Exercises on characteristic functions

Exercise [3.3.10]

1. Denoting by $\Phi_X(\theta)$ the ch.f. of $X$, since $X$ and $\tilde{X}$ are i.i.d., the ch.f. of $-\tilde{X}$ is $\Phi_X(-\theta) = \Phi_X(\theta)$. Hence, by Lemma 3.3.8,
   $$\Phi_Z(\theta) = \Phi_X(\theta)\Phi_X(\theta) = |\Phi_X(\theta)|^2 \geq 0.$$  

2. If $U = X - \tilde{X}$ for some i.i.d. $X$ and $\tilde{X}$, then by part (a), its ch.f. $\Phi_U(\theta)$ must be a real-valued non-negative function. Recall that the ch.f. of the uniform random variable on $(a,b)$ is $\Phi_U(\theta) = e^{i\theta(a+b)/2} \sin(c\theta)/(c\theta)$ for $c = (b-a)/2$ (see Example 3.3.7). This function is real-valued only when $a = -b$ and even then $\sin(b\theta) = -1$ for $\theta = 3\pi/(2b) > 0$, leading to the stated conclusion.

Exercise [3.3.20]

1. Recall Example 3.3.7 showing that the Uniform Distribution on $(-1,1)$, which is of bounded probability density function, has the ch.f. $\sin(\theta)/\theta$. Clearly, $\int_0^\pi (|\sin(\theta)|/|\theta|) d\theta = \infty$ (consider $\theta \in [\pi n + \pi/4, \pi n + 3\pi/4]$, $n = 0, 1, 2, \ldots$ for which $|\sin(\theta)| \geq 1/\sqrt{2}$).

2. Recall Example 3.3.13 showing that the Cauchy distribution has the ch.f. $\exp(-|\theta|)$ which is not differentiable at $\theta = 0$.

Exercise [3.3.21]

Combining Lemma 3.3.8 and Example 3.3.7, we deduce that
   $$\Phi_{S_n}(\theta) = (\sin(\theta/\theta))^n.$$  

For any $n \geq 2$, the integral $\int (|\sin(\theta)|/|\theta|)^n d\theta$ is finite. Thus, by the inversion formula (3.3.7), the r.v. $S_n$ has the bounded continuous probability density function
   $$f_{S_n}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\theta s} (\sin(\theta/\theta))^n d\theta = \frac{1}{\pi} \int_0^\infty \cos(\theta s)(\sin(\theta/\theta))^n d\theta$$  

with the latter identity due to the fact that $(\sin(\theta)/\theta)$ is invariant under the change of variable $\theta \mapsto -\theta$. Since $S_n \leq n$, the continuous p.d.f. $f_{S_n}(\cdot)$ must be identically zero for $s > n$, yielding the stated conclusion that $\int_0^\infty \cos(\theta s)(\sin(\theta/\theta))^n d\theta = 0$ for all $s > n \geq 2$.

An exercise on weak convergence of measures

1. Indeed $|A_n| = \binom{n}{n/2}$ is finite and
   $$\nu_n = \frac{1}{|A_n|} \sum_{\xi \in A_n} \delta_\xi$$  

   (1)
(this identity can be checked on the $\pi$-system $\mathcal{P} = \{N_\ell(\omega) : \ell \in \mathbb{N}, \omega \in \Omega\}$). Each $\delta_\xi$ is a probability measure, hence $\nu_n$ is a probability measure.

2. We claim that $\nu_n$ converges weakly to the uniform measure $\nu_\infty$, defined by

$$\nu_\infty(N_\ell(\omega)) = \frac{1}{2^\ell}. \quad (2)$$

In order to prove this fact, we will show that, for any bounded continuous function $h : \{0, 1\}^\mathbb{N} \to \mathbb{R}$, $\lim_{n \to \infty} \nu_n(h) = \nu_\infty(h)$. Let us start by a function $h$ measurable on $\sigma(\{N_\ell(\omega) : \omega \in \Omega\})$, i.e. depending only on the first $\ell$ coordinates of $\omega$. For such a function we have $h(\omega) = h_\ell(\omega_1)$. Therefore

$$\nu_n(h) = \sum_{\xi_1} h_\ell(\xi_1) \nu_n(\{\omega : \omega_1^\ell = \xi_1\}). \quad (3)$$

But, letting $k = \xi_1 + \cdots + \xi_\ell$, we have

$$\lim_{n \to \infty} \nu_n(\{\omega : \omega_1^\ell = \xi_1^\ell\}) = \lim_{n \to \infty} \left(\frac{n}{n/2}\right)^{-1} \left(\frac{n - \ell}{n/2 - k}\right) = \frac{1}{2^\ell}, \quad (4)$$

where the last equality is a straightforward application of Stirling formula. This proves the claim for $h \in m\sigma(\{N_\ell(\omega) : \omega \in \Omega\})$.

Consider now a general bounded continuous function $h$, and let for $\ell \in \mathbb{N}$, $\hat{h}_\ell \in m\sigma(\{N_\ell(\omega) : \omega \in \Omega\})$ be defined by $\hat{h}_\ell(\omega) = h(\omega_1^\ell, 0, 0, 0 \ldots)$. By Fact 1, we have, for any probability measure $\mu$, $|\mu(h) - \mu(\hat{h}_\ell)| \leq \int |h(\omega) - \hat{h}_\ell(\omega)| d\mu(\omega) \leq \delta(\ell)$. Therefore

$$|\nu_n(h) - \nu_\infty(h)| \leq |\nu_n(h) - \nu_n(\hat{h}_\ell)| + |\nu_n(\hat{h}_\ell) - \nu_\infty(\hat{h}_\ell)| + |\nu_\infty(h) - \nu_\infty(\hat{h}_\ell)| \quad (5)$$

$$\leq |\nu_n(\hat{h}_\ell) - \nu_\infty(\hat{h}_\ell)| + 2\delta(\ell). \quad (6)$$

By letting $n \to \infty$, and using the above result, we get

$$\lim_{n \to \infty} |\nu_n(h) - \nu_\infty(h)| \leq 2\delta(\ell). \quad (7)$$

The thesis follows because $\ell$ can be taken arbitrarily large.