
\|z_t - \pi\|_{\text{TV}} \leq \mathbb{P}[Z_t \neq Y_t] &\leq \mathbb{P}[\tau > t/2] + \mathbb{P}[X_{t/2} \neq 0] \\
&\leq \sum_{i=0}^{\infty} \pi(k) \mathbb{P}[\tau > t/2 | Z_0 = 0, Y_0 = k] + 1.3^{-t/2+1} \\
&\leq \sum_{k=0}^{\infty} \pi(k) (1 - (1 - e^{-t/2})^k) + 1.3^{-t/2+1} \\
&\leq \sum_{k=0}^{\beta m} k e^{-t/2} + \sum_{i=\beta m}^{\infty} \pi(i) + 1.3^{-t/2+1} \\
&\leq \beta^2 m^2 e^{-t/2} + 6e^{-(\beta-1)m/3} + 1.3^{-t/2+1}.
\end{aligned}$$

where in the second to last equation we used equation (I.4) and in the last equation we used [Proposition 16](#). Letting  $\beta = 4 \log(18/\epsilon)$  and  $t = 8 \log(4m/\epsilon)$  implies that  $\|z_t - \pi\|_{\text{TV}} \leq \epsilon$  which proves [Theorem 7](#).

## C Greedy and Patient: Performance Analysis

In this section we upper bound  $\mathbf{L}(\text{Greedy})$  and  $\mathbf{L}(\text{Patient})$  as a function of  $d$ . Note that the result that  $\mathbf{L}(\text{Greedy}) \geq \frac{1}{2d+1}$  is simply a corollary of the lower-bound on the class of DAG algorithms.

We prove the following three theorems.<sup>21</sup>

**Theorem 9.** *For any  $\epsilon \geq 0$  and  $T > 0$ ,*

$$\mathbf{L}(\text{Greedy}) \leq \frac{\log(2)}{d} + \frac{\tau_{\text{mix}}(\epsilon)}{T} + 6\epsilon + O\left(\frac{\log(m/d)}{\sqrt{dm}}\right), \quad (\text{C.1})$$

where  $\tau_{\text{mix}}(\epsilon) \leq 2 \log(m/d) \log(2/\epsilon)$ .

**Theorem 10.** *For any  $\epsilon > 0$  and  $T > 0$ ,*

$$\mathbf{L}(\text{Patient}) \leq \max_{z \in [1/2, 1]} \left( z + \tilde{O}(1/\sqrt{m}) \right) e^{-zd} + \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{\epsilon m}{d^2} + 2/m, \quad (\text{C.2})$$

where  $\tau_{\text{mix}}(\epsilon) \leq 8 \log(m) \log(4/\epsilon)$ .

---

<sup>21</sup>We use the operators  $O$  and  $\tilde{O}$  in the standard way. That is,  $f(m) = O(g(m))$  iff there exists a positive real number  $N$  and a real number  $m_0$  such that  $|f(m)| \leq N|g(m)|$  for all  $m \geq m_0$ .  $\tilde{O}$  is similar but ignores logarithmic factors, i.e.  $f(m) = \tilde{O}(g(m))$  iff  $f(m) = O(g(m) \log^k g(m))$  for some  $k$ .

Setting  $\epsilon$  small enough such that  $\epsilon m \rightarrow 0$  (e.g.  $\epsilon = 1/m^2$  or  $\epsilon = 2^{-m}$ ) implies the second part of [Theorem 1](#). This is because when  $m$  grows and for  $\epsilon m \rightarrow 0$ , the loss of the Patient algorithm is upper bounded by  $\max_{z \in [1/2, 1]} z e^{-zd}$ , which is maximized at  $z = 1/2$  for  $d \geq 2$ . Hence,  $\mathbf{L}(\text{Patient}) \leq e^{-d/2}/2$ .

We will prove [Theorem 9](#) in [Subsection C.1](#) and [Theorem 10](#) in [Subsection C.2](#). Note that the limit results of [Section 3](#) are derived by taking limits from [Equation C.1](#) and [Equation C.2](#) (as  $T, m \rightarrow \infty$ ).

## C.1 Loss of the Greedy Algorithm

In this part we upper bound  $\mathbf{L}(\text{Greedy})$ . We crucially exploit the fact that  $Z_t$  is a Markov Chain and has a unique stationary distribution,  $\pi : \mathbb{N} \rightarrow \mathbb{R}_+$ . Our proof proceeds in three steps: First, we show that  $\mathbf{L}(\text{Greedy})$  is bounded by a function of the expected pool size. Second, we show that the stationary distribution is highly concentrated around some point  $k^*$ , which we characterize. Third, we show that  $k^*$  is close to the expected pool size.

Let  $\zeta := \mathbb{E}_{Z \sim \mu} [Z]$  be the expected size of the pool under the stationary distribution of the Markov Chain on  $Z_t$ . First, observe that if the Markov Chain on  $Z_t$  is mixed, then the agents perish at the rate of  $\zeta$ , as the pool is almost always an empty graph under the Greedy algorithm. Roughly speaking, if we run the Greedy algorithm for a sufficiently long time then Markov Chain on size of the pool mixes and we get  $\mathbf{L}(\text{Greedy}) \simeq \frac{\zeta}{m}$ . This observation is made rigorous in the following lemma. Note that as  $T$  and  $m$  grow, the first three terms become negligible.

**Lemma 11.** *For any  $\epsilon > 0$ , and  $T > 0$ ,*

$$\mathbf{L}(\text{Greedy}) \leq \frac{\tau_{\text{mix}}(\epsilon)}{T} + 6\epsilon + \frac{1}{m}2^{-6m} + \frac{\mathbb{E}_{Z \sim \pi} [Z]}{m}.$$

The theorem is proved in the [Appendix F.1](#).

The proof of the above lemma involves lots of algebra, but the intuition is as follows: The  $\frac{\mathbb{E}_{Z \sim \pi} [Z]}{m}$  term is the loss *under* the stationary distribution. This is equal to  $\mathbf{L}(\text{Greedy})$  with two approximations: First, it takes some time for the chain to transit to the stationary distribution. Second, even when the chain mixes, the distribution of the chain is not *exactly* equal to the stationary distribution. The  $\frac{\tau_{\text{mix}}(\epsilon)}{T}$  term provides an upper bound for the loss associated with the first approximation, and the term  $(6\epsilon + \frac{1}{m}2^{-6m})$  provides an upper bound for the loss associated with the second approximation.

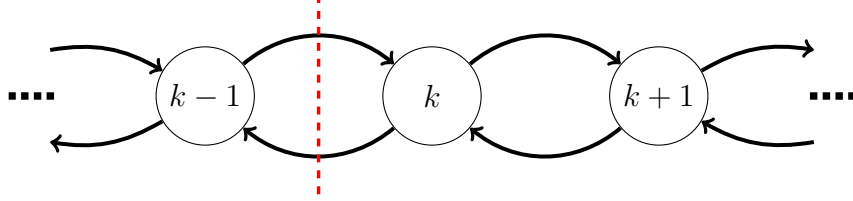


Figure 5: An illustration of the transition paths of the  $Z_t$  Markov Chain under the Greedy algorithm

Given [Lemma 11](#), in the rest of the proof we just need to get an upper bound for  $\mathbb{E}_{Z \sim \pi} [Z]$ . Unfortunately, we do not have any closed form expression of the stationary distribution,  $\pi(\cdot)$ . Instead, we use the balance equations of the Markov Chain defined on  $Z_t$  to characterize  $\pi(\cdot)$  and upper bound  $\mathbb{E}_{Z \sim \pi} [Z]$ .

Let us rigorously define the transition probability operator of the Markov Chain on  $Z_t$ . For any pool size  $k$ , the Markov Chain transits only to the states  $k+1$  or  $k-1$ . It transits to state  $k+1$  if a new agent arrives and the market-maker cannot match her (i.e., the new agent does not have any edge to the agents currently in the pool) and the Markov Chain transits to the state  $k-1$  if a new agent arrives and is matched or an agent currently in the pool gets critical. Thus, the transition rates  $r_{k \rightarrow k+1}$  and  $r_{k \rightarrow k-1}$  are defined as follows,

$$r_{k \rightarrow k+1} := m \left(1 - \frac{d}{m}\right)^k \quad (\text{C.3})$$

$$r_{k \rightarrow k-1} := k + m \left(1 - \left(1 - \frac{d}{m}\right)^k\right). \quad (\text{C.4})$$

In the above equations we used the fact that agents arrive at rate  $m$ , they perish at rate 1 and the probability of an acceptable transaction between two agents is  $d/m$ .

Let us write down the balance equation for the above Markov Chain (see equation [\(B.3\)](#) for the full generality). Consider the cut separating the states  $0, 1, 2, \dots, k-1$  from the rest (see [Figure 5](#) for an illustration). It follows that,

$$\pi(k-1)r_{k-1 \rightarrow k} = \pi(k)r_{k \rightarrow k-1}. \quad (\text{C.5})$$

Now, we are ready to characterize the stationary distribution  $\pi(\cdot)$ . In the following proposition we show that there is a number  $k^* \leq \log(2)m/d$  such that under the stationary distribution, the size of the pool is highly concentrated in an interval of length  $O(\sqrt{m/d})$  around  $k^*$ .<sup>22</sup>

<sup>22</sup>In this paper,  $\log x$  refers to the natural log of  $x$ .

**Proposition 12.** *There exists  $m/(2d+1) \leq k^* < \log(2)m/d$  such that for any  $\sigma > 1$ ,*

$$\mathbb{P}_\pi \left[ k^* - \sigma \sqrt{2m/d} \leq Z \leq k^* + \sigma \sqrt{2m/d} \right] \geq 1 - O(\sqrt{m/d})e^{-\sigma^2}.$$

*Proof.* Let us define  $f : \mathbb{R} \rightarrow \mathbb{R}$  as an interpolation of the difference of transition rates over the reals,

$$f(x) := m(1 - d/m)^x - (x + m(1 - (1 - d/m)^x)).$$

In particular, observe that  $f(k) = r_{k \rightarrow k+1} - r_{k \rightarrow k-1}$ . The above function is a decreasing convex function over non-negative reals. We define  $k^*$  as the unique root of this function. Let  $k_{\min}^* := m/(2d+1)$  and  $k_{\max}^* := \log(2)m/d$ . We show that  $f(k_{\min}^*) \geq 0$  and  $f(k_{\max}^*) \leq 0$ . This shows that  $k_{\min}^* \leq k^* < k_{\max}^*$ .

$$\begin{aligned} f(k_{\min}^*) &\geq -k_{\min}^* - m + 2m(1 - d/m)^{k_{\min}^*} \geq 2m \left( 1 - \frac{k_{\min}^* d}{m} \right) - k_{\min}^* - m = 0, \\ f(k_{\max}^*) &\leq -k_{\max}^* - m + 2m(1 - d/m)^{k_{\max}^*} \leq -k_{\max}^* - m + 2me^{-(k_{\max}^*)d/m} = -k_{\max}^* \leq 0. \end{aligned}$$

In the first inequality we used equation (I.4) from [Appendix I](#).

It remains to show that  $\pi$  is highly concentrated around  $k^*$ . In the following lemma, we show that stationary probabilities decrease geometrically.

**Lemma 13.** *For any integer  $k \geq k^*$*

$$\frac{\pi(k+1)}{\pi(k)} \leq e^{-(k-k^*)d/m}.$$

*And, for any  $k \leq k^*$ ,  $\pi(k-1)/\pi(k) \leq e^{-(k^*-k+1)d/m}$ .*

This has been proved in [Subsection F.2](#).

By repeated application of the above lemma, for any integer  $k \geq k^*$ , we get<sup>23</sup>

$$\pi(k) \leq \frac{\pi(k)}{\pi(\lceil k^* \rceil)} \leq \exp \left( -\frac{d}{m} \sum_{i=\lceil k^* \rceil}^{k-1} (i - k^*) \right) \leq \exp(-d(k - k^* - 1)^2/2m). \quad (\text{C.6})$$

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<sup>23</sup> $\lceil k^* \rceil$  indicates the smallest integer larger than  $k^*$ .



We are almost done. For any  $\sigma > 0$ ,

$$\begin{aligned} \sum_{k=k^*+1+\sigma\sqrt{2m/d}}^{\infty} \pi(k) &\leq \sum_{k=k^*+1+\sigma\sqrt{2m/d}}^{\infty} e^{-d(k-k^*-1)^2/2m} = \sum_{k=0}^{\infty} e^{-d(k+\sigma\sqrt{2m/d})^2/2m} \\ &\leq \frac{e^{-\sigma^2}}{\min\{1/2, \sigma\sqrt{d/2m}\}} \end{aligned}$$

The last inequality uses equation (I.1) from Appendix I. We can similarly upper bound  $\sum_{k=0}^{k^*-\sigma\sqrt{2m/d}} \pi(k)$ .  $\square$

**Proposition 12** shows that the probability that the size of the pool falls outside an interval of length  $O(\sqrt{m/d})$  around  $k^*$  drops exponentially fast as the market size grows. We also remark that the upper bound on  $k^*$  becomes tight as  $d$  goes to infinity.

The following lemma exploits **Proposition 12** to show that the expected value of the pool size under the stationary distribution is close to  $k^*$ .

**Lemma 14.** *For  $k^*$  as in **Proposition 12**,*

$$\mathbb{E}_{Z \sim \pi} [Z] \leq k^* + O(\sqrt{m/d} \log(m/d)).$$

This has been proved in **Subsection F.3**.

Now, **Theorem 9** follows immediately by **Lemma 11** and **Lemma 14** because we have

$$\frac{\mathbb{E}_{Z \sim \pi} [Z]}{m} \leq \frac{1}{m}(k^* + O(\sqrt{m} \log m)) \leq \frac{\log(2)}{d} + o(1)$$

## C.2 Loss of the Patient Algorithm

Let  $\pi : \mathbb{N} \rightarrow \mathbb{R}_+$  be the unique stationary distribution of the Markov Chain on  $Z_t$ , and let  $\zeta := \mathbb{E}_{Z \sim \pi} [Z]$  be the expected size of the pool under that distribution.

Once more our proof strategy proceeds in three steps. First, we show that  $\mathbf{L}(\text{Patient})$  is bounded by a function of  $\mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^{Z-1}]$ . Second, we show that the stationary distribution of  $Z_t$  is highly concentrated around some point  $k^*$ . Third, we use this concentration result to produce an upper bound for  $\mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^{Z-1}]$ .

By **Proposition 6**, at any point in time  $G_t$  is an Erdős-Rényi random graph. Thus, once an agent becomes critical, he has no acceptable transactions with probability  $(1 - d/m)^{Z_t-1}$ . Since each agent becomes critical with rate 1, if we run Patient for a sufficiently long time, then  $\mathbf{L}(\text{Patient}) \approx \frac{\zeta}{m}(1 - d/m)^{\zeta-1}$ . The following lemma makes the above discussion rigorous.

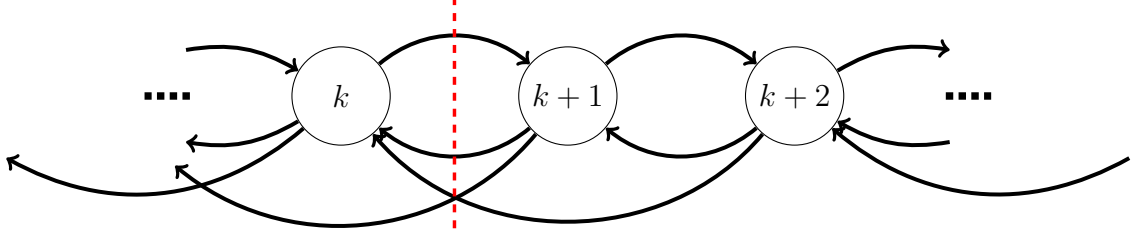


Figure 6: An illustration of the transition paths of the  $Z_t$  Markov Chain under the Patient Algorithm

**Lemma 15.** For any  $\epsilon > 0$  and  $T > 0$ ,

$$\mathbf{L}(\text{Patient}) \leq \frac{1}{m} \mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^{Z-1}] + \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{\epsilon m}{d^2}.$$

*Proof.* See Appendix F.4. □

So in the rest of the proof we just need to lower bound  $\mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^{Z-1}]$ . As in the Greedy case, we do not have a closed form expression for the stationary distribution,  $\pi(\cdot)$ . Instead, we use the balance equations of the Markov Chain on  $Z_t$  to show that  $\pi$  is highly concentrated around a number  $k^*$  where  $k^* \in [m/2, m]$ .

Let us start by defining the transition probability operator of the Markov Chain on  $Z_t$ . For any pool size  $k$ , the Markov Chain transits only to states  $k + 1$ ,  $k - 1$ , or  $k - 2$ . The Markov Chain transits to state  $k + 1$  if a new agent arrives, to the state  $k - 1$  if an agent gets critical and the planner cannot match him, and it transits to state  $k - 2$  if an agent gets critical and the planner matches him.

Remember that agents arrive with the rate  $m$ , they become critical with the rate of 1 and the probability of an acceptable transaction between two agents is  $d/m$ . Thus, the transition rates  $r_{k \rightarrow k+1}$ ,  $r_{k \rightarrow k-1}$ , and  $r_{k \rightarrow k-2}$  are defined as follows,

$$r_{k \rightarrow k+1} := m \tag{C.7}$$

$$r_{k \rightarrow k-1} := k \left(1 - \frac{d}{m}\right)^{k-1} \tag{C.8}$$

$$r_{k \rightarrow k-2} := k \left(1 - \left(1 - \frac{d}{m}\right)^{k-1}\right). \tag{C.9}$$

Let us write down the balance equation for the above Markov Chain (see equation (B.3) for the full generality). Consider the cut separating the states  $0, 1, 2, \dots, k$  from the rest (see

Figure 6 for an illustration). It follows that

$$\pi(k)r_{k \rightarrow k+1} = \pi(k+1)r_{k+1 \rightarrow k} + \pi(k+1)r_{k+1 \rightarrow k-1} + \pi(k+2)r_{k+2 \rightarrow k} \quad (\text{C.10})$$

Now we can characterize  $\pi(\cdot)$ . We show that under the stationary distribution, the size of the pool is highly concentrated around a number  $k^*$  where  $k^* \in [m/2 - 2, m - 1]$ . Remember that under the Greedy algorithm, the concentration was around  $k^* \in [\frac{m}{2d+1}, \frac{\log(2)m}{d}]$ , whereas here it is at least  $m/2$ .

**Proposition 16** (Patient Concentration). *There exists a number  $m/2 - 2 \leq k^* \leq m - 1$  such that for any  $\sigma \geq 1$ ,*

$$\mathbb{P}_\pi \left[ k^* - \sigma\sqrt{4m} \leq Z \right] \geq 1 - 2\sqrt{m}e^{-\sigma^2}, \quad \mathbb{P} \left[ Z \leq k^* + \sigma\sqrt{4m} \right] \geq 1 - 8\sqrt{m}e^{-\frac{\sigma^2\sqrt{m}}{2\sigma+\sqrt{m}}}.$$

*Proof Overview.* The proof idea is similar to Proposition 12. First, let us rewrite (C.10) by replacing transition probabilities from (C.7), (C.8), and (C.9):

$$m\pi(k) = (k+1)\pi(k+1) + (k+2)\left(1 - \left(1 - \frac{d}{m}\right)^{k+1}\right)\pi(k+2) \quad (\text{C.11})$$

Let us define a continuous  $f : \mathbb{R} \rightarrow \mathbb{R}$  as follows,

$$f(x) := m - (x+1) - (x+2)\left(1 - \left(1 - \frac{d}{m}\right)^{x+1}\right). \quad (\text{C.12})$$

It follows that

$$f(m-1) \leq 0, f(m/2-2) > 0,$$

so  $f(\cdot)$  has a root  $k^*$  such that  $m/2 - 2 < k^* < m$ . In the rest of the proof we show that the states that are far from  $k^*$  have very small probability in the stationary distribution, which completes the proof of Proposition 16. This part of the proof involves lots of algebra and is essentially very similar to the proof of the Proposition 12. We refer the interested reader to the Subsection F.5 for the complete proof of this last step.  $\square$

Since the stationary distribution of  $Z_t$  is highly concentrated around  $k^* \in [m/2 - 2, m - 1]$  by the above proposition, we derive the following upper bound for  $\mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^Z]$ , which is proved in the Appendix F.6.

**Lemma 17.** For any  $d \geq 0$  and sufficiently large  $m$ ,

$$\mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^Z] \leq \max_{z \in [m/2, m]} (z + \tilde{O}(\sqrt{m}))(1 - d/m)^z + 2.$$

Now [Theorem 10](#) follows immediately by combining [Lemma 15](#) and [Lemma 17](#).

## D Performance of the Optimum Algorithms

In this section we lower-bound the loss of the optimum solutions. In particular, we prove the following theorems.

**Theorem 18.** If  $m > 2d$ , then for any  $T > 0$

$$\mathbf{L}(\text{OPT}) \geq \frac{1}{2d + 1 + d^2/m}.$$

**Theorem 19.** If  $m > 2d$ , then for any  $T > 0$ ,

$$\mathbf{L}(\text{OMN}) \geq \frac{e^{-d-d^2/m}}{d + 1 + d^2/m}$$

Before proving the above theorems, it is useful to study the evolution of the system in the case of the *inactive* algorithm, i.e., where the algorithm does nothing and no agents ever get matched. We later use this analysis in this section, as well as [Appendix B](#) and [Appendix C](#).

We adopt the notation  $\tilde{A}_t$  and  $\tilde{Z}_t$  to denote the agents in the pool and the pool size in this case. Observe that by definition for any matching algorithm and any realization of the process,

$$Z_t \leq \tilde{Z}_t. \tag{D.1}$$

Using the above equation, in the following fact we show that for any matching algorithm  $\mathbb{E}[Z_t] \leq m$ .

**Proposition 20.** For any  $t_0 \geq 0$ ,

$$\mathbb{P}[\tilde{Z}_{t_0} = \ell] \leq \frac{m^\ell}{\ell!}.$$

Therefore,  $\tilde{Z}_t$  is distributed as a Poisson random variable of rate  $m(1 - e^{-t_0})$ , so

$$\mathbb{E} \left[ \tilde{Z}_{t_0} \right] = (1 - e^{-t_0})m.$$

*Proof.* Let  $K$  be a random variable indicating the number agents who enter the pool in the interval  $[0, t_0]$ . By Bayes rule,

$$\mathbb{P} \left[ \tilde{Z}_{t_0} = \ell \right] = \sum_{k=0}^{\infty} \mathbb{P} \left[ \tilde{Z}_{t_0} = \ell, K = k \right] = \sum_{k=0}^{\infty} \mathbb{P} \left[ \tilde{Z}_{t_0} = \ell | K = k \right] \cdot \frac{(mt_0)^k e^{-mt_0}}{k!},$$

where the last equation follows by the fact that arrival rate of the agents is a Poisson random variable of rate  $m$ .

Now, conditioned on the event that an agent  $a$  arrives in the interval  $[0, t_0]$ , the probability that she is in the pool at time  $t_0$  is at least,

$$\mathbb{P} [X_{a_i} = 1] = \int_{t=0}^{t_0} \frac{1}{t_0} \mathbb{P} [s(a_i) \geq t_0 - t] dt = \frac{1}{t_0} \int_{t=0}^{t_0} e^{t-t_0} dt = \frac{1 - e^{-t_0}}{t_0}.$$

Therefore, conditioned on  $K = k$ , the distribution of the number of agents at time  $t_0$  is a Binomial random variable  $B(k, p)$ , where  $p := (1 - e^{-t_0})/t_0$ . Let  $\mu = m(1 - e^{-t_0})$ , we have

$$\begin{aligned} \mathbb{P} \left[ \tilde{Z}_{t_0} = \ell \right] &= \sum_{k=\ell}^{\infty} \binom{k}{\ell} \cdot p^\ell \cdot (1-p)^{k-\ell} \frac{(mt_0)^k e^{-mt_0}}{k!} \\ &= \sum_{k=\ell}^{\infty} \frac{m^k e^{-mt_0}}{\ell! (k-\ell)!} (1 - e^{-t_0})^\ell (t_0 - 1 + e^{-t_0})^{k-\ell} \\ &= \frac{m^\ell e^{-mt_0} \mu^\ell}{\ell!} \sum_{k=\ell}^{\infty} \frac{(mt_0 - \mu)^{k-\ell}}{(k-\ell)!} = \frac{\mu^\ell e^{-\mu}}{\ell!}. \end{aligned}$$

Substituting  $\mu = m(1 - e^{-t_0})$  into the above equation and noting that  $(1 - e^{-t_0}) < 1$  proves the claim.  $\square$

## D.1 Loss of OPT

In this section, we prove [Theorem 18](#). Let  $\zeta$  be the expected pool size of the OPT,

$$\zeta := \mathbb{E}_{t \sim \text{unif}[0, T]} [Z_t]$$

Since OPT does not know  $A_t^c$ , each critical agent perishes with probability 1. Therefore,

$$\mathbf{L}(\text{OPT}) = \frac{1}{m \cdot T} \mathbb{E} \left[ \int_{t=0}^T Z_t dt \right] = \frac{\zeta T}{mT} = \zeta/m. \quad (\text{D.2})$$

To finish the proof we need to lower bound  $\zeta$  by  $m/(2d + 1 + d^2/m)$ . We provide an indirect proof by showing a lower-bound on  $\mathbf{L}(\text{OPT})$  which in turn lower-bounds  $\zeta$ .

The key idea is to lower-bound the probability that an agent does not have any acceptable transactions throughout her sojourn, and this directly gives a lower-bound on  $\mathbf{L}(\text{OPT})$  as those agents cannot be matched under any algorithm, so they will all perish, except those who belong to  $A_T$ .<sup>24</sup> Since a conservative upper bounds for  $\mathbb{E}[A_T]$  is  $m$ , we will then have that the expected number of perished agents is at least  $\mathbb{P}[N(a) = \emptyset] (mT - m)$ , so  $\mathbf{L}(\text{OPT}) \geq \mathbb{P}[N(a) = \emptyset] (1 - 1/T)$ . Since we are main stating our results for large values of  $T$ , we continue this proof by taking the limit and assuming that  $1 - 1/T \simeq 1$ , and then will discuss how this will change the final result when we include it explicitly.

Fix an agent  $a \in A$ . Say  $a$  enters the market at a time  $t_0 \sim \text{unif}[0, T]$ , and  $s(a) = t$ , we can write

$$\mathbb{P}[N(a) = \emptyset] \geq \int_{t=0}^{\infty} \mathbb{P}[s(a) = t] \cdot \mathbb{E}[(1 - d/m)^{|A_{t_0}|}] \cdot \mathbb{E}[(1 - d/m)^{|A_{t_0, t+t_0}^n|}] dt \quad (\text{D.3})$$

To see the above, note that  $a$  does not have any acceptable transactions, if she doesn't have any neighbors upon arrival, and none of the new agents that arrive during her sojourn are not connected to her. Using the Jensen's inequality, we have

$$\begin{aligned} \mathbb{P}[N(a) = \emptyset] &\geq \int_{t=0}^{\infty} e^{-t} \cdot (1 - d/m)^{\mathbb{E}[Z_{t_0}]} \cdot (1 - d/m)^{\mathbb{E}[|A_{t_0, t+t_0}^n|]} dt \\ &= \int_{t=0}^{\infty} e^{-t} \cdot (1 - d/m)^{\zeta} \cdot (1 - d/m)^{mt} dt \end{aligned}$$

The last equality follows by the fact that  $\mathbb{E}[|A_{t_0, t+t_0}^n|] = mt$ . Since  $1 - x - e^{-x-x^2} = 0$  has a solution at  $x \simeq 0.6838$  and is non-negative for smaller values of  $x$ , we have that  $1 - d/m \geq e^{-d/m - d^2/m^2}$  for  $d/m < 0.5$ ,

$$\mathbf{L}(\text{OPT}) \geq \mathbb{P}[N(a) = \emptyset] \geq e^{-\zeta(d/m + d^2/m^2)} \int_{t=0}^{\infty} e^{-t(1+d+d^2/m)} dt \geq \frac{1 - \zeta(1 + d/m)d/m}{1 + d + d^2/m} \quad (\text{D.4})$$

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<sup>24</sup>We thank the anonymous referee who pointed this last point about agents in  $A_T$  to us.

Putting (D.2) and (D.4) together, for  $\beta := \zeta d/m$  we get

$$\mathbf{L}(\text{OPT}) \geq \max\left\{\frac{1 - \beta(1 + d/m)}{1 + d + d^2/m}, \frac{\beta}{d}\right\} \geq \frac{1}{2d + 1 + 2d^2/m}$$

where the last inequality follows by letting  $\beta = \frac{d}{2d+1+2d^2/m}$  be the minimizer of the middle expression.

If we do not drop the  $1 - 1/T$  term, then the only change would be that the first term inside the max function has an additional  $1 - 1/T$  term. This changes the final bound to  $\mathbf{L}(\text{OPT}) \geq \frac{1}{(1+\frac{T}{T-1})d+1+(1+\frac{T}{T-1})d^2/m}$ , which goes to  $\frac{1}{2d+1+2d^2/m}$  as  $T$  grows.

## D.2 Loss of OMN

In this section, we prove [Theorem 19](#). This demonstrates that no expansion of the planner's information can yield a faster-than-exponential decrease in losses.

The proof is very similar to [Theorem 18](#). Let  $\zeta$  be the expected pool size of the OMN,

$$\zeta := \mathbb{E}_{t \sim \text{unif}[0, T]} [Z_t].$$

By (D.1) and [Proposition 20](#),

$$\zeta \leq \mathbb{E}_{t \sim \text{unif}[0, T]} [\tilde{Z}_t] \leq m.$$

Note that (D.2) does not hold in this case because the Omniscient algorithm knows the set of critical agents at time  $t$ .

Now, fix an agent  $a \in A$ , and let us lower-bound the probability that  $N(a) = \emptyset$ . Say  $a$  enters the market at time  $t_0 \sim \text{unif}[0, T]$  and  $s(a) = t$ , then

$$\begin{aligned} \mathbb{P}[N(a) = \emptyset] &= \int_{t=0}^{\infty} \mathbb{P}[s(a) = t] \cdot \mathbb{E}[(1 - d/m)^{Z_{t_0}}] \cdot \mathbb{E}[(1 - d/m)^{|A_{t_0, t+t_0}^a|}] dt \\ &\geq \int_{t=0}^{\infty} e^{-t} (1 - d/m)^{\zeta + mt} dt \geq \frac{e^{-\zeta(1+d/m)d/m}}{1 + d + d^2/m} \geq \frac{e^{-d-d^2/m}}{1 + d + d^2/m}. \end{aligned}$$

where the first inequality uses the Jensen's inequality and the second inequality uses the fact that when  $d/m < 0.5$ ,  $1 - d/m \geq e^{-d/m - d^2/m^2}$ .

### D.3 Loss of $\text{OPT}^c$

Here we prove [Theorem 2](#) and bound  $\mathbf{L}(\text{OPT}^c)$ .

$\text{OPT}^c$  matches agents (to non-critical partners) if and only if they are critical. Suppose we run  $\text{OPT}^c$  from time 0 to  $T$ . Take any realization of the stochastic process: All critical agents depart. A non-critical agent departs if and only if he is matched. Thus, the number of non-critical agents who are matched is equal to the number of departures (denoted  $\#depart$ ) minus the number of agents who become critical (denoted  $\#critical$ ). Each matched pair consists of one critical agent and one non-critical agent, so the total number of matched agents is equal to  $2(\#depart - \#critical)$ . By the definition of loss,

$$\mathbf{L}(\text{OPT}^c) = \frac{\mathbb{E}[|A - \text{OPT}^c(T)|]}{mT} - \frac{\mathbb{E}[|A_T|]}{mT} \geq \frac{mT - 2\mathbb{E}[\#depart - \#critical]}{mT} - \frac{1}{T} \quad (\text{D.5})$$

Note that  $\mathbb{E}[\#depart] \leq \mathbb{E}[|A|] = mT$ . Therefore,

$$\mathbf{L}(\text{OPT}^c) \geq \frac{2\mathbb{E}[\#critical]}{mT} - 1 - \frac{1}{T} \quad (\text{D.6})$$

As in [Subsection D.1](#),  $\mathbb{E}[\#critical] = \mathbb{E}\left[\int_{t=0}^T Z_t dt\right] = \zeta^c T$ , where  $\zeta^c$  is the expected pool size at  $t$  for  $t$  drawn uniformly at random from  $[0, T]$ . Thus,

$$\mathbf{L}(\text{OPT}^c) \geq \frac{2\zeta^c}{m} - 1 - \frac{1}{T} \quad (\text{D.7})$$

Finally, for any ALG,  $\mathbf{L}(\text{ALG}) \geq \mathbf{L}(\text{OPT}^c) \geq \frac{2\zeta^c}{m} - 1 - \frac{1}{T}$ , which leads to:

$$\frac{m}{2}\left(1 + \frac{1}{T} + \mathbf{L}(\text{ALG})\right) \geq \zeta^c \quad (\text{D.8})$$

By the same arguments as in [Subsection D.2](#),  $\mathbf{L}(\text{OPT}^c) \geq \frac{e^{-\zeta^c(1+d/m)d/m}}{1+d+d^2/m}$ . Substituting for  $\zeta^c$  and taking  $m, T \rightarrow \infty$  finishes the proof of [Theorem 2](#).

## E Incentive-Compatible Mechanisms

In this section we design a dynamic mechanism to elicit the departure times of agents. As alluded to in [Section 5](#), we assume that agents only have statistical knowledge about the rest of the market: That is, each agent knows the market parameters  $(m, d, 1)$ , her own status (present, critical, perished), and the details of the dynamic mechanism that the



market-maker is executing. Agents do not observe the graph  $G_t$  and their prior belief is the stationary distribution.

Each agent  $a$  chooses a *mixed strategy*, that is she reports getting critical at an infinitesimal time  $[t, t + dt]$  with rate  $c_a(t)dt$ . In other words, each agent  $a$  has a clock that ticks with rate  $c_a(t)$  at time  $t$  and she reports criticality when the clock ticks. We assume each agent's strategy function,  $c_a(\cdot)$  is *well-behaved*, i.e., it is non-negative, continuously differentiable and continuously integrable. Note that since the agent can only observe the parameters of the market  $c_a(\cdot)$  can depend on any parameter in our model but this function is constant in different sample paths of the stochastic process.

A *strategy profile*  $\mathcal{C}$  is a vector of well-behaved functions for each agent in the market, that is,  $\mathcal{C} = [c_a]_{a \in A}$ . For an agent  $a$  and a strategy profile  $\mathcal{C}$ , let  $\mathbb{E}[u_{\mathcal{C}}(a)]$  be the expected utility of  $a$  under the strategy profile  $\mathcal{C}$ . Note that for any  $\mathcal{C}, a$ ,  $0 \leq \mathbb{E}[u_{\mathcal{C}}(a)] \leq 1$ . Given a strategy profile  $\mathcal{C} = [c_a]_{a \in A}$ , let  $\mathcal{C} - c_a + \tilde{c}_a$  denote a strategy profile same as  $\mathcal{C}$  but for agent  $a$  who is playing  $\tilde{c}_a$  rather than  $c_a$ . The following definition introduces our solution concept.

**Definition 5.** A strategy profile  $\mathcal{C}$  is a strong  $\epsilon$ -Nash equilibrium if for any agent  $a$  and any well-behaved function  $\tilde{c}_a(\cdot)$ ,

$$1 - \mathbb{E}[u_{\mathcal{C}}(a)] \leq (1 + \epsilon)(1 - \mathbb{E}[u_{\mathcal{C} - c_a + \tilde{c}_a}]).$$

Note that the solution concept we are introducing here is slightly different from the usual definition of an  $\epsilon$ -Nash equilibrium, where the condition is either  $\mathbb{E}[u_{\mathcal{C}}(a)] \geq \mathbb{E}[u_{\mathcal{C} - c_a + \tilde{c}_a}] - \epsilon$ , or  $\mathbb{E}[u_{\mathcal{C}}(a)] \geq (1 - \epsilon)\mathbb{E}[u_{\mathcal{C} - c_a + \tilde{c}_a}]$ . The reason that we are using  $1 - \mathbb{E}[u_{\mathcal{C}}(a)]$  as a measure of distance is because we know that under Patient( $\alpha$ ) algorithm,  $\mathbb{E}[u_{\mathcal{C}}(a)]$  is very close to 1, so  $1 - \mathbb{E}[u_{\mathcal{C}}(a)]$  is a lower-order term. Thus, this definition restricts us to a stronger equilibrium concept, which requires us to show that in equilibrium agents can neither increase their utilities, *nor* the lower-order terms associated with their utilities by a factor of more than  $\epsilon$ .

Throughout this section let  $k^* \in [m/2 - 2, m - 1]$  be the root of (C.12) as defined in Proposition 16, and let  $\beta := (1 - d/m)^{k^*}$ . In this section we show that if  $r$  (the discount rate) is no more than  $\beta$ , then the strategy vector  $c_a(t) = 0$  for all agents  $a$  and  $t$  is an  $\epsilon$ -mixed strategy Nash equilibrium for  $\epsilon$  very close to zero. In other words, if all other agents are truthful, an agent's utility from being truthful is almost as large as any other strategy.

**Theorem 21.** *If the market is at stationary and  $r \leq \beta$ , then  $c_a(t) = 0$  for all  $a, t$  is a strong  $O(d^4 \log^3(m)/\sqrt{m})$ -Nash equilibrium for Patient-Mechanism.*

The proof of the above theorem is involved but the basic idea is very easy. If an agent reports getting critical at the time of arrival she will receive a utility of  $1 - \beta$ . On the other hand, if she is truthful (assuming  $r = 0$ ) she will receive about  $1 - \beta/2$ . In the course of the proof we show that by choosing any strategy vector  $c(\cdot)$  the expected utility of an agent interpolates between these two numbers, so it is maximized when she is truthful.

The precise proof of the theorem is based on [Lemma 22](#). In this lemma, we upper-bound the utility of an agent for any arbitrary strategy, given that all other agents are truthful.

**Lemma 22.** *Let  $Z_0$  be in the stationary distribution. Suppose  $a$  enters the market at time 0. If  $r < \beta$ , and  $10d^4 \log^3(m) \leq \sqrt{m}$ , then for any well-behaved function  $c(\cdot)$ ,*

$$\mathbb{E}[u_c(a)] \leq \frac{2(1 - \beta)}{2 - \beta + r} + O\left(d^4 \log^3(m)/\sqrt{m}\right)\beta,$$

*Proof.* In this section, we present the full proof of [Lemma 22](#). We prove the lemma by writing a closed form expression for the utility of  $a$  and then upper-bounding that expression.

In the following claim we study the probability  $a$  is matched in the interval  $[t, t + \epsilon]$  and the probability that it leaves the market in that interval.

**Claim 23.** *For any time  $t \geq 0$ , and  $\epsilon > 0$ ,*

$$\mathbb{P}[a \in M_{t,t+\epsilon}] = \epsilon \cdot \mathbb{P}[a \in A_t] (2 + c(t)) \mathbb{E}[1 - (1 - d/m)^{Z_t} | a \in A_t] \pm O(\epsilon^2) \quad (\text{E.1})$$

$$\mathbb{P}[a \notin A_{t+\epsilon}, a \in A_t] = \mathbb{P}[a \in A_t] (1 - \epsilon(1 + c(t)) + \mathbb{E}[1 - (1 - d/m)^{Z_t-1} | a \in A_t]) \pm O(\epsilon^2) \quad (\text{E.2})$$

*Proof.* The claim follows from two simple observations. First,  $a$  becomes critical in the interval  $[t, t + \epsilon]$  with probability  $\epsilon \cdot \mathbb{P}[a \in A_t] (1 + c(t))$  and if he is critical he is matched with probability  $\mathbb{E}[(1 - (1 - d/m)^{Z_t-1} | a \in A_t)]$ . Second,  $a$  may also get matched (without getting critical) in the interval  $[t, t + \epsilon]$ . Observe that if an agent  $b \in A_t$  where  $b \neq a$  gets critical she will be matched with  $a$  with probability  $(1 - (1 - d/m)^{Z_t-1}) / (Z_t - 1)$ . Therefore, the probability that  $a$  is matched at  $[t, t + \epsilon]$  without getting critical is

$$\begin{aligned} \mathbb{P}[a \in A_t] \cdot \mathbb{E}\left[\epsilon \cdot (Z_t - 1) \frac{1 - (1 - d/m)^{Z_t-1}}{Z_t - 1} | a \in A_t\right] \\ = \epsilon \cdot \mathbb{P}[a \in A_t] \mathbb{E}[1 - (1 - d/m)^{Z_t-1} | a \in A_t] \end{aligned}$$

The claim follows from simple algebraic manipulations. □

We need to study the conditional expectation  $\mathbb{E} [1 - (1 - d/m)^{Z_t-1} | a \in A_t]$  to use the above claim. This is not easy in general; although the distribution of  $Z_t$  remains stationary, the distribution of  $Z_t$  conditioned on  $a \in A_t$  can be a very different distribution. So, here we prove simple upper and lower bounds on  $\mathbb{E} [1 - (1 - d/m)^{Z_t-1} | a \in A_t]$  using the concentration properties of  $Z_t$ . By the assumption of the lemma  $Z_t$  is at stationary at any time  $t \geq 0$ . Let  $k^*$  be the number defined in [Proposition 16](#), and  $\beta = (1 - d/m)^{k^*}$ . Also, let  $\sigma := \sqrt{6 \log(8m/\beta)}$ . By [Proposition 16](#), for any  $t \geq 0$ ,

$$\begin{aligned}
\mathbb{E} [1 - (1 - d/m)^{Z_t-1} | a \in A_t] &\leq \mathbb{E} [1 - (1 - d/m)^{Z_t-1} | Z_t < k^* + \sigma\sqrt{4m}, a \in A_t] \\
&\quad + \mathbb{P} [Z_t \geq k^* + \sigma\sqrt{4m} | a \in A_t] \\
&\leq 1 - (1 - d/m)^{k^* + \sigma\sqrt{4m}} + \frac{\mathbb{P} [Z_t \geq k^* + \sigma\sqrt{4m}]}{\mathbb{P} [a \in A_t]} \\
&\leq 1 - \beta + \beta(1 - (1 - d/m)^{\sigma\sqrt{4m}}) + \frac{8\sqrt{m}e^{-\sigma^2/3}}{\mathbb{P} [a \in A_t]} \\
&\leq 1 - \beta + \frac{2\sigma d\beta}{\sqrt{m}} + \frac{\beta}{m^2 \cdot \mathbb{P} [a \in A_t]} \tag{E.3}
\end{aligned}$$

In the last inequality we used [\(I.4\)](#) and the definition of  $\sigma$ . Similarly,

$$\begin{aligned}
\mathbb{E} [1 - (1 - d/m)^{Z_t-1} | a \in A_t] &\geq \mathbb{E} [1 - (1 - d/m)^{Z_t-1} | Z_t \geq k^* - \sigma\sqrt{4m}, a \in A_t] \\
&\quad \cdot \mathbb{P} [Z_t \geq k^* - \sigma\sqrt{4m} | a \in A_t] \\
&\geq (1 - (1 - d/m)^{k^* - \sigma\sqrt{4m}}) \frac{\mathbb{P} [a \in A_t] - \mathbb{P} [Z_t < k^* - \sigma\sqrt{4m}]}{\mathbb{P} [a \in A_t]} \\
&\geq 1 - \beta - \beta((1 - d/m)^{-\sigma\sqrt{4m}} - 1) - \frac{2\sqrt{m}e^{-\sigma^2}}{\mathbb{P} [a \in A_t]} \\
&\geq 1 - \beta - \frac{4d\sigma\beta}{\sqrt{m}} - \frac{\beta^3}{m^3 \cdot \mathbb{P} [a \in A_t]} \tag{E.4}
\end{aligned}$$

where in the last inequality we used [\(I.4\)](#), the assumption that  $2d\sigma \leq \sqrt{m}$  and the definition of  $\sigma$ .

Next, we write a closed form upper-bound for  $\mathbb{P} [a \in A_t]$ . Choose  $t^*$  such that  $\int_{t=0}^{t^*} (2 + c(t))dt = 2 \log(m/\beta)$ . Observe that  $t^* \leq \log(m/\beta) \leq \sigma^2/6$ . Since  $a$  leaves the market with

rate at least  $1 + c(t)$  and at most  $2 + c(t)$ , we can write

$$\frac{\beta^2}{m^2} = \exp\left(-\int_{t=0}^{t^*} (2 + c(t))dt\right) \leq \mathbb{P}[a \in A_{t^*}] \leq \exp\left(-\int_{t=0}^{t^*} (1 + c(t))dt\right) \leq \frac{\beta}{m} \quad (\text{E.5})$$

Intuitively,  $t^*$  is a moment where the expected utility of that  $a$  receives in the interval  $[t^*, \infty)$  is negligible, i.e., in the best case it is at most  $\beta/m$ .

By [Claim 23](#) and [\(E.4\)](#), for any  $t \leq t^*$ ,

$$\begin{aligned} \frac{\mathbb{P}[a \in A_{t+\epsilon}] - \mathbb{P}[a \in A_t]}{\epsilon} &\leq -\mathbb{P}[a \in A_t] \left(2 + c(t) - \beta - \frac{4d\sigma\beta}{\sqrt{m}} - \frac{\beta^3}{m^3 \cdot \mathbb{P}[a \in A_t]} \pm O(\epsilon)\right) \\ &\leq -\mathbb{P}[a \in A_t] \left(2 + c(t) - \beta - \frac{5d\sigma\beta}{\sqrt{m}} \pm O(\epsilon)\right) \end{aligned}$$

where in the last inequality we used [\(E.5\)](#). Letting  $\epsilon \rightarrow 0$ , for  $t \leq t^*$ , the above differential equation yields,

$$\mathbb{P}[a \in A_t] \leq \exp\left(-\int_{\tau=0}^t \left(2 + c(\tau) - \beta - \frac{5d\sigma\beta}{\sqrt{m}}\right) d\tau\right) \leq \exp\left(-\int_{\tau=0}^t (2 + c(\tau) - \beta) d\tau\right) + \frac{2d\sigma^3\beta}{\sqrt{m}}. \quad (\text{E.6})$$

where in the last inequality we used  $t^* \leq \sigma^2/6$ ,  $e^x \leq 1 + 2x$  for  $x \leq 1$  and lemma's assumption  $5d\sigma^2 \leq \sqrt{m}$ .

Now, we are ready to upper-bound the utility of  $a$ . By [\(E.5\)](#) the expected utility that  $a$  gains after  $t^*$  is no more than  $\beta/m$ . Therefore,

$$\begin{aligned} \mathbb{E}[u_c(a)] &\leq \frac{\beta}{m} + \int_{t=0}^{t^*} (2 + c(t)) \mathbb{E}[1 - (1 - d/m)^{Z_t-1} | a \in A_t] \mathbb{P}[a \in A_t] e^{-rt} dt \\ &\leq \frac{\beta}{m} + \int_{t=0}^{t^*} (2 + c(t)) ((1 - \beta) \mathbb{P}[a \in A_t] + \beta/\sqrt{m}) e^{-rt} dt \\ &\leq \frac{\beta}{m} + \int_{t=0}^{t^*} (2 + c(t)) \left( (1 - \beta) \exp\left(-\int_{\tau=0}^t (2 + c(\tau) - \beta) d\tau\right) + \frac{3d\sigma^3\beta}{\sqrt{m}} \right) e^{-rt} dt \\ &\leq \frac{2d\sigma^5}{\sqrt{m}} \beta + \int_{t=0}^{\infty} (1 - \beta) (2 + c(t)) \exp\left(-\int_{\tau=0}^t (2 + c(\tau) - \beta) d\tau\right) e^{-rt} dt. \end{aligned}$$

In the first inequality we used equation [\(E.3\)](#), in second inequality we used equation [\(E.6\)](#), and in the last inequality we use the definition of  $t^*$ . We have finally obtained a closed form upper-bound on the expected utility of  $a$ .

Let  $U_c(a)$  be the right hand side of the above equation. Next, we show that  $U_c(a)$  is maximized by letting  $c(t) = 0$  for all  $t$ . This will complete the proof of [Lemma 22](#). Let  $c$

be a function that maximizes  $U_c(a)$  which is not equal to zero. Suppose  $c(t) \neq 0$  for some  $t \geq 0$ . We define a function  $\tilde{c} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and we show that if  $r < \beta$ , then  $U_{\tilde{c}}(a) > U_c(a)$ . Let  $\tilde{c}$  be the following function,

$$\tilde{c}(\tau) = \begin{cases} c(\tau) & \text{if } \tau < t, \\ 0 & \text{if } t \leq \tau \leq t + \epsilon, \\ c(\tau) + c(\tau - \epsilon) & \text{if } t + \epsilon \leq \tau \leq t + 2\epsilon, \\ c(\tau) & \text{otherwise.} \end{cases}$$

In words, we push the mass of  $c(\cdot)$  in the interval  $[t, t + \epsilon]$  to the right. We remark that the above function  $\tilde{c}(\cdot)$  is not necessarily continuous so we need to smooth it out. The latter can be done without introducing any errors and we do not describe the details here. Let  $S := \int_{\tau=0}^t (1 + c(\tau) + \beta) d\tau$ . Assuming  $\tilde{c}'(t) \ll 1/\epsilon$ , we have

$$\begin{aligned} U_{\tilde{c}}(a) - U_c(a) &\geq -\epsilon \cdot c(t)(1 - \beta)e^{-S}e^{-rt} + \epsilon \cdot c(t)(1 - \beta)e^{-S - \epsilon(2 - \beta)}e^{-r(t + \epsilon)} \\ &\quad + \epsilon(1 - \beta)(2 + c(t + \epsilon))(e^{-S - \epsilon(2 - \beta)}e^{-r(t + \epsilon)} - e^{-S - \epsilon(2 + c(t) - \beta)}e^{-r(t + \epsilon)}) \\ &= -\epsilon^2 \cdot c(t)(1 - \beta)e^{-S - rt}(2 - \beta + r) + \epsilon^2(1 - \beta)(2 + c(t + \epsilon))e^{-S - rt}c(t) \\ &\geq \epsilon^2 \cdot (1 - \beta)e^{-S - rt}c(t)(\beta - r). \end{aligned}$$

Since  $r < \beta$  by the lemma's assumption, the maximizer of  $U_c(a)$  is the all zero function. Therefore, for any well-behaved function  $c(\cdot)$ ,

$$\begin{aligned} \mathbb{E}[u_c(a)] &\leq \frac{2d\sigma^5}{\sqrt{m}}\beta + \int_{t=0}^{\infty} 2(1 - \beta) \exp\left(-\int_{\tau=0}^t (2 - \beta) d\tau\right) e^{-rt} dt \\ &\leq O\left(\frac{d^4 \log^3(m)}{\sqrt{m}}\right)\beta + \frac{2(1 - \beta)}{2 - \beta + r}. \end{aligned}$$

In the last inequality we used that  $\sigma = O(\sqrt{\log(m/\beta)})$  and  $\beta \leq e^{-d}$ . This completes the proof of [Lemma 22](#). □

The proof of [Theorem 21](#) follows simply from the above analysis.

*Proof of [Theorem 21](#).* All we need to do is to lower-bound the expected utility of an agent  $a$  if she is truthful. We omit the details as they are essentially similar. So, if all agents are

truthful,

$$\mathbb{E}[u(a)] \geq \frac{2(1-\beta)}{2-\beta+r} - O\left(\frac{d^4 \log^3(m)}{\sqrt{m}}\right)\beta.$$

This shows that the strategy vector corresponding to truthful agents is a strong  $O(d^4 \log^3(m)/\sqrt{m})$ -Nash equilibrium.  $\square$

## F Proofs from Appendix C

### F.1 Proof of Lemma 11

*Proof.* By Proposition 20,  $\mathbb{E}[Z_t] \leq m$  for all  $t$ , so

$$\begin{aligned} \mathbf{L}(\text{Greedy}) &= \frac{1}{m \cdot T} \mathbb{E} \left[ \int_{t=0}^T Z_t dt \right] = \frac{1}{mT} \int_{t=0}^T \mathbb{E}[Z_t] dt \\ &\leq \frac{1}{mT} m \cdot \tau_{\text{mix}}(\epsilon) + \frac{1}{mT} \int_{t=\tau_{\text{mix}}(\epsilon)}^T \mathbb{E}[Z_t] dt \quad (\text{F.1}) \end{aligned}$$

where the second equality uses the linearity of expectation. Let  $\tilde{Z}_t$  be the number of agents in the pool at time  $t$  when we do not match any pair of agents. By (D.1),

$$\mathbb{P}[Z_t \geq i] \leq \mathbb{P}[\tilde{Z}_t \geq i].$$

Therefore, for  $t \geq \tau_{\text{mix}}(\epsilon)$ ,

$$\begin{aligned} \mathbb{E}[Z_t] &= \sum_{i=1}^{\infty} \mathbb{P}[Z_t \geq i] \leq \sum_{i=0}^{6m} \mathbb{P}[Z_t \geq i] + \sum_{i=6m+1}^{\infty} \mathbb{P}[\tilde{Z}_t \geq i] \\ &\leq \sum_{i=0}^{6m} (\mathbb{P}_{Z \sim \pi}[Z \geq i] + \epsilon) + \sum_{i=6m+1}^{\infty} \sum_{\ell=i}^{\infty} \frac{m^\ell}{\ell!} \\ &\leq \mathbb{E}_{Z \sim \pi}[Z] + \epsilon 6m + \sum_{i=6m+1}^{\infty} 2 \frac{m^i}{i!} \\ &\leq \mathbb{E}_{Z \sim \pi}[Z] + \epsilon 6m + \frac{4m^{6m}}{(6m)!} \leq \mathbb{E}_{Z \sim \pi}[Z] + \epsilon 6m + 2^{-6m}. \quad (\text{F.2}) \end{aligned}$$

where the second inequality uses  $\mathbb{P}[\tilde{Z}_t = \ell] \leq m^\ell / \ell!$  of [Proposition 20](#) and the last inequality follows by the Stirling's approximation<sup>25</sup> of  $(6m)!$ . Putting [\(F.1\)](#) and [\(F.2\)](#) proves the lemma.  $\square$

## F.2 Proof of Lemma 13

*Proof.* For  $k \geq k^*$ , by [\(C.3\)](#), [\(C.4\)](#), [\(C.5\)](#),

$$\frac{\pi(k)}{\pi(k+1)} = \frac{(k+1) + m(1 - (1 - d/m)^{k+1})}{m(1 - d/m)^k} = \frac{k - k^* + 1 - m(1 - d/m)^{k+1} + 2m(1 - d/m)^{k^*}}{m(1 - d/m)^k}$$

where we used the definition of  $k^*$ . Therefore,

$$\frac{\pi(k)}{\pi(k+1)} \geq -(1 - d/m) + \frac{2}{(1 - d/m)^{k-k^*}} \geq \frac{1}{(1 - d/m)^{k-k^*}} \geq e^{-(k^*-k)d/m}$$

where the last inequality uses  $1 - x \leq e^{-x}$ . Multiplying across the inequality yields the claim. Similarly, we can prove the second conclusion. For  $k \leq k^*$ ,

$$\begin{aligned} \frac{\pi(k-1)}{\pi(k)} &= \frac{k - k^* - m(1 - d/m)^k + 2m(1 - d/m)^{k^*}}{m(1 - d/m)^{k-1}} \\ &\leq -(1 - d/m) + 2(1 - d/m)^{k^*-k+1} \leq (1 - d/m)^{k^*-k+1} \leq e^{-(k^*-k+1)d/m}, \end{aligned}$$

where the second to last inequality uses  $k \leq k^*$ .  $\square$

## F.3 Proof of Lemma 14

*Proof.* Let  $\Delta \geq 0$  be a parameter that we fix later. We have,

$$\mathbb{E}_{Z \sim \pi} [Z] \leq k^* + \Delta + \sum_{i=k^*+\Delta+1}^{\infty} i\pi(i). \tag{F.3}$$

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<sup>25</sup>Stirling's approximation states that

$$n! \geq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

By equation (C.6),

$$\begin{aligned}
\sum_{i=k^*+\Delta+1}^{\infty} i\pi(i) &= \sum_{i=\Delta+1}^{\infty} e^{-d(i-1)^2/2m}(i+k^*) \\
&= \sum_{i=\Delta}^{\infty} e^{-di^2/2m}(i-1) + \sum_{i=\Delta}^{\infty} e^{-di^2/2m}(k^*+2) \\
&\leq \frac{e^{-d(\Delta-1)^2/2m}}{d/m} + (k^*+2) \frac{e^{-d\Delta^2/2m}}{\min\{1/2, d\Delta/2m\}}, \tag{F.4}
\end{aligned}$$

where in the last step we used equations (I.1) and (I.2). Letting  $\Delta := 1 + 2\sqrt{m/d} \log(m/d)$  in the above equation, the right hand side is at most 1. The lemma follows from (F.3) and the above equation.  $\square$

## F.4 Proof of Lemma 15

*Proof.* By linearity of expectation,

$$\mathbf{L}(\text{Patient}) = \frac{1}{m \cdot T} \mathbb{E} \left[ \int_{t=0}^T Z_t (1 - d/m)^{Z_t-1} dt \right] = \frac{1}{m \cdot T} \int_{t=0}^T \mathbb{E} [Z_t (1 - d/m)^{Z_t-1}] dt.$$

We can then write

$$\begin{aligned}
\mathbf{L}(\text{Patient}) &\leq \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{1}{m \cdot T} \int_{t=\tau_{\text{mix}}(\epsilon)}^T \sum_{i=0}^{\infty} (\pi(i) + \epsilon) i (1 - d/m)^{i-1} dt \\
&\leq \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{\mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^{Z-1}]}{m} + \frac{\epsilon m}{d^2}
\end{aligned}$$

where the first inequality is using the fact that for any  $t \geq 0$ ,  $\mathbb{E} [Z_t (1 - d/m)^{Z_t-1}] \leq \mathbb{E} [Z_t] \leq \mathbb{E} [\tilde{Z}_t] \leq m$  and the last inequality uses the identity  $\sum_{i=0}^{\infty} i(1 - d/m)^{i-1} = m^2/d^2$ , which can be proved by taking derivative of the standard geometric sum identity  $\sum_{i=0}^{\infty} (1 - x)^i = 1/x$  (for  $x < 1$ ).  $\square$

## F.5 Proof of Proposition 16

Let us first rewrite what we derived in the proof overview of this proposition in the main text. The balance equations of the Markov chain associated with the Patient algorithm can be written as follows by replacing transition probabilities from (C.7), (C.8), and (C.9) in



(C.10):

$$m\pi(k) = (k+1)\pi(k+1) + (k+2)\left(1 - \left(1 - \frac{d}{m}\right)^{k+1}\right)\pi(k+2) \quad (\text{F.5})$$

Now define a continuous  $f : \mathbb{R} \rightarrow \mathbb{R}$  as follows,

$$f(x) := m - (x+1) - (x+2)(1 - (1 - d/m)^{x+1}). \quad (\text{F.6})$$

It follows that

$$f(m-1) \leq 0, f(m/2-2) > 0,$$

which means that  $f(\cdot)$  has a root  $k^*$  such that  $m/2 - 2 < k^* < m$ . In the rest of the proof we show that the states that are far from  $k^*$  have very small probability in the stationary distribution

In order to complete the proof of [Proposition 16](#), we first prove the following useful lemma.

**Lemma 24.** *For any integer  $k \leq k^*$ ,*

$$\frac{\pi(k)}{\max\{\pi(k+1), \pi(k+2)\}} \leq e^{-(k^*-k)/m}.$$

*Similarly, for any integer  $k \geq k^*$ ,  $\frac{\min\{\pi(k+1), \pi(k+2)\}}{\pi(k)} \leq e^{-(k-k^*)/(m+k-k^*)}$ .*

*Proof.* For  $k \leq k^*$ , by equation (C.11),

$$\begin{aligned} \frac{\pi(k)}{\max\{\pi(k+1), \pi(k+2)\}} &\leq \frac{(k+1) + (k+2)(1 - (1 - d/m)^{k+1})}{m} \\ &\leq \frac{(k - k^*) + (k^* + 1) + (k^* + 2)(1 - (1 - d/m)^{k^*+1})}{m} \\ &= 1 - \frac{k^* - k}{m} \leq e^{-(k^*-k)/m}, \end{aligned}$$

where the last equality follows by the definition of  $k^*$  and the last inequality uses  $1 - x \leq e^{-x}$ .

The second conclusion can be proved similarly. For  $k \geq k^*$ ,

$$\begin{aligned} \frac{\min\{\pi(k+1), \pi(k+2)\}}{\pi(k)} &\leq \frac{m}{(k+1) + (k+2)(1 - (1 - d/m)^{k+1})} \\ &\leq \frac{m}{(k - k^*) + (k^* + 1) + (k^* + 2)(1 - (1 - d/m)^{k^*+1})} \\ &= \frac{m}{m + k - k^*} = 1 - \frac{k - k^*}{m + k - k^*} \leq e^{-(k-k^*)/(m+k-k^*)}. \end{aligned}$$

where the equality follows by the definition of  $k^*$ .  $\square$

Now, we use the above claim to upper-bound  $\pi(k)$  for values  $k$  that are far from  $k^*$ . First, fix  $k \leq k^*$ . Let  $n_0, n_1, \dots$  be sequence of integers defined as follows:  $n_0 = k$ , and  $n_{i+1} := \arg \max\{\pi(n_i + 1), \pi(n_i + 2)\}$  for  $i \geq 1$ . It follows that,

$$\pi(k) \leq \prod_{i:n_i \leq k^*} \frac{\pi(n_i)}{\pi(n_{i+1})} \leq \exp\left(-\sum_{i:n_i \leq k^*} \frac{k^* - n_i}{m}\right) \leq \exp\left(-\sum_{i=0}^{(k^*-k)/2} \frac{2i}{m}\right) \leq e^{-(k^*-k)^2/4m}, \quad (\text{F.7})$$

where the second to last inequality uses  $\sum_{i=0}^{(k^*-k)/2} \frac{2i}{m} \leq -\sum_{i:n_i \leq k^*} \frac{k^* - n_i}{m}$ . To see why, note that the two sums have equal number of terms, and equal first and last terms. For other terms, the second sum is  $\frac{k^* - (k+2)}{m} + \frac{k^* - (k+4)}{m} + \dots$ , whereas the first sum is  $\frac{k^* - n_1}{m} + \frac{k^* - n_2}{m} + \dots$ . The inequality follows from observing that  $|n_i - n_{i-1}| \leq 2$ .

Now, fix  $k \geq k^* + 2$ . In this case we construct the following sequence of integers,  $n_0 = \lfloor k^* + 2 \rfloor$ , and  $n_{i+1} := \arg \min\{\pi(n_i + 1), \pi(n_i + 2)\}$  for  $i \geq 1$ . Let  $n_j$  be the largest number in the sequence that is at most  $k$  (observe that  $n_j = k - 1$  or  $n_j = k$ ). We upper-bound  $\pi(k)$  by upper-bounding  $\pi(n_j)$ ,

$$\begin{aligned} \pi(k) &\leq \frac{m \cdot \pi(n_j)}{k} \leq 2 \prod_{i=0}^{j-1} \frac{\pi(n_i)}{\pi(n_{i+1})} \leq 2 \exp\left(-\sum_{i=0}^{j-1} \frac{n_i - k^*}{m + n_i - k^*}\right) \\ &\leq 2 \exp\left(-\sum_{i=0}^{(j-1)/2} \frac{2i}{m + k - k^*}\right) \leq 2 \exp\left(\frac{-(k - k^* - 1)^2}{4(m + k - k^*)}\right). \end{aligned} \quad (\text{F.8})$$

To see the first inequality note that if  $n_j = k$ , then there is nothing to show; otherwise we have  $n_j = k - 1$ . In this case by equation (C.11),  $m\pi(k - 1) \geq k\pi(k)$ . The second to last inequality uses the fact that  $|n_i - n_{i+1}| \leq 2$ .

We are almost done. The proposition follows from (F.8) and (F.7). First, for  $\sigma \geq 1$ , let  $\Delta = \sigma\sqrt{4m}$ , then by equation (I.1)

$$\sum_{i=0}^{k^* - \Delta} \pi(i) \leq \sum_{i=\Delta}^{\infty} e^{-i^2/4m} \leq \frac{e^{-\Delta^2/4m}}{\min\{1/2, \Delta/4m\}} \leq 2\sqrt{m}e^{-\sigma^2}.$$

Similarly,

$$\begin{aligned} \sum_{i=k^*+\Delta}^{\infty} \pi(i) &\leq 2 \sum_{i=\Delta+1}^{\infty} e^{-(i-1)^2/4(i+m)} \leq 2 \sum_{i=\Delta}^{\infty} e^{-i/(4+\sqrt{4m}/\sigma)} \\ &\leq 2 \frac{e^{-\Delta/(4+\sqrt{4m}/\sigma)}}{1 - e^{-1/(4+\sqrt{4m})}} \leq 8\sqrt{m} e^{\frac{-\sigma^2\sqrt{m}}{2\sigma+\sqrt{m}}} \end{aligned}$$

This completes the proof of [Proposition 16](#).

## F.6 Proof of Lemma 17

*Proof.* Let  $\Delta := 3\sqrt{m} \log(m)$ , and let  $\beta := \max_{z \in [m/2-\Delta, m+\Delta]} z(1-d/m)^z$ .

$$\mathbb{E}_{Z \sim \pi} [Z(1-d/m)^Z] \leq \beta + \sum_{i=0}^{m/2-\Delta-1} \frac{m}{2} \pi(i)(1-d/m)^i + \sum_{i=m+\Delta}^{\infty} i \pi(i)(1-d/m)^m \quad (\text{F.9})$$

We upper bound each of the terms in the right hand side separately. We start with upper bounding  $\beta$ . Let  $\Delta' := 4(\log(2m) + 1)\Delta$ .

$$\begin{aligned} \beta &\leq \max_{z \in [m/2, m]} z(1-d/m)^z + m/2(1-d/m)^{m/2}((1-d/m)^{-\Delta} - 1) + (1-d/m)^m \Delta \\ &\leq \max_{z \in [m/2, m]} (z + \Delta' + \Delta)(1-d/m)^z + 1. \end{aligned} \quad (\text{F.10})$$

To see the last inequality we consider two cases. If  $(1-d/m)^{-\Delta} \leq 1 + \Delta'/m$  then the inequality obviously holds. Otherwise, (assuming  $\Delta' \leq m$ ),

$$(1-d/m)^{\Delta} \leq \frac{1}{1 + \Delta'/m} \leq 1 - \Delta'/2m,$$

By the definition of  $\beta$ ,

$$\beta \leq (m + \Delta)(1-d/m)^{m/2-\Delta} \leq 2m(1 - \Delta'/2m)^{m/2\Delta-1} \leq 2me^{\Delta'/4\Delta-1} \leq 1.$$

It remains to upper bound the second and the third term in [\(F.9\)](#). We start with the second term. By [Proposition 16](#),

$$\sum_{i=0}^{m/2-\Delta-1} \pi(i) \leq \frac{1}{m^{3/2}}. \quad (\text{F.11})$$



Figure 7: If  $a_2$  gets critical in the above graph, Smart Patient and Patient algorithms make two different decisions.

where we used equation (I.1). On the other hand, by equation (F.8)

$$\sum_{i=m+\Delta}^{\infty} i\pi(i) \leq e^{-\Delta/(2+\sqrt{m})} \left( \frac{m}{1 - e^{-1/(2+\sqrt{m})}} + \frac{2\Delta + 4}{1/(2 + \sqrt{m})^2} \right) \leq \frac{1}{\sqrt{m}}. \quad (\text{F.12})$$

where we used equation (I.3).

The lemma follows from (F.9), (F.10), (F.11) and (F.12).  $\square$

## G Exploiting the Network Structure: Smart Patient

Since our results distinguish between exponential and fractional small losses, one may suspect that in extremely sparse graphs the upper bound for  $\mathbf{L}(\text{Patient})$  leaves room for improvement by considering the network structure. For instance, one could modify the Patient algorithm so that when a critical agent has multiple compatible partners, it breaks ties in favor of partners with lower degree. We call such algorithm the “smart Patient” algorithm:

**Definition 6** (Smart Patient Algorithm). *If an agent  $a$  becomes critical at time  $t$ , match the agent to the neighbor who has the lowest degree, breaking ties randomly.*

The same example that we used in the paper clarifies the difference between Patient and Smart Patient algorithms. Let  $G_t$  be the graph shown in Figure 7, and let  $a_2 \in A_t^c$ , i.e.,  $a_2$  is critical at time  $t$ . Observe that it is strictly better to match  $a_2$  to  $a_1$  as opposed to  $a_3$ , and that’s what Smart Patient does. Nevertheless, since the Patient algorithm makes decisions that depend only on the immediate neighbors of the agent it is trying to match, it cannot differentiate between  $a_1$  and  $a_3$  and will choose either of them with equal probability.

Figure 8 depicts the performance of Smart Patient and Patient algorithms, for different values of  $d$ . As  $d$  increases, the difference between the two becomes tiny, showing that waiting to thicken the market is a first-order concern (by comparing the two with Greedy). But for smaller values of  $d$ , Smart Patient beats Patient by a reasonable margin, showing that in sparse graphs, network considerations can matter.

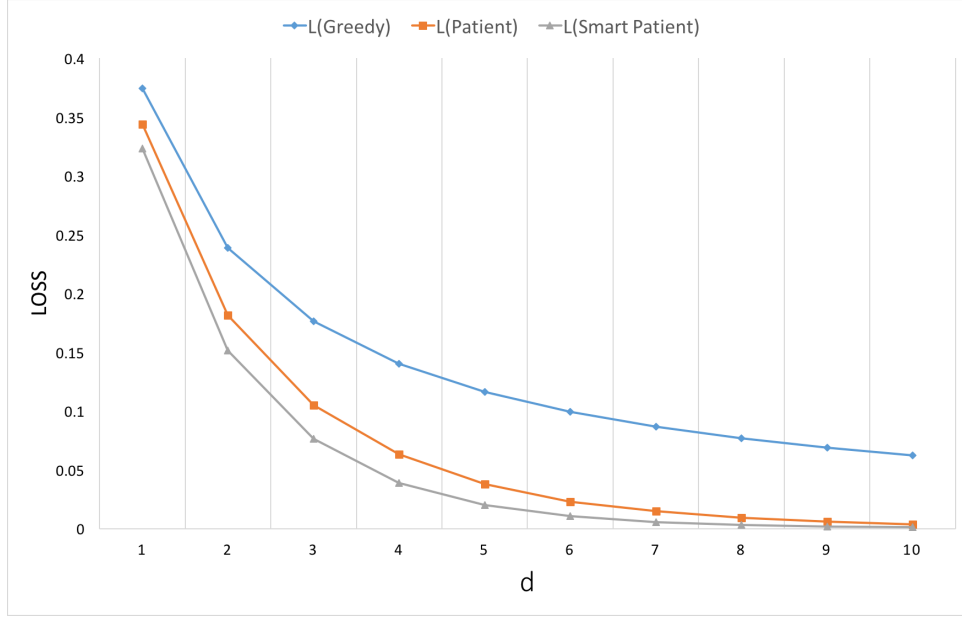


Figure 8: Smart Patient versus Patient algorithm

## H Small Market Simulations

In [Proposition 12](#) and [Proposition 16](#), we prove that the Markov chains of the Greedy and Patient algorithms are highly concentrated in intervals of size  $O(\sqrt{m/d})$  and  $O(\sqrt{m})$ , respectively. These intervals are plausible concentration bounds when  $m$  is relatively large. In fact, most of our theoretical results are interesting when markets are relatively large. Therefore, it is natural to ask: What if  $m$  is relatively small? And what if the  $d$  is not small relative to  $m$ ?

Figure 9 depicts the simulation results of our model for small  $m$  and small  $T$ . We simulated the market for  $m = 20$  and  $T = 100$  periods, repeated this process for 500 iterations, and computed the average loss for the Greedy, Patient, and the Omniscient algorithms. As it is clear from the simulation results, the loss of the Patient algorithm is lower than the Greedy for any  $d$ , and in particular, when  $d$  increases, the Patient algorithm's performance gets closer and closer to the Omniscient algorithm, whereas the Greedy algorithm's loss remains far above both of them.

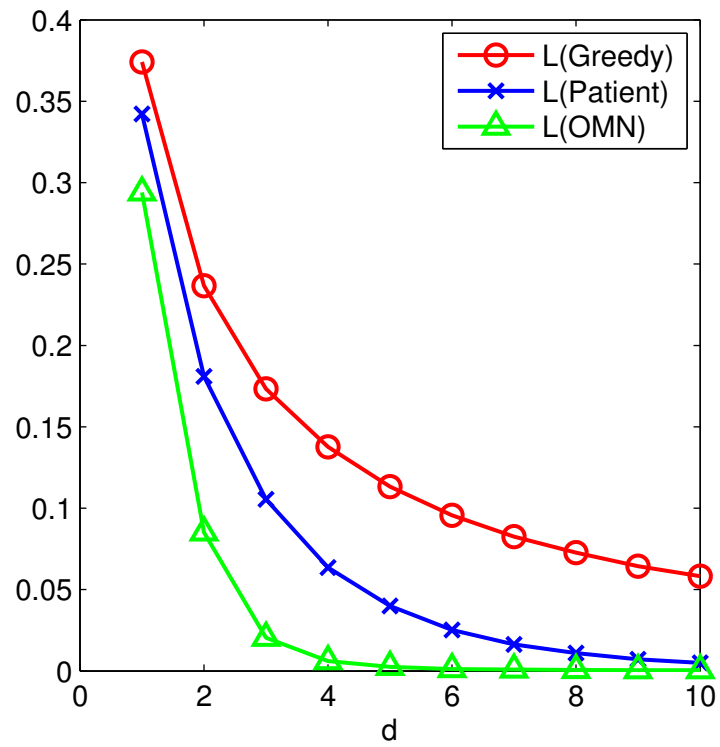


Figure 9: Simulated Losses for  $m = 20$ . For very small market sizes and even for relatively large values of  $d$ , the Patient algorithm outperforms the Greedy Algorithm.

# I Auxiliary Inequalities

In this section we prove several inequalities that are used throughout the paper. For any  $a, b \geq 0$ ,

$$\begin{aligned} \sum_{i=a}^{\infty} e^{-bi^2} &= \sum_{i=0}^{\infty} e^{-b(i+a)^2} \leq \sum_{i=0}^{\infty} e^{-ba^2-2iab} = e^{-ba^2} \sum_{i=0}^{\infty} (e^{-2ab})^i \\ &= \frac{e^{-ba^2}}{1 - e^{-2ab}} \leq \frac{e^{-ba^2}}{\min\{ab, 1/2\}}. \end{aligned} \quad (\text{I.1})$$

The last inequality can be proved as follows: If  $2ab \leq 1$ , then  $e^{-2ab} \leq ab$ , otherwise  $e^{-2ab} \leq 1/2$ .

For any  $a, b \geq 0$ ,

$$\sum_{i=a}^{\infty} (i-1)e^{-bi^2} \leq \int_{a-1}^{\infty} xe^{-bx^2} dx = \frac{-1}{2b} e^{-bx^2} \Big|_{a-1}^{\infty} = \frac{e^{-b(a-1)^2}}{2b}. \quad (\text{I.2})$$

For any  $a \geq 0$  and  $0 \leq b \leq 1$ ,

$$\sum_{i=a}^{\infty} ie^{-bi} = e^{-ba} \sum_{i=0}^{\infty} (i+a)e^{-bi} = e^{-ba} \left( \frac{a}{1 - e^{-b}} + \frac{1}{(1 - e^{-b})^2} \right) \leq \frac{e^{-ba}(2ba + 4)}{b^2}. \quad (\text{I.3})$$

The Bernoulli inequality states that for any  $x \leq 1$ , and any  $n \geq 1$ ,

$$(1 - x)^n \geq 1 - xn. \quad (\text{I.4})$$

Here, we prove for integer  $n$ . The above equation can be proved by a simple induction on  $n$ . It trivially holds for  $n = 0$ . Assuming it holds for  $n$  we can write,

$$(1 - x)^{n+1} = (1 - x)(1 - x)^n \geq (1 - x)(1 - xn) = 1 - x(n + 1) + x^2n \geq 1 - x(n + 1).$$