1. Problem 1
(a) To show \( \sigma(\mathcal{H}) \subseteq \mathcal{B}_{\mathbb{R}^2} \), it suffices to show \( \mathcal{H} \subseteq \mathcal{B}_{\mathbb{R}^2} \). Given \( A = [a, b] \times [c, d] \in \mathcal{H} \), where \( a < b, c < d \in \mathbb{R} \), we let \( A_n = (a - 1/n, b) \times (c, d) \). Note that \( A_n \downarrow A \). Since \( A_n \in \mathcal{B}_{\mathbb{R}^2} \), by closure of countable intersection for \( \sigma \)-algebra, we have \( A \in \mathcal{B}_{\mathbb{R}^2} \) as well. Inequality is implied by part (b). \( \sigma(\mathcal{H}) = \mathcal{B}_{\mathbb{R}^2} \) and clearly by the definition, \( \mathcal{B}_{\mathbb{R}^2} \neq \mathcal{B}_{\mathbb{R}^2} \).

(b) We first show that \( \mathcal{B}_{\mathbb{R}^2} \) is a \( \sigma \)-algebra.

i. \( \mathbb{R} \times \mathbb{R} \in \mathcal{B}_{\mathbb{R}^2} \);

ii. Suppose \( A \in \mathcal{B}_{\mathbb{R}^2} \). We want to show \( A^c = R(A) \). In fact, \( (x, y) \in A^c \iff (x, y) \notin R(A) \iff (x, y) \notin A \). The first equivalence is by \( A = R(A) \). The second and last are by definition of 'reflection' set. Thus, \( A^c = R(A) \).

iii. Suppose \( A_1, \ldots, A_n, \ldots \in \mathcal{B}_{\mathbb{R}^2} \). We want to show \( \bigcup_{i=1}^{\infty} A_i = R(\bigcup_{i=1}^{\infty} A_i) \). In fact, \( (x, y) \in \bigcup_{i=1}^{\infty} A_i \iff (x, y) \in A_k \) for some \( k \in \mathbb{N} \iff (x, y) \in R(A_k) \) for some \( k \in \mathbb{N} \iff (x, y) \in A_k \) for some \( k \in \mathbb{N} \iff (x, y) \in \bigcup_{i=1}^{\infty} A_i \). Thus, \( \bigcup_{i=1}^{\infty} A_i = R(\bigcup_{i=1}^{\infty} A_i) \).

Therefore, \( \mathcal{B}_{\mathbb{R}^2} \) is a \( \sigma \)-algebra. We clearly have \( \mathcal{H} \subseteq \mathcal{B}_{\mathbb{R}^2} \), and thus \( \sigma(\mathcal{H}) \subseteq \mathcal{B}_{\mathbb{R}^2} \).

Next we prove the other direction. Let \( \mathcal{R} = \{(a, b) \times [c, d] : a, b, c, d \in \mathbb{R} \text{ and } a < b, c < d \} \) be the collection of non-empty rectangles. First note that \( \mathcal{B}_{\mathbb{R}^2} = \sigma(\mathcal{R}) \). For a collection \( \mathcal{G} \) and a set \( L \), we define the intersection \( \mathcal{G} \cap L = \{G \cap L : G \in \mathcal{G} \} \). In the rest of the problem, we let \( L = \{(x, y) \in \mathbb{R}^2 : x \geq y \} \).

Let \( \mathcal{G} = \{A \in \mathcal{B}_{\mathbb{R}^2} : A \cap L = C \cap L \text{ for some } C \in \sigma(\mathcal{H}) \} \). We first show that \( \mathcal{R} \subseteq \mathcal{G} \). In fact, for any \( R = [a, b] \times [c, d] \), \( R \cap L = \{(a, b) \times [c, d] \cap ((c, a) \times [c, a])^c \cap ((d, b) \times [d, b])^c \cap L \} \). This implies \( R \in \mathcal{G} \) and thus \( \mathcal{R} \subseteq \mathcal{G} \). It is easy to verify that \( \mathcal{G} \) is a \( \sigma \)-algebra. Then as a result, \( \mathcal{B}_{\mathbb{R}^2} = \sigma(\mathcal{R}) \subseteq \mathcal{G} \). Thus, for any \( A \in \mathcal{B}_{\mathbb{R}^2} \), \( A \cap L = C \cap L \) for some \( C \in \sigma(\mathcal{H}) \).

For any \( A \in \mathcal{B}_{\mathbb{R}^2} \), we can write \( A = (A \cap L) \cup (A \cap R(L)) \). Since \( A \in \mathcal{B}_{\mathbb{R}^2} \), there exists \( H \in \sigma(\mathcal{H}) \) such that \( A \cap L = H \cap L \). Note that \( A \cap R(L) = R(A) \cap R(L) = R(A \cap L) = R(H \cap L) = R(H) \cap R(L) = H \cap R(L) \), where the last equality is by \( \sigma(\mathcal{H}) \subseteq \mathcal{B}_{\mathbb{R}^2} \). Thus, \( A = (H \cap L) \cup (A \cap R(L)) = H \cap (L \cup R(L)) = H \in \sigma(\mathcal{H}) \). Since \( A \) is arbitrary, we have \( \mathcal{B}_{\mathbb{R}^2} \subseteq \sigma(\mathcal{H}) \).

Therefore, \( \sigma(\mathcal{H}) = \mathcal{B}_{\mathbb{R}^2} \).

2. Problem 2
(a) Let \( g(x) = \sum_{n=1}^{\infty} n^{-\alpha} |f(nx)| \). By Fubini's theorem,
\[
\int g(x)\lambda(dx) = \int \left( \sum_{n=1}^{\infty} n^{-\alpha} |f(nx)| \right) \lambda(dx) = \sum_{n=1}^{\infty} n^{-\alpha} \int |f(nx)|\lambda(dx).
\]
Note that \( \int |f(nx)|\lambda(dx) = (1/n) \int |f(x)|\lambda(dx) \) by change of variables formula. We have
\[
\int g(x)\lambda(dx) = \sum_{n=1}^{\infty} n^{-(\alpha+1)} \int |f(x)|\lambda(dx) = \left( \int |f(x)|\lambda(dx) \right) \left( \sum_{n=1}^{\infty} n^{-(\alpha+1)} \right) < \infty,
\]
where the last inequality is because the series is convergent for \( \alpha > 0 \) and \( f \) is an integrable function. Note \( g \geq 0 \). \( \int g(x)\lambda(dx) < \infty \) implies that \( g < \infty \) a.e., and further implies that \( \sum_{n=1}^{\infty} n^{-\alpha} f(nx) \) is
well defined a.e. Since $|\sum_{n=1}^{\infty} n^{-\alpha} f(nx)| \leq \sum_{n=1}^{\infty} n^{-\alpha} |f(nx)|$ and the right hand side is integrable, we have $\sum_{n=1}^{\infty} n^{-\alpha} f(nx)$ is integrable as well.

(b) By part (a), we know $\sum_{n=1}^{\infty} n^{-\alpha} f(nx)$ is convergent a.e. On each $x$ where this series converges, we necessarily have $\lim_{n \to \infty} n^{-\alpha} f(nx) = 0$. This implies $\lim_{n \to \infty} n^{-\alpha} f(nx) = 0$ a.e..

3. Problem 3

(a) By definition, we have
\[
\int |T_a f(x)|^p \lambda(dx) = \int |f(x - a)|^p \lambda(dx) = \int |f(x)|^p \lambda(dx),
\]
where the last equality is due to change of variable and invariance of Lebesgue measure under translation. Thus we have $||T_a f||_p = ||f||_p$.

(b) Define $g_M(x) \equiv f(x) I(|x| \leq M)$. By triangular inequality it is sufficient to show that, for $M$ large enough, $\|g_M - f\|_p \leq \epsilon/2$ and $\|g_M - f\|_p \leq \epsilon/2$. For the first quantity, note that
\[
\limsup_{M \to \infty} \|g_M - f\|_p = \limsup_{M \to \infty} \inf_{M \to \infty} |f(x)|^p I(|x| > M) \lambda(dx) = 0,
\]
where the last equality follows from dominated convergence. For the second term
\[
\limsup_{M \to \infty} \|g_M - f\|_p = \limsup_{M \to \infty} \inf_{M \to \infty} |f(x)|^p I(|x| \leq M) \lambda(dx) \leq \limsup_{M \to \infty} \inf_{M \to \infty} |f(x)|^p I(|f(x)| > M) \lambda(dx) \leq \epsilon/2
\]
with the last equality following again by dominated convergence.

(c) Note that continuous functions with compact support are dense in $L_p(\mathbb{R})$. For any $\epsilon > 0$, there exists a continuous function $f_\epsilon$ with compact support such that $\|f - f_\epsilon\|_p < \epsilon$. Note that for continuous function $f_\epsilon$, we have $T_a f_\epsilon \to f_\epsilon$ as $a \to 0$. Using triangle inequality, we have
\[
\|f - T_a f\|_p \leq \|f - f_\epsilon\|_p + \|f_\epsilon - T_a f_\epsilon\|_p + \|T_a f_\epsilon - T_a f\|_p = 1 + II + III.
\]
The first term $I < \epsilon$ by our choice of $f_\epsilon$. The third term $III < \epsilon$ by invariance under translation. We look at the second term. Since $f_\epsilon$ is a continuous function with compact support $S$, there exists $M < \infty$ such that $|f_\epsilon| \leq M$ and $0 < u < \infty$ such that $S \subseteq [-u, u]$. Hence, if $|a| \leq 1$, $|f_\epsilon - T_a f_\epsilon| \leq 2M \|f_\epsilon\|_{L_p([-u-1, u+1])}$, and $\int |f_\epsilon - T_a f_\epsilon|^p \lambda(dx) \leq (2M)^p (2u + 2) < \infty$. We can thus apply dominated convergence theorem as $a \to 0$,
\[
\lim_{a \to 0} \int |f_\epsilon - T_a f_\epsilon|^p \lambda(dx) = \int \lim_{a \to 0} |f_\epsilon - T_a f_\epsilon|^p \lambda(dx) = 0.
\]
Hence, $\|f_\epsilon - T_a f_\epsilon\|_p \to 0$. This further implies that
\[
\limsup_{a \to 0} \|f - T_a f\|_p \leq 2\epsilon
\]
for any $\epsilon > 0$. Let $\epsilon \to 0$, and we have $\limsup_{a \to 0} \|f - T_a f\|_p = 0$. Therefore,
\[
\lim_{a \to 0} \|f - T_a f\|_p = 0.
\]

4. Problem 4

Throughout this problem we will represent a number in $[0, 1)$ by its non-terminating decimal expansion.
4.1. **Part a.** We wish to establish that $X_n$ is a random variable for each $n \geq 1$. We first note that it suffices to prove that the maps $E$ and $O$ are measurable (since the composition of measurable maps is measurable). Let $\omega_k$ denote a function which maps a point $\omega \in [0, 1)$ to the $k^{th}$ digit in its non-terminating decimal expansion — thus $\omega_k(\omega)$ is the $k^{th}$ digit in the decimal expansion of a point $\omega$. We will first establish that for all $k \geq 1$, $\omega_k : [0, 1) \to \{0, \cdots, 9\}$ is measurable. For $k = 1$, we have,

$$\{\omega : \omega_1(\omega) = i\} = \left[\frac{i}{10}, \frac{i+1}{10}\right).$$

Thus, $\omega_1$ is measurable, and its distribution is uniform on the set $\{0, 1, 2, \cdots, 9\}$. Again, we have, for $i, j \in \{0, \cdots, 9\}$, we have,

$$\{\omega : \omega_1(\omega) = i, \omega_2(\omega) = j\} = \left[\frac{i}{10} + \frac{j}{10}, \frac{i+1}{10} + \frac{j+1}{10}\right).$$

Taking an union over $i \in \{0, 1, \cdots, 9\}$ it follows that the events $\{\omega_2(\omega) = j\}$ are Borel measurable for each $j \in \{0, 1, 2, \cdots, 9\}$. Thus $\omega_2$ is measurable. Further, the above representation shows that

$$\mathbb{P}[\omega_1 = i, \omega_2 = j] = \frac{1}{100} = \mathbb{P}[\omega_1 = i] \mathbb{P}[\omega_2 = j].$$

Thus $\omega_2$ is uniformly distributed on the set $\{0, \cdots, 9\}$. Let $\omega_1, \omega_2 \in \{0, \cdots, 9\}$ are mutually independent. A direct extension of the ideas shows that the sequence $\{\omega_1, \omega_2, \cdots\}$ is i.i.d. with uniform distribution on the set $\{0, 1, \cdots, 9\}$. Now, we will show that $O$ and $E$ are measurable. We have, for each $n \geq 1$, $g_n = \sum_{k=1}^{n} \frac{\omega_k - 1}{10^k}$ is measurable. Further, it is easy to see that $\lim_n g_n(\omega)$ exists for each $\omega \in [0, 1)$. Thus $O = \lim_{n \to \infty} g_n$ is measurable. The same argument may be used to conclude that $E$ is measurable.

4.2. **Part b.** We have established that the sequence $\{\omega_1, \omega_2, \cdots\}$ is i.i.d. We simply observe that $X_n$ are functions of disjoint sets of independent random variables and thus independent.

4.3. **Part c.** It is easy to see that the $X_n$ are identically distributed. Thus it suffices to determine the distribution of $X_1$. Further, we see that for any $a = \sum_{i=1}^{m} a_i/10^i$ and $b = \sum_{j=1}^{n} b_j/10^j$ with $a_i, b_j \in \{0, 1, \cdots, 9\}$ and $a < b$,

$$\mathbb{P}[X_1 \in [a,b]] = (b-a).$$

Further, any $a, b \in [0, 1)$ may be approximated by points of the form $\sum_{i=1}^{n} a_i/10^i$. Thus we can conclude that $X_1$ is uniformly distributed on $[0, 1)$.

5. **Problem 5**

5.1. **Part (a).** Let $C = \{\mu : \int g \, d\mu \geq c \}, g : \mathbb{R} \to \mathbb{R}$ non-negative, measurable, $c \in \mathbb{R}$. Let $A \in C$. We will show that $T_f^{-1}(A) \in \mathcal{F}$ for all $A \in C$. To this end, let $A = \{\mu : \int g \, d\mu \geq c \}$ for some fixed $g$ non-negative measurable and $c \in \mathbb{R}$. Thus

$$T_f^{-1}(A) = \{\nu : T_f(\nu) \in A\} = \{\nu : \int g \circ f \, d\nu \geq c\} \in \mathcal{F},$$

as the function $g \circ f$ is non-negative measurable. Thus $T_f$ is measurable for all $f$ measurable.

5.2. **Part (b).** We first look at the case $n = 1$. In this case, for any non-negative measurable function $h$, we have,

$$F_h^{-1}([c, \infty)) = \{\mu : \int h \, d\mu \geq c\} \in \mathcal{F}.$$
For a general $n > 1$, if $h(x_1, \ldots, x_n) = \prod f_i(x_i)$, then $F_h = \prod F_{f_i}$ which is measurable. Now, let $S = h : \mathbb{R}^n \to \mathbb{R} : F_h$ is measurable. Then we note that $S$ contains all functions of the form $\prod f_i(x_i)$ and thus contains the indicators of $(-\infty, a_1] \times \cdots \times (-\infty, a_n]$ which is a $\pi$-system generating the Borel sigma algebra on $\mathbb{R}^n$. Further, we note that

(a) $S$ contains the identity function.
(b) $S$ is a vector space.
(c) $S$ is closed under increasing point wise limits of non-negative functions.

Thus the proof is completed by an application of the Monotone Convergence Theorem.