Stats 310A Midterm Solutions

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Problem 1

(a) Let $\mathcal{G}_0 = \{\cup_{i\in I_k} J_i | \forall I_k \in \mathcal{F}\}$, the sets contains one element($\{k\}$) is not in $\mathcal{G}_0$. It can be shown that $\mathcal{G} \subseteq \mathcal{G}_0$:

(1) $\mathcal{G}_0$ is a σ-algebra: (a) $\cup_{i\in\mathbb{N}} J_i \in \mathcal{G}_0$. (b)For all $B = \cup_{i\in I_k} J_i \in \mathcal{G}_0$, $B^c = \cup_{i\in I_k^c} J_i \in \mathcal{G}_0$. (c) If $B_k = \cup_{i\in I_k} J_i \in \mathcal{G}_0$, for k = 1,2,... then $\cup_k B_k = \cup_{i\in\cup_k I_k} J_i \in \mathcal{G}_0$.

(2) $J_k \in \mathcal{G}_0$, for all $k \in \mathbb{N}$.

As a result, the sets contains one element($\{k\}$) is not in $\mathcal{G}$, however, it is in $\mathcal{F} \Rightarrow \mathcal{G} \neq \mathcal{F}$.

(b) See (a), all sets contain only one element is not measurable on $\mathcal{G}$.

(c) For any function f we consider, let $f_N(\omega) = \sum_{n=0}^{N} f(n) I_{\{n\}}(\omega)$, by definition:

$$\mu(f_N) = \sum_{n=1}^{N} f(n)\mu_n,\ \forall N < \infty \tag{1}$$

As for all N,$f_N \leq f$, and $f_N \uparrow f$, by Monotone convergence theorem, we have

$$\mu(f) = \lim_{N\to\infty} \mu(f_N) = \sum_{n=0}^{\infty} \mu_n f(n) \tag{2}$$

Problem 2 It is obvious that $\mathcal{B}^{\mathbb{Q}} \subseteq 2^{\mathbb{Q}}$, and we just need to show the other way around. Because $\mathbb{Q}$ is countable, and for all $S \in \mathbb{Q}$, we can write it as a countable collection of singleton: $S = \cup_i \{q_i\}$, and $\{q_i\} \in B^{\mathbb{Q}}$ as it is closed. As a result, we have that, for all $S \in \mathbb{Q}$, $S \in \mathcal{B}^{\mathbb{Q}} \Rightarrow 2^{\mathbb{Q}} \subseteq B^{\mathbb{Q}}$. This ends the proof.

Problem 3

(a) For any $A \in \mathscr{B}$, by Komogorov extension theorem we have

$$\begin{aligned}
P(X_i \in A) &= P(X \in \mathbb{R} \times .. \times A ... \times \mathbb{R}) \\
&= \int_{\mathbb{R}\times..\times A...\times\mathbb{R}} f(x_1, ..., x_i, ..., x_n)\lambda_n(dx_1, ..., dx_n) \\
&= \int_{\mathbb{R}\times...\times\mathbb{R}\times A} f(x_1, ..., x_i, ..., x_n)\lambda_n(dx_1, .., dx_{i-1}, dx_{i+1}., dx_n, dx_i) \\
&= \int_A \left(\int_{\mathbb{R}\times...\times\mathbb{R}} f(x_1, ..., x, ..., x_n)\lambda_{n-1}(dx_1, .., dx_{i-1}, dx_{i+1}., dx_n) \right) \lambda_1(dx)
\end{aligned}$$

by consistency of Lesbegue measure.

By the given definition of density, we conclude that
$f_i(x) = \int_{\mathbb{R}^{n-1}} f(x_1, ..., x, ..., x_n)\lambda_{n-1}(dx_1, .., dx_{i-1}, dx_{i+1}., dx_n)$.

(b) By theorem in the book, we only need to show the equality on any generating set in $\mathbb{R}^n$. We proved that $\mathscr{B}_{R^n} = \sigma(A_1 \times ... \times A_n)$ for $A_1, ..., A_n \in \mathscr{B}$. Hence, for any measurable set $A_1, A_2, ..., A_n$, since $X_1, ..., X_n$ are mutually independent, we have

$$\begin{aligned} P(X \in A_1 \times ... \times A_n) = P(X_1 \in A_1, ..., X_n \in A_n) = \prod_{i=1}^n P(X_i \in A_i) = \prod_{i=1}^n \int_{A_i} f_i(x_i)\lambda_1(dx_i) \\ = \int_{A_1} ... \int_{A_n} f_1(x_1)...f_n(x_n)\lambda_1(dx_1)...\lambda_1(dx_n) \\ = \int_A f_1(x_1)...f_n(x_n)\lambda_n(dx_1, ..., dx_n) \text{ by property of Lesbegue measure.} \end{aligned}$$

Since this is true for any $A_1 \times ... \times A_n$, and by the definition of density we can conclude that the density of $(X_1, ..., X_n)$ exists and $f(X_1, ..., X_n) = f_1(X_1)...f_n(X_n)$.

(c) Counter example: $X_1 \sim U(0,1), X_2 \sim U(0,1)$ and $X_1 \overset{a.e.}{=} X_2$. Clearly P_{X_1}, P_{X_2} both have density $f(x) = \mathbf{1}(x \in [0,1])$. However, the support of (X_1, X_2) is $A = \{\omega = (\omega_1, \omega_2) : \omega_1 = \omega_2, 0 \le \omega_1, \omega_2 \le 1\}$. Thus, A is a line, and has Lesbegue measure 0 in dimension 2. If (X_1, X_2) has a density in dimension 2, then $P(\Omega) = P(A) = \int_A f(X_1, X_2)\lambda_2(dx_1, dx_2) = 0$ for some density f. However, $P(\Omega)$ should be 1. Thus, we get a contradiction, which implies (X_1, X_2) has no density in dimension 2.

Problem 4

(a) Define $X_n(\omega) = \sum_{k=1}^n \omega_k \lambda^{k-1}$. As $(\omega_1, \dots, \omega_n)$ follows the uniform distribution in $\{\pm 1\}^n$, the coordinates $\omega_1, \dots, \omega_n$ are mutually independent, giving that

$$\mathbb{E}[e^{i\xi X_n(\omega)}] = \prod_{k=1}^n \mathbb{E}[e^{i\xi\omega_k\lambda^{k-1}}] = \prod_{k=1}^n \frac{1}{2}\left(e^{i\lambda^{k-1}} + e^{-i\lambda^{k-1}}\right) = \prod_{k=1}^n \cos(\lambda^{k-1}\xi).$$

As $\lambda \in (0,1)$, we have for all $\omega \in \Omega$ that

$$|X_n(\omega) - X(\omega)| = \left|\sum_{k=n+1}^\infty \omega_k\lambda^{k-1}\right| \le \sum_{k=n+1}^\infty \lambda^{k-1} = \frac{\lambda^n}{1-\lambda} \to 0 \quad \text{as } n \to \infty.$$

So X_n converges pointwise (and hence almost surely) to X. Consequently, $\exp(i\xi X_n) \overset{a.s.}{\to} \exp(i\xi X)$. As $|\exp(i\xi X_n)| \le 1$, applying the dominated convergence theorem, we get

$$\mathbb{E}[e^{i\xi X}] = \lim_{n\to\infty} \mathbb{E}[e^{i\xi X_n}] = \lim_{n\to\infty} \prod_{k=1}^n \cos(\lambda^{k-1}\xi) = \prod_{k=1}^\infty \cos(\lambda^{k-1}\xi),$$

where the last limit exists automatically. To analytically justify the limit exists, use the inequality $|\cos(x) - 1| \le x^2$ for small x.

(b) We show that the shifted r.v. $\widetilde{\omega} = (\omega_2, \omega_3, \dots)$ has the same distribution as ω. Indeed, marginalizing over ω_1, the distribution over cylindrical sets is exactly the same as ω, hence by the Kolmogorov extension theorem, such an extension must be unique, and so $\widetilde{\omega} \stackrel{d}{=} \omega$. Further, as ω_1 is independent of any finite subcollection of $\{\omega_k\}_{k \geq 2}$, ω_1 is independent of $\widetilde{\omega}$. We thus have

$$\begin{aligned}
F_X(x) &= \mathbb{P}(X \leq x) \\
&= \mathbb{P}(\omega_1 = 1)\mathbb{P}\left(\sum_{n=2}^{\infty} \omega_n \lambda^{n-1} \leq x - 1\right) + \mathbb{P}(\omega_1 = -1)\mathbb{P}\left(\sum_{n=2}^{\infty} \omega_n \lambda^{n-1} \leq x - 1\right) \\
&= \frac{1}{2}\left(\mathbb{P}\left(\sum_{n=1}^{\infty} \widetilde{\omega}_n \lambda^{n-1} \leq \frac{x-1}{\lambda}\right) + \mathbb{P}\left(\sum_{n=1}^{\infty} \widetilde{\omega}_n \lambda^{n-1} \leq \frac{x+1}{\lambda}\right)\right) \\
&= \frac{1}{2}\left(F_X\left(\frac{x-1}{\lambda}\right) + F_X\left(\frac{x+1}{\lambda}\right)\right).
\end{aligned}$$

(c) This amounts to showing that the Cantor distribution does not have a density. First, note that X is supported on $A_0 = [-1/2, 1/2]$. We define closed sets $A_1, A_2, \dots$ as $A_1 = A_0 \setminus (-1/6, 1/6)$ and sequentially

$$A_{n+1} = \{A_n \text{ with the middle one third taken out from each sub-interval}\}.$$

We show that $X \in A_n$ almost surely for all A_n. Indeed, we have

$$|X| \geq |\omega_1/3| - \left|\sum_{n=2}^{\infty} \omega_n (1/3)^n\right| \geq \frac{1}{3} - \frac{1/9}{1 - 1/3} = \frac{1}{6},$$

which shows that $X \notin (-1/6, 1/6)$ and hence $X \in A_1$. To show the result for A_n, we condition on the value of $(\omega_1, \dots, \omega_n)$ (2^n possibilities) and use the same argumnent to show that it cannot belong to the middle one third of any sub-interval left.

Now, we have $\lambda(A_{n+1}) = 2/3\lambda(A_n)$ and hence $\lambda(A_n) = (2/3)^n \to 0$. Let $A = \lim_{n\to\infty} A_n = \bigcap_n A_n$, then $\lambda(A) = 0$. However, as $\mathbb{P}(X \in A_n) = 1$, taking the limit gives $\mathbb{P}(X \in A) = 1$. If X has density f w.r.t. λ, we then have

$$1 = \mathbb{P}(X \in A) = \int_A f(x) d\lambda(x) = 0,$$

a contradiction. So X does not have a density.