On the deletion channel with small deletion probability

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***Abstract*—The deletion channel is the simplest point-to-point communication channel that models lack of synchronization. Despite significant effort, little is known about its capacity, and even less about optimal coding schemes. In this paper we intiate a new systematic approach to this problem, by demonstrating that capacity can be computed in a series expansion for small deletion probability. We compute two leading terms of this expansion, and show that capacity is achieved, up to this order, by i.i.d. uniform random distribution of the input.**

We think that this strategy can be useful in a number of capacity calculations.

I. Introduction

The (binary) deletion channel accepts bits as inputs, and deletes each transmitted bit independently with probability d. Computing or providing systematic approximations to its capacity is one of the outstanding problems in information theory [1]. An important motivation comes from the need to understand synchronization errors and optimal ways to cope with them.

In this paper we suggest a new approach. We demonstrate that capacity can be computed in a series expansion for small deletion probability, by computing the first two orders of such an expansion. Our main result is the following.

Theorem I.1. *Let $C(d)$ be the capacity of the deletion channel with deletion probability d. Then, for any $\epsilon > 0$,*

$$C(d) = 1 + d\log d - A_1\, d + O(d^{3/2-\epsilon}) \qquad (1)$$

where $A_1 \equiv \log(2e) - \sum_{l=1}^{\infty} 2^{-l-1} l \log l$. Further, the iid Bernoulli$(1/2)$ process achieves capacity up to corrections of order $O(d^{3/2-\epsilon})$.

Logarithms here (and in the rest of the paper) are understood to be in base 2. The constant A_1 can be easily evaluated to yield $A_1 \approx 1.154163765$. While one might be skeptical about the concrete meaning of asymtotic expansions of the type (1), they often prove surprisingly accurate. For instance at 10% deletion probability, Eq. (1) is off the best lower bound proved in [5] by about 0.010 bits. More importantly they provide useful design insight. For instance, the above result shows that Bernoulli$(1/2)$ is an excellent starting point for the optimal input distribution. Next terms in expansion indicate how to systematically modify the input distribution for larger d [2].

We think that the strategy adopted here might be useful in other information theory problems. The underlying philosophy is that whenever capacity is known for a specific value of the channel parameter, and the corresponding optimal input distribution is unique and well characterized, it should be possible to compute an asymptotic expansion around that value. In the present context the special channel is the perfect channel, i.e. the deletion channel with deletion probability $d = 0$. The corresponding input distribution is the iid Bernoulli$(1/2)$ process.

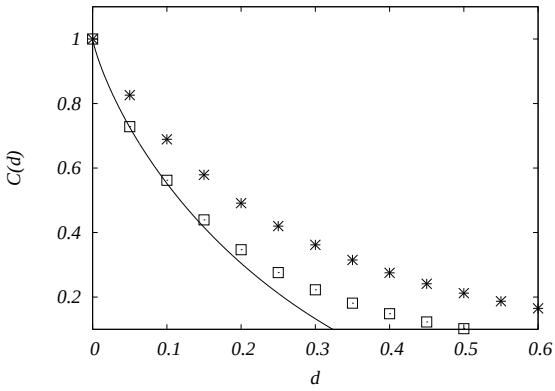

Fig. 1. Comparison of the asymptotic formula (1) (continuous line) with upper bounds from [6] (stars $*$) and lower bounds from [5] (squares, $\Box$).

A. Related work

Dobrushin [3] proved a coding theorem for the deletion channel, and other channels with synchronization errors. He showed that the maximum rate of reliable communication is given by the maximal mutual information per bit, and proved that this can be achieved through a random coding scheme. This characterization has so far found limited use in proving concrete estimates. An important exception is provided by the work of Kirsch and Drinea [4] who use Dobrushin coding theorem to prove lower bounds on the capacity of channels with deletions and duplications. We will also use Dobrushin theorem in a crucial way, although most of our effort will be devoted to proving upper bounds on the capacity.

Several capacity bounds have been developed over the last few years, following alternative approaches, and are surveyed in [1]. In particular, it has been proved that $C(d) = \Theta(1-d)$ as $d \to 1$. However determining the asymptotic behavior in this limit (i.e. finding a constant B_1 such that $C(d) = B_1(1-d) + o(1-d)$) is an open problem. When applied to the small d regime, none of the known upper bounds actually captures the correct behavior (1). As we show in the present paper, this

behavior can be controlled exactly.

When this paper was nearing submission, a preprint by Kalai, Mitzenmacher and Sudan [7] was posted online, proving a statement analogous to Theorem I.1. The result of [7] is however not the same as in Theorem I.1: only the $d \log d$ term of the series is proved in [7]. Further, the two proofs are based on very different approaches.

II. Preliminaries

For the reader's convenience, we restate here some known results that we will use extensively, along with with some definitions and auxiliary lemmas.

Consider a sequence of channels $\{W_n\}_{n\geq 1}$, where W_n allows exactly n inputs bits, and deletes each bit inependently with probability d. The output of W_n for input X^n is a binary vector denoted by $Y(X^n)$. The length of $Y(X^n)$ is a binomial random variable. We want to find maximum rate at which we can send information over this sequence of channels with vanishingly small error probability.

The following characterization follows from [3].

Theorem II.1. *Let*

$$C_n = \frac{1}{n} \max_{p_{X^n}} I(X^n; Y(X^n)) \tag{2}$$

Then, the following limit exists

$$C = \lim_{n\to\infty} C_n = \inf_{n\geq 1} C_n \,, \tag{3}$$

and is equal to the capacity of the deletion channel.

Proof: This is just a reformulation of Theorem 1 in [3], to which we add the remark $C = \inf_{n\geq 1} C_n$, which is of independent interest. In order to prove this fact, consider the channel W_{m+n}, and let $X^{m+n} = (X_1^m, X_{m+1}^{m+n})$ be its input. The channel W_{m+n} can be realized as follows. First the input is passed through a channel $\widetilde{W}_{m+n}$ that introduces deletions independently in the two strings X_1^m and X_{m+1}^{m+n} and outputs $\widetilde{Y}(X_1^{m+n}) \equiv (Y(X_1^m), |, Y(X_{m+1}^{m+n}))$ where $|$ is a marker. Then the marker is removed.

This construction proves that W_{m+n} is physically degraded with respect to $\widetilde{W}_{m+n}$, whence

$$\begin{aligned}(m+n)C_{m+n} &\leq \max_{p_{X^{m+n}}} I(X^{m+n}; \widetilde{Y}(X_1^{m+n})) \\ &\leq mC_m + nC_n \,.\end{aligned}$$

Here the last inequality follows from the fact that $\widetilde{W}_{m+n}$ is the product of two independent channels, and hence the mutual information is maximized by a product input distribution.

Therefore the sequence $\{nC_n\}_{n\geq 1}$ is superadditive, and the claim follows from Fekete's lemma. ∎

A last useful remark is that, in computing capacity, we can assume $(X_1, \dots, X_n)$ to be n consecutive coordinates of a stationary ergodic process.

Lemma II.2. *Let $\mathbb{X} = \{X_i\}_{i\in\mathbb{Z}}$ be a stationary and ergodic process, with X_i taking values in $\{0,1\}$. Then the limit $I(\mathbb{X}) = \lim_{n\to\infty} \frac{1}{n} I(X^n; Y(X^n))$ exists and*

$$C = \max_{\mathbb{X} \text{ stat. erg.}} I(\mathbb{X}) \,. \tag{4}$$

Proof: Take any stationary $\mathbb{X}$, and let $I_n = I(X^n; Y(X^n))$. Notice that $Y(X_1^n) - X_1^n - X_{n+1}^{n+m} - Y(X_{n+1}^{n+m})$ form a Markov chain. Define $\widetilde{Y}(X^{n+m})$ as in the proof of Theorem II.1. As before we have $I_{n+m} \leq I(X^{n+m}, \widetilde{Y}(X^{n+m})) \leq I(X_1^m; \widetilde{Y}(X_1^m)) + I(X_{m+1}^{m+n}; Y(X_{m+1}^{m+n})) = I_m + I_n$. (the last identity follows by stationarity of $\mathbb{X}$). Thus $I_{m+n} \leq I_n + I_m$ and the limit $\lim_{n\to\infty} I_n/n$ exists by Fekete's lemma, and is equal to $\inf_{n\geq 1} I_n/n$.

Clearly, $I_n \leq C_n$ for all n. Fix any $\varepsilon > 0$. We will construct a process $\mathbb{X}$ such that

$$I_N/N \geq C - \varepsilon \qquad \forall\, N > N_0(\varepsilon) \,, \tag{5}$$

thus proving our claim.

Fix n such that $C_n \geq C - \varepsilon/2$. Construct $\mathbb{X}$ with iid blocks of length n with common distribution $p^*(n)$ that achieves the supremum in the definition of C_n. In order to make this process stationary, we make the first complete block to the right of the position 0 start at position s uniformly random in $\{1, 2, \dots, n\}$. We call the position s the offset. The resulting process is clearly stationary and ergodic.

Now consider $N = kn + r$ for some $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-1\}$. The vector X_1^N contains at least $k-1$ complete blocks of size n, call them $X(1), X(2), \dots, X(k-1)$ with $X(i) \sim p^*(n)$. The block $X(1)$ starts at position s. There will be further $r + k - s + 1$ bits at the end, so that $X_1^N = (X_1^{s-1}, X(1), X(2), \dots, X(k-1), X_{s+kn}^N)$. Abusing notation, we write $Y(i)$ for $Y(X(i))$. Given the output Y, we define $\widetilde{Y} = (Y(X_1^{s-1})|Y(1)|Y(2)|\dots|Y(k-1)|Y(X_{s+kn}^N))$, by introducing k sychronization symbols $|$. There are at most $(n+1)^k$ possibilities for $\widetilde{Y}$ given Y (corresponding to potential placements of synchronization symbols). Therefore we have

$$\begin{aligned}H(Y) &= H(\widetilde{Y}) - H(\widetilde{Y}|Y) \\ &\geq H(\widetilde{Y}) - \log((n+1)^k) \\ &\geq (k-1)H(Y(1)) - k\log(n+1) \,,\end{aligned}$$

where we used the fact that the $(X(i), Y(i))$'s are iid. Further

$$H(Y|X^N) \leq H(\widetilde{Y}|X^N) \leq (k-1)H(Y(1)|X(1)) + 2n \,,$$

where the last term accounts for bits outside the blocks. We conclude that

$$\begin{aligned}I(X^N; Y(X^N)) &= H(Y) - H(Y|X^N) \\ &\geq (k-1)nC_n - k\log(n+1) - 2n \\ &\geq N(C_n - \varepsilon/2)\end{aligned}$$

provided $\log(n+1)/n < \varepsilon/8$, $N > N_0 \equiv 10n/\varepsilon$. Since $C_n \geq C - \varepsilon/2$, this in turn implies Eq. (5). ∎

III. Proof of the main theorem: Outline

In this section we provide the proof of Theorem I.1. We defer the proof of several technical lemmas to the next section.

The first step consists in proving achievability by estimating $I(\mathbb{X})$ for the iid Bernoulli$(1/2)$ process.

Lemma III.1. *Let $\mathbb{X}^*$ be the iid* Bernoulli$(1/2)$ *process. For any $\epsilon > 0$, we have*

$$I(\mathbb{X}^*) = 1 + d\log d - A_1\, d + O(d^{2-\epsilon})\,. \tag{6}$$

Lemma II.2 allows us to restrict our attention to stationary ergodic processes in proving the converse. In light of Lemma III.1, we can further restrict consideration to processes $\mathbb{X}$ satisfying $I(\mathbb{X}) > 1+2d\log d$ and hence $H(\mathbb{X}) > 1+2d\log d$ (here and below, for a process $\mathbb{X}$, we denote by $H(\mathbb{X})$ its *entropy rate*).

Given a (possibly infinite) binary sequence, a *run* of 0's (of 1's) is a maximal subsequence of consecutive 0's (1's), i.e. an subsequence of 0's bordered by 1's (respectively, of 1's bordered by 0's). Denote by $\mathcal{S}$ the set of all stationary ergodic processes and by $\mathcal{S}_L$ the set of stationary ergodic processes such that, with probability one, no run has length larger than L. The next lemma shows that we don't lose much by restricting ourselves to $\mathcal{S}_{L^*}$ for large enough L^*.

Lemma III.2. *For any $\epsilon > 0$ there exists $d_0 = d_0(\epsilon) > 0$ such that the following happens for all $d < d_0$. For any $\mathbb{X} \in \mathcal{S}$ such that $H(\mathbb{X}) > 1 + (3/2)d\log d$ and for any $L^* > d^{-3/4}$, there exists $\mathbb{X}_{L^*} \in \mathcal{S}_{L^*}$ such that $H(\mathbb{X}_{L^*}) > 1 + 2d\log d$ and*

$$I(\mathbb{X}) \le I(\mathbb{X}_{L^*}) + d^{1/2-\epsilon}(L^*)^{-1}\log L^*\,. \tag{7}$$

We are left with the problem of bounding $I(\mathbb{X})$ from above for all $\mathbb{X} \in \mathcal{S}_{L^*}$. The next lemma establishes such a bound.

Lemma III.3. *For any $\epsilon > 0$ there exists $d_0 = d_0(\epsilon) > 0$ such that the following happens. For any $L_* \in \mathbb{N}$ and any $\mathbb{X} \in \mathcal{S}_{L_*}$ such that $H(\mathbb{X}) > 1 + 2d\log d$, if $d < d_0(\epsilon)$, then*

$$I(\mathbb{X}) \le 1 + d\log d - A_1 d + d^{2-\epsilon}(1 + d^{1/2}L_*)\,. \tag{8}$$

Proof of Theorem I.1: Lemma III.1 shows achievability. The converse follows from Lemmas III.2 and III.3 with $L^* = \lfloor 1/d \rfloor$. ∎

IV. Proofs of the Lemmas

In Section IV-A we characterize any stationary ergodic $\mathbb{X}$ in terms of its 'bit perspective' and 'block perspective' run-length distributions, and show that these distributions must be close to the distributions obtained for the iid Bernoulli$(1/2)$ process. In Section IV-B we construct a modified deletion process that allows accurate estimation of $H(Y|X^n)$ in the small d limit. Finally, in Section IV-C we present proofs of the Lemmas quoted in Section III using the tools developed.

We will often write X_a^b for the random vector $(X_a, X_{a+1}, \dots, X_b)$ where the X_i's are distributed according to the process $\mathbb{X}$.

A. Characterization in terms of runs

Consider a stationary ergodic process $\mathbb{X}$. Without loss of generality we can assume that almost surely all runs have finite length (by ergodicity and stationarity this only excludes the constant 0 and constant 1 proceses). Let L_0 be the length of the run containing position 0 in $\mathbb{X}$. Let L_1 be the length of first run of to occur to the right of position 0 in $\mathbb{X}$ and, in general, let L_i be the length of the i-th run to the right of position 0. Let $p_{L,\mathbb{X}}$ denote the limit of the empirical distribution of $L_1, L_2, \dots, L_K$, as $K \to \infty$. By ergodicity $p_{L,\mathbb{X}}$ is a well defined probablity distribution on $\mathbb{N}$. Note that the empirical distribution of run lengths in X_1^n also converges to $p_{L,\mathbb{X}}$ almost surely, since the first and last run do not matter in the limit. We call $p_{L,\mathbb{X}}$ the *block-perspective* run length distribution for obvious reasons, and use L to denote a random variable drawn according to $p_{L,\mathbb{X}}$.

It is not hard to see that, for any $l \ge 1$,

$$\mathbb{P}(L_0 = l) = \frac{l p_{L,\mathbb{X}}(l)}{\mathbb{E}[L]} \tag{9}$$

In other words L_0 is distributed according to the size biased version of $p_{L,\mathbb{X}}$. We call this the *bit perspective* run length distribution, and shall often drop the subscript $\mathbb{X}$ when clear from the context. Notice that since L_0 is a well defined and almost surely finite, we have $\mathbb{E}[L] < \infty$.

If $L_0^+, L_1, \dots, L_K$ are the run lengths in the block X_0^n, it is clear that $H(X_0^n) \le 1 + H(L_1, \dots, L_{K_n}, K_n)$ (where one bit is needed to remove the $0, 1$ ambiguity). By ergodicity $K_n/n \to 1/\mathbb{E}[L]$ almost surely as $n \to \infty$. This also implies $H(K_n)/n \to 0$. Futher, conditional on $K_n = k$, $H(L_1, \dots, L_{K_n}, K_n | K_n = k) \le k\, H(L)$. If $H(\mathbb{X})$ is the entropy rate of the process $\mathbb{X}$, by taking the $n \to \infty$ limit, it is easy to deduce that

$$H(\mathbb{X}) \le \frac{H(L)}{\mathbb{E}[L]}\,, \tag{10}$$

with equality if and only if $\mathbb{X}$ consists of iid runs with common distribution p_L.

For convenience of notation, define $\mu(\mathbb{X}) \equiv \mathbb{E}[L]$. We know that given $\mathbb{E}[L] = \mu$, the probability distribution with largest possible entropy $H(L)$ is geometric with mean μ, i.e. $p_L(l) = (1-1/\mu)^{l-1}1/\mu$ for all $l \ge 1$, leading to

$$\frac{H(L)}{\mathbb{E}[L]} \le -\big(1-\frac{1}{\mu}\big)\log\big(1-\frac{1}{\mu}\big) - \frac{1}{\mu}\log\frac{1}{\mu} \equiv h(1/\mu)\,. \tag{11}$$

Here we introduced the notation $h(p) = -p\log p - (1-p)\log(1-p)$ for the binary entropy function.

In light of Lemma III.1 we can restrict ourselves to $H(\mathbb{X}) > 1 + B\, d\log d$ for $B > 1$. Using this, we are able to obtain sharp bounds on p_L and $\mu(\mathbb{X})$.

Lemma IV.1. *For any $B > 0$ there exists $d_0 > 0$ such that, for any $\mathbb{X} \in \mathcal{S}$ with $H(\mathbb{X}) > 1 + Bd\log d$,*

$$|\mu(\mathbb{X}) - 2| \le \sqrt{50B\ \ d\log(1/d)} \tag{12}$$

for all $d < d_0$.

Proof: By Eqs. (10) and (11), we have $h(1/\mu) \ge 1 + Bd\log d$. By Pinsker's inequality $h(p) \le 1 - (1-2p)^2/(2\ln 2)$, and therefore $|1 - (2/\mu)|^2 \le (2B\ln 2)d\log(1/d)$. The claim follows from simple calculus. ∎

Lemma IV.2. *For all $B > 0$ there exists $K' < \infty$ such that, for any $\mathbb{X} \in \mathcal{S}$ with $H(\mathbb{X}) > 1 + Bd\log d$, and any $d < 1/2$,*

$$\sum_{l=1}^{\infty} \left| p_L(l) - \frac{1}{2^l} \right| \le K' \sqrt{d \log(1/d)}\,. \tag{13}$$

Proof: Let $p_L^*(l) = 1/2^l,\ l \ge 1$ and recall that $\mu(\mathbb{X}) = \mathbb{E}[L] = \sum_{l\ge 1} p_L(l) l$. An explicit calculation yields

$$H(p_L^*) - H(p_L) = D(p_L||p_L^*) + (2 - \mu(\mathbb{X}))\,. \tag{14}$$

Now, by Pinsker's inequality,

$$D(p_L||p_L^*) \ge \frac{2}{\ln 2} ||p_L - p_L^*||_{\rm TV}^2 \tag{15}$$

Combining Lemma IV.1, and Eqs. (10), (14) and (15), we get the desired result. ∎

Lemma IV.3. *For all $B > 0$ there exists $K'' < \infty$ such that, for any $\mathbb{X} \in \mathcal{S}$ with $H(\mathbb{X}) > 1 + Bd\log d$, and any $d < 1/2$,*

$$\sum_{l=1}^{\infty} \left| \mathbb{P}(L_0 = l) - \frac{l}{2^{l+1}} \right| \le K'' \sqrt{d(\log(1/d))^3}\,. \tag{16}$$

Proof: Let $l_0 = \lfloor -\log(K' \sqrt{d\log(1/d)}) \rfloor$. It follows from Lemma IV.2 that

$$\sum_{l=1}^{l_0} \left| p_L(l) - \frac{1}{2^l} \right| \le K' \sqrt{d \log(1/d)}\,, \tag{17}$$

which in turn implies

$$\sum_{l=0}^{l_0} l p_L(l) \ge \sum_{l=0}^{l_0-1} \frac{l}{2^l}\,. \tag{18}$$

Summing the geometric series, we find that there exists a constant $K_1 < \infty$ such that

$$\sum_{l=l_0}^{\infty} \frac{l}{2^l} = (l_0+1) 2^{1-l_0} \le K_1 \sqrt{d(\log d)^3}\,. \tag{19}$$

Using the identity $\sum_{l=0}^{\infty} l\, 2^{-l} = 2$, together with Eqs. (18) and (19), we get

$$\sum_{l=0}^{l_0} l p_L(l) \ge 2 - K_1 \sqrt{d(\log(1/d))^3}\,. \tag{20}$$

Combining this result with Lemma IV.1, we conclude (eventually enlarging the constant K_1)

$$\sum_{l=l_0+1}^{\infty} l p_L(l) \le 2K_1 \sqrt{d(\log(1/d))^3}\,. \tag{21}$$

Using this result together with Eq. (19), we get

$$\sum_{l=l_0+1}^{\infty} |l p_L(l) - \frac{l}{2^l}| \le 4K_1 \sqrt{d(\log(1/d))^3}\,. \tag{22}$$

From a direct application of Lemma IV.2 it follows that there exists a constant $K_2 < \infty$, such that

$$\sum_{l=1}^{l_0} \left| l p_L(l) - \frac{l}{2^l} \right| \le K_2 \sqrt{d(\log(1/d))^3}\,. \tag{23}$$

and therefore summing Eqs. (23) and (22)

$$\sum_{l=1}^{\infty} \left| \frac{l p_L(l)}{2} - \frac{l}{2^{l+1}} \right| \le 2(K_1+K_2) \sqrt{d(\log(1/d))^3}\,. \tag{24}$$

We know that $\mathbb{P}(L_0 = l) = l p_L(l)/\mu(\mathbb{X})$. The proof is completed by using Eq. (24) and bounding $\mu(\mathbb{X})$ with the Lemma IV.1. ∎

B. A modified deletion process

We define an auxiliary sequence of channels $\widehat{W}_n$ whose output –denoted by $\widehat{Y}(X^n)$– is obtained by modifying the deletion channel output in the following way. If a 'extended run' (i.e. a run along with one additional bit at each end of the run) undergoes more than one deletion under the deletion channel, then all deletions *within* such run are reversed i.e. the corresponding bits are *present* in $\widehat{Y}(X^n)$.

Formally, we construct this sequence of channels as follows when the input is a stationary process $\mathbb{X}$. Let $\mathbb{D}$ be an iid Bernoulli(d) process, independent of $\mathbb{X}$, with D_1^n being the n-bit vector that contains a 1 if and only if the corresponding bit in X^n is deleted by the channel W_n. We define $\widehat{\mathbb{D}}(\mathbb{D},\mathbb{X})$ to be the process containing a subset of the 1s in $\mathbb{D}$. The process $\widehat{\mathbb{D}}$ is obtained by deterministically flipping some of the 1s in $\mathbb{D}$ as described above. The output of the channel $\widehat{W}_n$ is simply defined by deleting from X^n those bits whose positions correspond to 1s in $\widehat{\mathbb{D}}$.

Notice that $(\mathbb{X},\mathbb{D},\widehat{\mathbb{D}})$ are jointly stationary. The sequence of channels W_n are defined from $\mathbb{D}$, and the coupled sequence of channels $\widehat{W}_n$ are defined from $\mathbb{D}$. We emphasize that $\widehat{\mathbb{D}}$ is a function of $(\mathbb{X},\mathbb{D})$. Let $\mathbb{Z} \equiv \mathbb{D} \oplus \widehat{\mathbb{D}}$ (where $\oplus$ is componentwise sum modulo 2). The process $\mathbb{Z}$ is stationary with $\mathbb{P}(Z_0 = 1) \equiv z = \mathbb{E}[d - d(1-d)^{L_0+1}] \le 2\, d^2\, \mathbb{E}[L_0]$. Note that $z = O(d^2)$ for $\mathbb{E}[L_0] < \infty$.

The following lemma shows the utility of the modified deletion process.

Lemma IV.4. *Consider any $\mathbb{X} \in \mathcal{S}$ such that $\mathbb{E}[L_0 \log L_0] < \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{n} H(\widehat{D}^n | X^n, \widehat{Y}^n) = d\,\mathbb{E}[\log L_0] - \delta\,, \tag{25}$$

where $0 \le \delta = \delta(d,\mathbb{X}) \le 2d^2 \mathbb{E}[L_0 \log L_0]$.

Proof: Fix a channel input x^n and any possible output $\widehat{y} = \widehat{y}(x^n)$ (i.e. an output that occurs with positive probability under $\widehat{W}_n$). The proof consists in estimating (the logarithm of) the number of realizations of $\widehat{D}^n$ that might lead to the input/ouput pair $(x^n, \widehat{y})$, and then taking the expectation over $(x^n, \widehat{y})$.

Proceeding from left to right, and using the constraint on $\widehat{\mathbb{D}}$, we can map unambiguously each run in $\widehat{y}$ to one or more runs in x^n, that gave rise to it through the deletion process. Consider a run of length ℓ in $\widehat{y}$. If there is a unique 'parent' run, it must have length ℓ or $\ell+1$. If the length of the parent run is ℓ, then no deletion occurred in this run, and hence the contribution to $H(\widehat{D}^n | x^n, \widehat{y})$ of such runs vanishes. If the length of the parent run is $\ell+1$, one bit was deleted by $\widehat{W}^n$

and each of the $\ell+1$ possibilities is equally likely, leading to a contribution $\log(\ell+1)$ to $H(\widehat{D}^n|x^n,\widehat{y})$.

Finally, if there are multiple parent runs of lengths $l_1, l_2, \dots, l_k$, they must be separated by single bits of taking the opposite value in x^n, all of which were deleted. It also must be the case that $\sum_{i=1}^k l_i = \ell$ i.e. there is no ambiguity in $\widehat{D}^n$. This also implies $l_1 < \ell$.

Notice that the three cases described corresponds to three different lengths for the run in $\widehat{y}$. This allows to sequentially associate runs in $\widehat{y}$ with runs in x^n, as claimed.

By the above argument, $H(\widehat{D}^n|x^n,\widehat{y}^n) = \sum_{r\in\mathcal{D}}\log(\ell_r)$ where $\mathcal{D}$ is the set of runs on which deletions did occurr, and ℓ_r are their lengths. Using the definition of $\widehat{\mathbb{D}}$, the sum can be expressed as $\sum_{i=1}^n \widehat{D}_i\log(\ell_{(i)})$, with $\ell_{(i)}$ the length of the run containing the i-th bit. Using the definition of $\widehat{\mathbb{D}}$, we get $\mathbb{P}(\widehat{D}_i=1) = d(1-d)^{\ell_{(i)}+1} \in (d-(\ell_{(i)}+1)d^2, d)$ (except for the last and first block in x^n, that can be disregarded). Taking expectation and letting $n\to\infty$ we get the claim. ∎

Corollary IV.5. *Under the assumptions of the last Lemma, and denoting by $h(p)$ the binary entropy function, we have*

$$\lim_{n\to\infty}\frac{1}{n}H(Y(X^n)|X^n) = h(d) - d\,\mathbb{E}[\log L_0] + \delta$$

where $-h(z) \le \delta = \delta(d,\mathbb{X}) \le 2d^2\mathbb{E}[L_0\log L_0] + h(z)$ and $z = d - \mathbb{E}[d(1-d)^{L_0+1}]$.

Proof: By definition, D^n is independent of X^n. We have, for $Y = Y(X^n)$,

$$\begin{aligned} H(Y|X^n) &= H(Y,D^n|X^n) - H(D^n|X^n,Y) \\ &= nh(d) - H(\widehat{D}^n|X^n,Y) + n\delta_1 \end{aligned}$$

with $|\delta_1(d,\mathbb{X})| < H(Z^n)/n \le h(z)$. In the second idequality we used the fact that D^n is a function of $(\widehat{D}^n, Z^n)$ and $\widehat{D}^n$ is a function of (D^n, Z^n). ∎

C. Proofs of Lemmas III.1, III.2 and III.3

Proof of Lemma III.1: Clearly, $\mathbb{X}^*$ has run length distribution $p_L(l) = 2^{-l}$, $l \ge 1$. Moreover, $Y(X^{*,n})$ is also a iid Bernoulli$(1/2)$ string of length $\sim$ Binomial$(n, 1-d)$. Hence, $H(Y) = n(1-d)+O(\log n)$. We now use the estimate of $H(Y|X^{*,n})$ from Corollary IV.5. We have $z = O(d^2)$ and $\mathbb{E}[L_0\log L_0] < \infty$, leading to

$$H(Y|X^{*,n}) = n(h(d) - d\,\mathbb{E}[\log L_0] + O(d^{2-\epsilon})) + o(n)\,.$$

Computing $H(Y) - H(Y|X^{*,n})$, we get the claim. ∎

Proof of Lemma III.2: We construct $\mathbb{X}_{L^*}$ by flipping a bit each time it is the (L^*+1)-th consecutive bit with the same value (either 0 or 1). The density of such bits in $\mathbb{X}$ is upper bounded by $\alpha = \mathbb{P}(L_0 > L^*)/L^*$. The expected fraction of bits in the channel output $Y_{L^*} = Y(X^n_{L^*})$ that have been flipped relative to $Y = Y(X^n)$ (output of the same channel realization with different input) is also at most α. Let $F = F(\mathbb{X}, Y)$ be the binary vector having the same length as Y, with a 1 wherever the corresponding bit in Y_{L^*} is flipped relative to Y, and 0s elsewhere. The expected fraction of 1's in F is α. Therefore

$$H(F) \le n(1-d)h(\alpha) + \log(n+1)\,. \tag{26}$$

Notice that Y is a deterministic function of (Y_{L^*}, F) and Y_{L^*} is a deterministic function of (Y, F), whence

$$|H(Y) - H(Y_{L^*})| \le H(F)\,. \tag{27}$$

Further, $Y - X^n - \mathbb{X} - \mathbb{X}_{L^*} - X^n_{L^*} - Y_{L^*}$ form a Markov chain, and $\mathbb{X}_{L^*}$, $X^n_{L^*}$ are deterministic functions of $\mathbb{X}$. Hence, $H(Y_{L^*}|X^n_{L^*}) = H(Y_{L^*}|\mathbb{X})$ and $H(Y|X^n) = H(Y|\mathbb{X})$. Therefore (the second step is analogous to Eq. (27))

$$\begin{aligned} &|H(Y_{L^*}|X^n_{L^*}) - H(Y|X^n)| = \\ &\qquad = |H(Y_{L^*}|\mathbb{X}) - H(Y|\mathbb{X})| \le H(F)\,. \end{aligned} \tag{28}$$

It follows from Lemma IV.3 and $L^* > \log(1/d)$ that $\alpha \le 2K''\sqrt{d(\log(1/d))^3}/L^*$ for sufficiently small d. Hence, $h(\alpha) \le d^{1/2-\epsilon}\log L^*/(2L^*)$ for $d < d_0(\epsilon)$, for some $d_0(\epsilon) > 0$. The result follows by combining Eqs. (26), (27) and (28) to bound $|I(\mathbb{X}) - I(\mathbb{X}_{L^*})|$. Finally, $|H(X^n) - H(X^n_{L^*})| \le nh(\alpha)$ exactly as before, so $h(\alpha) \le (1/2)d\log d$ for sufficiently small d yields $H(\mathbb{X}_{L^*}) > 1 + 2d\log d$. ∎

Proof of Lemma III.3: We know that $Y(X^n)$ contains Binomial$(n, 1-d)$ bits, leading immediately to

$$H(Y) \le n(1-d) + \log(n+1) \tag{29}$$

We use the lower bound on $H(Y|X^{*,n})$ from Corollary IV.5. We have $z \le 2d^2\mathbb{E}[L_0]$. It follows from Lemma IV.3 that $\mathbb{E}[L_0] \le K_1(1+\sqrt{d(\log d)^3}L^*)$, leading to $h(z) \le d^{2-\epsilon}(1 + d^{1/2}L^*)$ for all $d < d_0$, where $d_0 = d_0(\epsilon) > 0$. Thus, we have the bound

$$H(Y|X^{*,n}) \ge n\{h(d) - \mathbb{E}[\log L_0] - d^{2-\epsilon}(1+d^{1/2}L^*)\}\,.$$

The result follows. ∎

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