Option 1: Exercises on measure spaces

Solve Exercises [1.1.4], [1.1.13], [1.1.21], [1.1.22] and [1.1.33] in Amir Dembo’s lecture notes.

Option 2: The Banach-Tarski paradox in one dimension

The objective of this homework is to prove a simplified version of the Banach-Tarski paradox, in the case of the real line. The definition of equidecomposable subsets of \( \mathbb{R} \) is provided below.

**Definition 1.** The sets \( A, B \in \mathbb{R} \) are (countably) equidecomposable if there exist countable partitions
\[
A = \bigcup_{i=1}^{\infty} A_i, \quad B = \bigcup_{i=1}^{\infty} B_i, \tag{1}
\]
and real numbers \( \{t_1, t_2, t_3 \ldots \} \) such that for every \( i \in \mathbb{N} \) \( A_i = B_i + t_i \) (+\( t_i \) here indicates translation by \( t_i \)).

It might also be useful to recall the Axiom of Choice.

**Axiom 2.** Let \( \Omega \) be a set and \( C = \{A_\alpha\}_{\alpha \in \Gamma} \) be a collection of nonempty subsets \( A_\alpha \subseteq \Omega \). Then there exists at least one choice function, i.e. a function \( f : C \to \Omega \) such that
\[
f(A) \in A, \tag{2}
\]
for each \( A \in C \).

First, we start with some useful remarks. Here \( A, B, A_i, B_i \) are subsets of \( \mathbb{R} \). Further, for \( t \in \mathbb{R} \), \( R_t : \mathbb{R} \to \mathbb{R} \) is the translation by \( t \): \( R_t(x) = x + t \).

**A1** We will say that a function \( f : A \to \mathbb{R} \) is a (countable) equidecomposition if there exists a countable partition \( A = \bigcup_{i=1}^{\infty} A_i \), and reals \( \{t_i\}_{i \in \mathbb{N}} \) such that \( f|_{A_i} = R_{t_i}|_{A_i} \) for each \( i \).

Show that \( A \) is equidecomposable with \( B \) if and only if there exists an equidecomposition \( f : A \to B \) which is bijective.

**A2** Let \( A' \subseteq A \) and \( B' \subseteq B \), and assume there exist bijective equidecompositions \( f : A \to B' \) and \( g : B \to A' \).

Construct an equidecomposition \( h : A \to B \), and prove that it is bijective.

[Hint: Let \( A^{(0)} \equiv A \setminus g(B) \), and \( A^{(*)} \equiv \bigcup_{n=0}^{\infty} (g \circ f)^n(A^{(0)}) \). Define \( h(x) = f(x) \) if \( x \in A^{(*)} \) and \( h(x) = g^{-1}(x) \) if \( x \in A \setminus A^{(*)} \).]
Next to the actual problem:

B1 Use the axiom of choice to show that there exists $C \subseteq [0, 1/2]$ such that
\[ R = \bigcup_{x \in C} \{ x + \mathbb{Q} \} . \] (3)

B2 Show that $\mathbb{Q} \cap [0, 1/2]$ is equidecomposable with $\mathbb{Q}$.

B3 Deduce that
\[ A \equiv \bigcup_{x \in C} \{ x + (\mathbb{Q} \cap [0, 1/2]) \} \subseteq [0, 1] \] (4)
is equidecomposable with $\mathbb{R}$.

B4 Use A1, A2 above to show that this implies that $[0, 1]$ is equidecomposable with $\mathbb{R}$. What does this result imply for measures on $\mathbb{R}$?