Stat 310A/Math 230A Theory of Probability

Practice Midterm Solutions

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Problem 1

Let λ_2 be the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$. We know already that it is invariant under translation i.e. that $\lambda_2(B + x) = \lambda_2(B)$ for any Borel set B and $x \in \mathbb{R}^2$ (whereby $B + x = \{y \in \mathbb{R}^2 \, : \, y - x \in B\}$).

(a) Show that it is invariant under rotations as well, i.e. that for any $\alpha \in [0, 2\pi]$, and any Borel set $B \subseteq \mathbb{R}^2$, $\lambda_2(\mathsf{R}(\alpha)B) = \lambda_2(B)$ (whereby $\mathsf{R}(\alpha)$ denotes a rotation by an angle α and $\mathsf{R}(\alpha)B = \{x \in \mathbb{R}^2 \, : \, \mathsf{R}(-\alpha)x \in B\}$).

Solution : Throughout the solution we will use the fact that $\lambda_2 = \lambda_1 \times \lambda_1$ whence we obtain the action of λ_2 on rectangles: $\lambda_2(A_1 \times A_2) = \lambda_1(A_1)\lambda_1(A_2)$. Also, for $J_1, J_2 \subseteq \mathbb{R}$ two intervals, let T_{J_1,J_2} be any triangle with two sides equal to J_1 (parallel to the first axis) and J_2 (equal to the second axis). from the addittivity of λ_2 it follows immediately that $\lambda_2(T_{J_1,J_2}) = |J_1| \cdot |J_2|/2$. (We use here the fact that for a segment $S = \{x_0 + x_1\lambda \, : \, \lambda \in [a, b]\}$, $x_0, x_1 \in \mathbb{R}^2$, $\lambda_2(S) = 0$, which can be proved by covering S with squares.)

Consider next a rectangle $A = [0, a) \times [0, b)$, and let $A' = \mathsf{R}(\alpha)A$. Using again addittivity it follows that, for $\beta = \pi/2 - \alpha$:

$$\begin{aligned}\lambda_2(A') &= (a\cos\alpha + b\cos\beta)(a\sin\alpha + b\sin\beta) - a^2\sin\alpha\cos\alpha - b^2\sin\beta\cos\beta \\ &= ab(\cos\alpha\sin\beta + \cos\beta\sin\alpha) = ab\sin(\alpha + \beta) = ab\,.\end{aligned}$$

Hence $\lambda_2(A) = \lambda_2(\mathsf{R}(\alpha)A)$ and by translation invariance this holds for any $A = [a_1, a_2) \times [b_1, b_2)$ (not necessarily with a corner at the origin).

Since the π-system $\mathcal{P} = \{A = [a_1, a_2) \times [b_1, b_2) \, : \, a_1 < a_2, b_1 < b_2\}$ generates the Borel σ algebra, and recalling that λ_2 is σ-finite, this proves the claim by Caratheodory uniqueness theorem.

(b) For $s \in \mathbb{R}_+$, and $B \subseteq \mathbb{R}^2$ Borel, let $sB \equiv \{x \in \mathbb{R}^2 \, : \, s^{-1}x \in B\}$. Prove that $\lambda_2(sB) = s^2\lambda_2(B)$.

Solution : The proof is analogous to the previous one. Let μ be the measure defined by $\mu(B) \equiv s^{-2}\lambda_2(sB)$. For $A = [a_1, a_2) \times [b_1, b_2)$, $a_1 < a_2, b_1 < b_2$, we have $sA = [sa_1, sa_2) \times [sb_1, sb_2)$, whence

$$\mu(A) = \frac{1}{s^2}\lambda_2(sA) = \frac{1}{s^2}(sa_2 - sa_1)(sb_2 - sb_1) = (a_2 - a_1)(b_2 - b_1) = \lambda_2(A)\,.$$

The claim follows by Caratheodory uniqueness theorem.

(c) For $r > 0$, $0 \le \alpha < \beta \le 2\pi$, let

$$C_{r,\alpha,\beta} \equiv \left\{x = (u\cos\theta, u\sin\theta) \, : \, u \in [0, r]\,, \theta \in [\alpha, \beta]\right\}. \tag{1}$$

Prove that $\lambda_2(C_{r,\alpha,\beta}) = (\beta - \alpha)r^2$.

Solution : Notice that $C_{r,\alpha,\beta} = rC_{1,\alpha,\beta}$. Therefore, by point (b) above, it is sufficient to prove the claim for $r = 1$. Further by invariance under rotation (point (a)), $\lambda_2(C_{1,\alpha,\beta}) = \lambda_2(C_{1,0,\beta-\alpha})$. It is therefore sufficient to show that $F(\theta) \equiv \lambda_2(C_{1,0,\theta}) = \theta/2$.

By covering $C_{1,0,\theta}$ with a triangle and inscribing a triangle in it we have

$$\frac{1}{2}\sin\theta\cos\theta \le F(\theta) \le \frac{1}{2}\,\tan\theta\,.$$

From these we have $F(\theta) = \theta/2 + O(\theta^2)$ as $\theta \to 0$. By addittivity of λ_2, and splitting $C_{1,0,\theta} = C_{1,0,\theta/n} \cup C_{1,\theta/n,2\theta/n} \cup \cdots \cup C_{1,\theta-\theta/n,\theta}$, we get

$$F(\theta) = nF(\theta/n) = \lim_{n\to\infty} nF(\theta/n) = \lim_{n\to\infty} n\Big[\frac{\theta}{2n} + O(\theta^2/n^2)\Big] = \frac{\theta}{2}\,.$$

This finishes the proof.

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d) Let $\Omega \equiv [0, 2\pi] \times [0, \infty)$, $g : \Omega \to \mathbb{R}_+$ be given by $g(\theta, r) = r$, and define ρ to be the measure on $(\Omega, \mathcal{B}_\Omega)$ with density g with respect to the Lebesgue measure.

For any function $f \in L_1(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}_2}, \lambda_2)$, let $\widehat{f} : \Omega \to \mathbb{R}$ be defined by $\widehat{f}(\theta, r) \equiv f(r\cos\theta, r\sin\theta)$. Prove that $f \in L_1(\Omega, \mathcal{B}_\Omega, \rho)$, and that

$$\int_\Omega \widehat{f}\ \mathrm{d}\rho = \int_{\mathbb{R}^2} f\ \mathrm{d}\lambda_2\,. \tag{2}$$

Solution : The proof follows by the Monotone Class Theorem. Denote by $\mathcal{H}$ the class of Borel functions such that (2) holds. Then (a) $1 \in \mathcal{H}$ since both sides are infinite; (b) If $h_1, h_2 \in \mathcal{H}$ then $c_1h_1 + c_2h_2 \in \mathcal{H}$ by linearity of the integral; (c) $\mathcal{H}$ is closed under limits from below by monotone convergence.

Finally, let $A = C_{r,\alpha,\beta}$. We claim that $f \equiv \mathbb{I}_A \in \mathcal{H}$. Indeed by point (c) above $\int_{\mathbb{R}^2} f\ \mathrm{d}\lambda_2$. On the other hand $\widehat{f}(\theta, u) = \mathbb{I}_{[\alpha,\beta]\times[0,r]}(\theta, u)$, whence $\int_\Omega \widehat{f}\ \mathrm{d}\rho = (\beta-\alpha)\int_0^r u\mathrm{d}\lambda_1(u) = (\beta-\alpha)r^2/2$. Therefore, for the π-system

$$\begin{aligned}\mathcal{P} &\equiv \big\{\widetilde{C}_{r,\alpha,\beta} :\ r \ge 0, 0 \le \alpha < \beta < 2\pi\big\}\\ \widetilde{C}_{r,\alpha,\beta} &\equiv \big\{x = (u\cos\theta, u\sin\theta)\ :\ u \in [0,r]\,, \theta \in [\alpha,\beta)\big\}\,,\end{aligned}$$

we have $\mathbb{I}_A \in \mathcal{H}$ for any $A \in \mathcal{P}$. The thesis is completed by noting that $\sigma(\mathcal{P})$ is the Borel σ-algebra (this is a standard argument).

Problem 2

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $\{A_n\}_{n\in\mathbb{N}}$ a sequence of measurable sets and $f \in L_1(\Omega, \mathcal{F}, \mu)$. Assume that

$$\lim_{n\to\infty}\int |\mathbb{I}_{A_n} - f|\,\mathrm{d}\mu = 0\,. \tag{3}$$

Prove that there exists $A \in \mathcal{F}$ such that $f = \mathbb{I}_A$ almost everywhere.

Solution : For $\epsilon > 0$, let A_ϵ be defined as

$$A_\epsilon \equiv \big\{\omega \in \Omega\ :\ \min(|f(\omega)|, |f(\omega) - 1|) \ge \epsilon\big\}\,.$$

Of course we have $|\mathbb{I}_{A_n} - f| \geq \epsilon\, \mathbb{I}_{A_\epsilon}$, whence

$$\mu(A_\epsilon) \leq \frac{1}{\epsilon} \int |\mathbb{I}_{A_n} - f| \, \mathrm{d}\mu \to 0\,.$$

Therefore

$$\mu(\{\omega\,:\,f(\omega) \notin \{0,1\}\}) = \mu(\cup_{k=1}^{\infty} A_{1/k}) = 0\,,$$

where the second identity follows since $A_{1/k}$ is an increasing sequence of sets. This finishes the proof.

Problem 3

Let $f_1, f_2 : [0,1] \to \mathbb{R}$ be two Borel functions with $f_1(x) \leq f_2(x)$ for all $x \in [0,1]$, and define $A \subseteq \mathbb{R}^2$ by

$$A \equiv \big\{(x,y) \in [0,1] \times \mathbb{R}\,:\, f_1(x) \leq y \leq f_2(x)\,\big\} \tag{4}$$

(a) Prove that A is a Borel set.

Solution : Indeed $A = A_1 \cap A_2^c$ where $A_a \equiv \big\{(x,y) \in [0,1] \times \mathbb{R}\,:\, f_a(x) \leq y\,\big\}$. To see that A_a is Borel, define $F_a : \mathbb{R}^2 \to \mathbb{R}$ by $F_a(x,y) = y - f_a(x)$. This is a Borel function (since it is the difference of Borel functions), and $A_a = F_a^{-1}([0,\infty))$, whence the claim follows.

(b) Denoting by λ_d the Lebesgue measure on $\mathbb{R}^d$, prove that

$$\lambda_2(A) = \int_{[0,1]} [f_2(x) - f_1(x)]\, \mathrm{d}\lambda_1(x)\,. \tag{5}$$

Solution : Applying Fubini's theorem to the non-negative Borel function $\mathbb{I}_A$ and the Lebesgue measure $\lambda_2 = \lambda_1 \times \lambda_1$, we have

$$\lambda_2(A) = \int \mathbb{I}_A \, \mathrm{d}\lambda_2(x,y) = \int_{[0,1]} \Big\{ \int_{\mathbb{R}} \mathbb{I}_{[f_1(x),f_2(x)]}(y) \mathrm{d}\lambda_1(y) \Big\} \mathrm{d}\lambda_1(x) = \int_{[0,1]} [f_2(x) - f_1(x)]\, \mathrm{d}\lambda_1(x)\,.$$

(c) For a Borel function $f : [0,1] \to \mathbb{R}$, and $y \in \mathbb{R}$, let

$$A_y \equiv \big\{x \in [0,1]\,:\, y = f(x)\,\big\}\,. \tag{6}$$

Prove that $\lambda_1(A_y) = 0$ for almost every y.

Solution : Let $A = \cup_{y \in \mathbb{R}} A_y$. Applying point (b) to $f_1 = f_2 = f$, we get $\lambda_2(A) = 0$. On the other hand

$$\lambda_2(A) = \int \mathbb{I}_A \, \mathrm{d}\lambda_2(x,y) = \int_{\mathbb{R}} \Big\{ \int_{[0,1]} \inf_{A_y}(x) \mathrm{d}\lambda_1(x) \Big\} \mathrm{d}\lambda_1(y) = \int_{\mathbb{R}} \lambda_1(A_y)\, \mathrm{d}\lambda_1(y)\,.$$

Since $\lambda_1(A_y) \geq 0$ and $\int_{\mathbb{R}} \lambda_1(A_y)\, \mathrm{d}\lambda_1(y) = 0$ it follows that $\lambda_1(A_y) = 0$ almost everywhere.

Problem 4

Let $\Omega = \{\mathsf{red}, \mathsf{blue}\}^{\mathbb{Z}^2}$ be the set of all possible ways to color the vertices of $\mathbb{Z}^2$ (the infinite 2-dimensional lattice) with two colors (red and blue). An element of this space is an assignment of colors $\omega : x \mapsto \omega_x \in \{\mathsf{red}, \mathsf{blue}\}$ for all $x \in \mathbb{Z}^2$.

Let A_x be the set of conficurations such that vertex x is red: $A_x = \{\omega : \omega_x = \mathsf{red}\}$, and consider the σ-algebra $\mathcal{F} \equiv \sigma(\{A_x : x \in \mathbb{Z}^2\})$.

Given a coloring ω, a *red cluster* R is a connected subset of red vertices. By 'connected' we mean that for any two vertices $x, y \in R$, there exists a nearest-neighbors path of red vertices connecting them (i.e. a sequence $x_1, x_2, \dots, x_n \in \mathbb{Z}^2$ such that $x_1 = x$, $x_n = y$, $\|x_{i+1} - x_i\| = 1$ and $\omega_{x_i} = \mathsf{red}$ for all i.

(a) Let $C \subseteq \Omega$ be the subset of configurations defined by

$$C = \big\{\omega : \omega \text{ contains a red cluster with infinitely many vertices } \big\} . \tag{7}$$

Prove that $C \in \mathcal{F}$.

Solution : Given integers $m < n$, let $C_{m,n}$ be the event that there exists a red cluster $R \subseteq \mathbb{Z}^2$ with at least one vertex $x \in R$ such that $\|x\|_\infty \le m$ and at least one vertex $x \in R$ such that $\|x\|_\infty \ge n$. Of course $C_{m,n} \in \mathcal{F}$ since membership in $C_{m,n}$ only depends $\{\omega_x : \|x\|_\infty \le n\}$.

Next consider $C_m \equiv \cap_{n=m+1}^\infty C_{m,n}$. This also is in $\mathcal{F}$ since is a countable intersection. Further C_m is the event that there exists an infinite red cluster with at least one vertex x such that $\|x\|_\infty \le m$. The proof is finished by noting that $C = \cup_{m=1}^\infty C_m$.

(b) Let $p \in [0, 1]$ be given and define $\mathbb{P}$ to be the probability measure on $(\Omega, \mathcal{F})$ such that the collection of events $\{A_x : x \in \mathbb{Z}^2\}$ are mutually independent, with $\mathbb{P}(A_x) = p$ for all $x \in \mathbb{Z}^2$.

Prove that either $\mathbb{P}(C) = 1$ or $\mathbb{P}(C) = 0$.

Solution : Let $X_\ell(\omega) = \omega_{x(\ell)}$ where $x(1), x(2), \dots$ is an ordering of the vertices of the two-dimensional lattice $\mathbb{Z}^2$ such that $\|x(\ell)\|_\infty$ is non-decreasing. Denote by $\mathcal{T}_\ell \equiv \sigma(X_\ell, X_{\ell+1}, \dots)$.

With the notation at the previous point, $C_m \in \mathcal{T}_\ell$ provided $m > \|x(\ell)\|_\infty$. As a consequence $C \in \mathcal{T} = \cap_\ell \mathcal{T}_\ell$. The proof is finished by applying Kolmogorov's 0-1 law.

Problem 5

Consider the measurable space $(\Omega, \mathcal{F})$, with: $\Omega = \{0, 1\}^{\mathbb{N}}$ the set of (infinite) binary sequences $\omega = (\omega_1, \omega_2, \omega_3, \dots)$; $\mathcal{F}$ the σ-algebra generated by cylindrical sets (equivalently the σ-algebra generated by sets of the type $A_{i,x} = \{\omega : \omega_i = x\}$ for $i \in \mathbb{N}$ and $x \in \{0, 1\}$).

Let $\mathbb{P}$ be the probability measure on $(\Omega, \mathcal{F})$ such that for all n

$$\mathbb{P}\big(\{\omega : (\omega_1, \dots, \omega_n) = (x_1, \dots, x_n)\}\big) = \frac{1}{2} \prod_{i=1}^{n-1} p_i(x_i, x_{i+1}) , \tag{8}$$

where

$$p_i(x_i, x_{i+1}) = \begin{cases} 1 - (1/i^2) & \text{if } x_i = x_{i+1}, \\ (1/i^2) & \text{otherwise.} \end{cases} \tag{9}$$

(a) Prove that a probability measure satisfying Eqs. (8) and (9) does indeed exist.

Solution : This follows by checking the hypotheses of Kolmogorov extension theorem, which is immediate

$$\sum_{x_n} \mathbb{P}\big(\{\omega\,:\,(\omega_1,\dots,\omega_n)=(x_1,\dots,x_n)\}\big) = \frac{1}{2}\prod_{i=1}^{n-2} p_i(x_i,x_{i+1})\sum_{x_n} p_{i-1}(x_{i-1},x_i)$$
$$= \mathbb{P}\big(\{\omega\,:\,(\omega_1,\dots,\omega_{n-1})=(x_1,\dots,x_{n-1})\}\,.$$

(b) Let $X_i(\omega)=\omega_i$ and consider the tail σ-algebra $\mathcal{T}=\cap_{n=1}^{\infty}\sigma(\{X_m\,:\,m\geq n\})$. Is $\mathcal{T}$ trivial? Prove your answer.

Solution : $\mathcal{T}$ is non-trivial.
Consider the events

$$A \equiv \big\{\omega\,:\,\lim\sup_{n\to\infty} X_n(\omega)=\lim\inf_{n\to\infty} X_n(\omega)\big\}\,,$$
$$A_0 \equiv \big\{\omega\,:\,\lim\sup_{n\to\infty} X_n(\omega)=\lim\inf_{n\to\infty} X_n(\omega)=0\big\}\,,$$
$$A_1 \equiv \big\{\omega\,:\,\lim\sup_{n\to\infty} X_n(\omega)=\lim\inf_{n\to\infty} X_n(\omega)=1\big\}\,.$$

Clearly $A, A_0, A_1 \in \mathcal{T}$. Further $A_0\cap A_1=\emptyset$ and $A_0\cup A_1=A$ whence by symmetry $\mathbb{P}(A_0)=\mathbb{P}(A_1)=\mathbb{P}(A)/2$. The claim follows by proving that $\mathbb{P}(A)=1$. To show this, consider the event $A^c=\{\omega\,:\,X_n(\omega)\neq X_{n+1}(\omega)$ infinitely often $\}$. Since

$$\sum_{n=1}^{\infty}\mathbb{P}\{X_n(\omega)\neq X_{n+1}(\omega)\}=\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty\,,$$

we have $\mathbb{P}(A^c)=0$ by Borel-Cantelli.
