Problem 1

Let \(\lambda_2\) be the Lebesgue measure on \((\mathbb{R}^2, B_{\mathbb{R}^2})\). We know already that it is invariant under translation i.e. that \(\lambda_2(B + x) = \lambda_2(B)\) for any Borel set \(B\) and \(x \in \mathbb{R}^2\) (whereby \(B + x = \{y \in \mathbb{R}^2 : y - x \in B\}\)).

(a) Show that it is invariant under rotations as well, i.e. that for any \(\alpha \in [0, 2\pi]\), and any Borel set \(B \subseteq \mathbb{R}^2\), \(\lambda_2(R(\alpha)B) = \lambda_2(B)\) (whereby \(R(\alpha)\) denotes a rotation by an angle \(\alpha\) and \(R(\alpha)B = \{x \in \mathbb{R}^2 : R(-\alpha)x \in B\}\)).

**Solution**: Throughout the solution we will use the fact that \(\lambda_2 = \lambda_1 \times \lambda_1\) whence we obtain the action of \(\lambda_2\) on rectangles: \(\lambda_2(A_1 \times A_2) = \lambda_1(A_1)\lambda_1(A_2)\). Also, for \(J_1, J_2 \subseteq \mathbb{R}\) two intervals, let \(T_{J_1, J_2}\) be any triangle with two sides equal to \(J_1\) (parallel to the first axis) and \(J_2\) (equal to the second axis). From the additivity of \(\lambda_2\) it follows immediately that \(\lambda_2(T_{J_1, J_2}) = |J_1| \cdot |J_2|/2\).

For any Borel set \(B \subseteq \mathbb{R}^2\), let \(\lambda_2(B)\) be the Lebesgue measure on \((\mathbb{R}^2, B_{\mathbb{R}^2})\). We know already that it is invariant under translation i.e. that \(\lambda_2(B + x) = \lambda_2(B)\) for any Borel set \(B\) and \(x \in \mathbb{R}^2\) (whereby \(B + x = \{y \in \mathbb{R}^2 : y - x \in B\}\)).

The proof is analogous to the previous one. Let \(\mu\) be the measure defined by \(\mu(B) \equiv s^{-2}\lambda_2(sB)\). For \(A = [a_1, a_2) \times [b_1, b_2)\), \(a_1 < a_2, b_1 < b_2\), we have \(sA = [sa_1, sa_2) \times [sb_1, sb_2)\), whence

\[
\mu(A) = \frac{1}{s^2} \lambda_2(sA) = \frac{1}{s^2} (sa_2 - sa_1)(sb_2 - sb_1) = (a_2 - a_1)(b_2 - b_1) = \lambda_2(A).
\]

The claim follows by Caratheodory uniqueness theorem.

(b) For \(s \in \mathbb{R}_+,\) and \(B \subseteq \mathbb{R}^2\) Borel, let \(sB \equiv \{x \in \mathbb{R}^2 : s^{-1}x \in B\}\). Prove that \(\lambda_2(sB) = s^2\lambda_2(B)\).

(c) For \(r > 0,\) \(0 \leq \alpha < \beta \leq 2\pi\), let

\[
C_{r,\alpha,\beta} \equiv \{x = (u \cos \theta, u \sin \theta) : u \in [0, r], \theta \in [\alpha, \beta]\}.
\]

Prove that \(\lambda_2(C_{r,\alpha,\beta}) = (\beta - \alpha)r^2\).
Problem 2

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $\{A_n\}_{n \in \mathbb{N}}$ a sequence of measurable sets and $f \in L_1(\Omega, \mathcal{F}, \mu)$. Assume that

$$\lim_{n \to \infty} \int |I_{A_n} - f| \, d\mu = 0. \quad (3)$$

Prove that there exists $A \in \mathcal{F}$ such that $f = I_A$ almost everywhere.

**Solution**: For $\epsilon > 0$, let $A_\epsilon$ be defined as

$$A_\epsilon \equiv \{ \omega \in \Omega : \min(|f(\omega)|, |f(\omega) - 1|) \geq \epsilon \}.$$
Of course we have \(|\mathbb{I}_{A_n} - f| \geq \epsilon\), whence
\[
\mu(A) \leq \frac{1}{\epsilon} \int |\mathbb{I}_{A_n} - f| \, d\mu \to 0.
\]
Therefore
\[
\mu(\{ \omega : f(\omega) \notin \{0, 1\}) = \mu(\bigcup_{k=1}^{\infty} A_{1/k}) = 0,
\]
where the second identity follows since \(A_{1/k}\) is an increasing sequence of sets. This finishes the proof.

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**Problem 3**

Let \(f_1, f_2 : [0, 1] \to \mathbb{R}\) be two Borel functions with \(f_1(x) \leq f_2(x)\) for all \(x \in [0, 1]\), and define \(A \subseteq \mathbb{R}^2\) by
\[
A = \{(x, y) \in [0, 1] \times \mathbb{R} : f_1(x) \leq y \leq f_2(x)\}
\]

(a) Prove that \(A\) is a Borel set.

**Solution**: Indeed \(A = A_1 \cap A_2^c\) where \(A_n = \{(x, y) \in [0, 1] \times \mathbb{R} : f_n(x) \leq y\}\). To see that \(A_n\) is Borel, define \(F_n : \mathbb{R}^2 \to \mathbb{R}\) by \(F_n(x, y) = y - f_n(x)\). This is a Borel function (since it is the difference of Borel functions), and \(A_n = F_n^{-1}((0, \infty))\), whence the claim follows.

(b) Denoting by \(\lambda_d\) the Lebesgue measure on \(\mathbb{R}^d\), prove that
\[
\lambda_2(A) = \int_{[0,1]} [f_2(x) - f_1(x)] \, d\lambda_1(x).
\]

**Solution**: Applying Fubini’s theorem to the non-negative Borel function \(\mathbb{I}_A\) and the Lebesgue measure \(\lambda_2 = \lambda_1 \times \lambda_1\), we have
\[
\lambda_2(A) = \int A \, d\lambda_2(x, y) = \int_{[0,1]} \left\{ \int_{\mathbb{R}} [f_2(x) - f_1(x)] \, d\lambda_1(y) \right\} \, d\lambda_1(x) = \int_{[0,1]} [f_2(x) - f_1(x)] \, d\lambda_1(x).
\]

(c) For a Borel function \(f : [0, 1] \to \mathbb{R}\), and \(y \in \mathbb{R}\), let
\[
A_y = \{ x \in [0, 1] : y = f(x) \}.
\]
Prove that \(\lambda_1(A_y) = 0\) for almost every \(y\).

**Solution**: Let \(A = \bigcup_{y \in \mathbb{R}} A_y\). Applying point (b) to \(f_1 = f_2 = f\), we get \(\lambda_2(A) = 0\). On the other hand
\[
\lambda_2(A) = \int A \, d\lambda_2(x, y) = \int_{\mathbb{R}} \left\{ \int_{[0,1]} \inf(x) \, d\lambda_1(x) \right\} \, d\lambda_1(y) = \int_{\mathbb{R}} \lambda_1(A_y) \, d\lambda_1(y).
\]
Since \(\lambda_1(A_y) \geq 0\) and \(\int_{\mathbb{R}} \lambda_1(A_y) \, d\lambda_1(y) = 0\) it follows that \(\lambda_1(A_y) = 0\) almost everywhere.
Problem 4

Let $Ω = \{\text{red, blue}\}^{Z^2}$ be the set of all possible ways to color the vertices of $Z^2$ (the infinite 2-dimensional lattice) with two colors (red and blue). An element of this space is an assignment of colors $ω : x \mapsto ω_x \in \{\text{red, blue}\}$ for all $x \in Z^2$.

Let $A_x$ be the set of configurations such that vertex $x$ is red: $A_x = \{ω : ω_x = \text{red}\}$, and consider the $σ$-algebra $F ≡ σ(\{A_x : x \in Z^2\})$.

Given a coloring $ω$, a red cluster $R$ is a connected subset of red vertices. By ‘connected’ we mean that for any two vertices $x, y \in R$, there exists a nearest-neighbors path of red vertices connecting them (i.e. a sequence $x_1, x_2, \ldots, x_n \in Z^2$ such that $x_1 = x$, $x_n = y$, $∥x_{i+1} − x_i∥ = 1$ and $ω_{x_i} = \text{red}$ for all $i$).

(a) Let $C \subseteq Ω$ be the subset of configurations defined by

$$C = \{ω : ω \text{ contains a red cluster with infinitely many vertices} \}.$$  \hspace{1cm} (7)

Prove that $C \in F$.

**Solution :** Given integers $m < n$, let $C_{m,n}$ be the event that there exists a red cluster $R \subseteq Z^2$ with at least one vertex $x \in R$ such that $∥x∥_∞ ≤ m$ and at least one vertex $x \in R$ such that $∥x∥_∞ ≥ n$. Of course $C_{m,n} \in F$ since membership in $C_{m,n}$ only depends $\{ω_x : ∥x∥_∞ ≤ n\}$.

Next consider $C_m ≡ ∩_{n=m+1}^{∞} C_{m,n}$. This also is in $F$ since is a countable intersection. Further $C_m$ is the event that there exists an infinite red cluster with at least one vertex $x$ such that $∥x∥_∞ ≤ m$. The proof is finished by noting that $C = ∪_{m=1}^{∞} C_m$.

(b) Let $p \in [0, 1]$ be given and define $P$ to be the probability measure on $(Ω, F)$ such that the collection of events $\{A_x : x \in Z^2\}$ are mutually independent, with $P(A_x) = p$ for all $x \in Z^2$.

Prove that either $P(C) = 1$ or $P(C) = 0$.

**Solution :** Let $X_ℓ(ω) = ω_x(ℓ)$ where $x(1), x(2), \ldots$ is an ordering of the vertices of the two-dimensional lattice $Z^2$ such that $∥x(ℓ)∥_∞$ is non-decreasing. Denote by $T_ℓ ≡ σ(X_ℓ, X_{ℓ+1}, \ldots)$. With the notation at the previous point, $C_m \in T_ℓ$ provided $m > ∥x(ℓ)∥_∞$. As a consequence $C \in T = ∩_ℓ T_ℓ$. The proof is finished by applying Kolmogorov’s 0-1 law.

Problem 5

Consider the measurable space $(Ω, F)$, with: $Ω = \{0,1\}^N$ the set of (infinite) binary sequences $ω = (ω_1, ω_2, ω_3, \ldots)$; $F$ the $σ$-algebra generated by cylindrical sets (equivalently the $σ$-algebra generated by sets of the type $A_{i,x} = \{ω : ω_i = x\}$ for $i \in N$ and $x \in \{0,1\}$).

Let $P$ be the probability measure on $(Ω, F)$ such that for all $n$

$$P(\{ω : (ω_1, \ldots, ω_n) = (x_1, \ldots, x_n)\}) = \frac{1}{2} \prod_{i=1}^{n-1} p_i(x_i, x_{i+1}) \hspace{1cm} (8)$$

where

$$p_i(x_i, x_{i+1}) = \begin{cases} 1 - (1/i^2) & \text{if } x_i = x_{i+1}, \\ (1/i^2) & \text{otherwise}. \end{cases} \hspace{1cm} (9)$$
(a) Prove that a probability measure satisfying Eqs. 8 and 9 does indeed exist.

**Solution:** This follows by checking the hypotheses of Kolmogorov extension theorem, which is immediate

\[
\sum_{x_n} \mathbb{P}\left( \{ \omega : (\omega_1, \ldots, \omega_n) = (x_1, \ldots, x_n) \} \right) = \frac{1}{2} \prod_{i=1}^{n-2} p_i(x_i, x_{i+1}) \sum_{x_n} p_{i-1}(x_{i-1}, x_i) \\
= \mathbb{P}\left( \{ \omega : (\omega_1, \ldots, \omega_{n-1}) = (x_1, \ldots, x_{n-1}) \} \right).
\]

(b) Let \( X_i(\omega) = \omega_i \) and consider the tail \( \sigma \)-algebra \( T = \cap_{n=1}^{\infty} \sigma(\{X_m : m \geq n\}) \). Is \( T \) trivial? Prove your answer.

**Solution:** \( T \) is non-trivial. Consider the events

\[
A \equiv \{ \omega : \limsup_{n \to \infty} X_n(\omega) = \liminf_{n \to \infty} X_n(\omega) \}, \\
A_0 \equiv \{ \omega : \limsup_{n \to \infty} X_n(\omega) = \liminf_{n \to \infty} X_n(\omega) = 0 \}, \\
A_1 \equiv \{ \omega : \limsup_{n \to \infty} X_n(\omega) = \liminf_{n \to \infty} X_n(\omega) = 1 \}.
\]

Clearly \( A, A_0, A_1 \in T \). Further \( A_0 \cap A_1 = \emptyset \) and \( A_0 \cup A_1 = A \) whence by symmetry \( \mathbb{P}(A_0) = \mathbb{P}(A_1) = \mathbb{P}(A)/2 \). The claim follows by proving that \( \mathbb{P}(A) = 1 \). To show this, consider the event \( A^c = \{ \omega : X_n(\omega) \neq X_{n+1}(\omega) \text{ infinitely often} \} \). Since

\[
\sum_{n=1}^{\infty} \mathbb{P}\{X_n(\omega) \neq X_{n+1}(\omega)\} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,
\]

we have \( \mathbb{P}(A^c) = 0 \) by Borel-Cantelli.