Option 1: Exercises on measure spaces

Solve Exercises [1.1.4], [1.1.13], [1.1.21], [1.1.22] and [1.1.33] in Amir Dembo’s lecture notes.

Option 2: The Banach-Tarski paradox in one dimension

The objective of this homework is to prove a simplified version of the Banach-Tarski paradox, in the case of the real line. The definition of equidecomposable subsets of $\mathbb{R}$ is provided below.

**Definition 1.** The sets $A, B \in \mathbb{R}$ are (countably) equidecomposable if there exist countable partitions

$$A = \bigcup_{i=1}^{\infty} A_i, \quad B = \bigcup_{i=1}^{\infty} B_i,$$

and real numbers $\{t_1, t_2, t_3 \ldots\}$ such that for every $i \in \mathbb{N}$ $A_i = B_i + t_i$ (+$t_i$ here indicates translation by $t_i$).

It might also be useful to recall the Axiom of Choice.

**Axiom 2.** Let $\Omega$ be a set and $\mathcal{C} = \{A_\alpha\}_{\alpha \in \Gamma}$ be a collection of nonempty subsets $A_\alpha \subseteq \Omega$. Then there exists at least one choice function, i.e. a function $f : \mathcal{C} \to \Omega$ such that

$$f(A) \in A,$$

for each $A \in \mathcal{C}$.

First, we start with some useful remarks. Here $A, B, A_i, B_i$ are subsets of $\mathbb{R}$. Further, for $t \in \mathbb{R}$, $R_t : \mathbb{R} \to \mathbb{R}$ is the translation by $t$: $R_t(x) = x + t$.

**A1** We will say that a function $f : A \to \mathbb{R}$ is a (countable) equidecomposition if there exists a countable partition $A = \bigcup_{i=1}^{\infty} A_i$, and reals $\{t_i\}_{i \in \mathbb{N}}$ such that $f|_{A_i} = R_{t_i}|_{A_i}$ for each $i$.

Show that $A$ is equidecomposable with $B$ if and only if there exists an equidecomposition $f : A \to B$ which is bijective.

**A2** Let $A' \subseteq A$ and $B' \subseteq B$, and assume there exist bijective equidecompositions $f : A \to B'$ and $g : B \to A'$.

Construct an equidecomposition $h : A \to B$, and prove that it is bijective.

[Hint: Let $A^{(0)} \equiv A \setminus g(B)$, and $A^{(\ldots)} \equiv \bigcup_{n=0}^{\infty} (g \circ f)^n(A^{(0)})$. Define $h(x) = f(x)$ if $x \in A^{(*)}$ and $h(x) = g^{-1}(x)$ if $x \in A \setminus A^{(*)}$.]
Next to the actual problem:

B1 Use the axiom of choice to show that there exists $C \subseteq [0, 1/2]$ such that the following is a partition

$$\mathbb{R} = \bigcup_{x \in C} \{x + \mathbb{Q}\}. \quad (3)$$

B2 Show that $\mathbb{Q} \cap [0, 1/2]$ is equidecomposable with $\mathbb{Q}$.

B3 Deduce that

$$A \equiv \bigcup_{x \in C} \{x + (\mathbb{Q} \cap [0, 1/2])\} \subseteq [0, 1] \quad \text{is equidecomposable with } \mathbb{R}.$$  \quad (4)

B4 Use A1, A2 above to show that this implies that $[0, 1]$ is equidecomposable with $\mathbb{R}$. What does this result imply for measures on $\mathbb{R}$?