STATS 310A

Final Solution

1. Problem 1

(a) $1 = P(X \in \mathbb{R}) = P(X \in \bigcup_{k=-\infty}^{\infty} [k\epsilon - \epsilon, k\epsilon + \epsilon]) \leq \sum_{k=-\infty}^{\infty} P(X \in [k\epsilon - \epsilon, k\epsilon + \epsilon])$. Hence, there exists $x_0 = k_0\epsilon$ such that $P(X \in [x_0 - \epsilon, x_0 + \epsilon]) > 0$.

(b) By the first part, there exists x_0 such that $P(X \in [x_0 - \epsilon/2, x_0 + \epsilon/2]) > 0$. Since X, Y are independent and identically distributed, this holds for Y, i.e. $P(Y \in [x_0 - \epsilon/2, x_0 + \epsilon/2]) > 0$. Then,

$$\begin{aligned} P(|X - Y| \leq \epsilon) &\geq P(X \in [x_0 - \epsilon/2, x_0 + \epsilon/2], Y \in [x_0 - \epsilon/2, x_0 + \epsilon/2]) \\ &= P(X \in [x_0 - \epsilon/2, x_0 + \epsilon/2]) \cdot P(Y \in [x_0 - \epsilon/2, x_0 + \epsilon/2]) \\ &> 0. \end{aligned}$$

2. Problem 2

Let $\mu = EX_1$. We can write

$$W_1(F_n, F) = \int_{-\infty}^{-M} |F_n(t) - F(t)|dt + \int_{-M}^{M} |F_n(t) - F(t)|dt + \int_{M}^{\infty} |F_n(t) - F(t)|dt,$$

where $M > 0$ is a non-random constant. For any $\epsilon > 0$, we choose M such that $M > |\mu|$ and

$$\int_{-\infty}^{-M} F(t)dt < \epsilon, \quad \int_{M}^{\infty} (1 - F(t))dt < \epsilon.$$

This is possible because $\int_{-\infty}^{0} F(t)dt = -EX_1\mathbf{1}_(X_1 \leq 0) < \infty$ and $\int_{0}^{\infty}(1 - F(t))dt = EX_1\mathbf{1}_(X_1 \geq 0) < \infty$. By Glivenko-Cantelli, we have

$$\limsup_{n\to\infty} \int_{-M}^{M} |F_n(t) - F(t)|dt \leq \limsup_{n\to\infty} (2M \sup_{x\in\mathrm{R}} |F_n(x) - F(x)|) = 0, \ a.s.$$

The first term

$$\begin{aligned} \int_{-\infty}^{-M} |F_n(t) - F(t)|dt &\leq \int_{-\infty}^{-M} F_n(t)dt + \int_{-\infty}^{-M} F(t)dt \\ &= \int_{-\infty}^{-M} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(t \geq X_i)dt + \epsilon \\ &= \frac{1}{n}\sum_{i=1}^{n}(-M - X_i)_+ + \epsilon. \end{aligned}$$

Since $E|X_1| < \infty$, by the strong law of large numbers, we have $(1/n)\sum_{i=1}^{n}(-M - X_i)_+ \to (-M - \mu)_+ = 0, \ a.s.$. This implies

$$\limsup_{n\to\infty} \int_{-\infty}^{-M} |F_n(t) - F(t)|dt \leq \epsilon, \ a.s.$$

For the third term, if we write the integrand as

$$\int_{M}^{\infty} |F_n(t) - F(t)|dt = \int_{M}^{\infty} |(1 - F_n(t)) - (1 - F(t))|dt,$$

we can follow the same argument above and have $\limsup_{n\to\infty} \int_M^\infty |F_n(t) - F(t)|dt \le \epsilon,\ a.s.$ Thus we have showed that

$$\limsup_{n\to\infty} \int_{-\infty}^{\infty} |F_n(t) - F(t)|dt \le 2\epsilon,\ a.s.$$

Since $\epsilon > 0$ is arbitrary, we therefore have

$$\lim_{n\to\infty} W_1(F_n, F) = \lim_{n\to\infty} \int_{-\infty}^{\infty} |F_n(t) - F(t)|dt = 0,\ a.s.$$

3. Problem 3

(a) The characteristic function for $X_n \sim N(a_n, b_n)$ is $\Phi_n(\theta) = \exp(ia_n\theta - (1/2)b_n\theta^2)$. Since $a_n \to a, b_n \to b$, we have

$$\Phi_n(\theta) \to \exp(ia\theta - (1/2)b\theta^2) = \Phi(\theta).$$

Note that $\Phi(\theta)$ is continuous at $\theta = 0$. By Levy's continuity theorem, $X_n \xrightarrow{d} X$.

(b) In converse, $X_n \xrightarrow{d} X$ implies $\Phi_n(\theta) = \exp(ia_n\theta - (1/2)b_n\theta^2) \to f(\theta) = \Phi(\theta)$. In particular, we have point-wise convergence

$$\exp(-(1/2)b_n\theta^2) = |\Phi_n(\theta)| \to |f(\theta)|.$$

If b_n is unbounded, we can take a subsequence b_{n_k} such that $b_{n_k} \to \infty$, and this implies $|f(\theta)| = 0,\ \theta \ne 0$. Note $|f(\theta)| = |E\cos(\theta X) + iE\sin(\theta X)|$, it further implies $E\cos(\theta X) = 0,\ \theta \ne 0$. Since $\cos(\theta X) \to 1$ as $\theta \to 0$ and $|\cos(\theta X)| \le 1$, we have by dominated convergence theorem that the limit is $1 = 0$. Contradiction! Thus, b_n must be bounded. By Levy's continuity theorem, the sequence $N(a_n, b_n)$ must be uniformly tight. This, together with boundedness of variance b_n easily implies boundedness of a_n. From bounded sequences a_n, b_n, we can take a subsequence a_{n_k}, b_{n_k} such that

$$a_{n_k} \to a, \quad b_{n_k} \to b \quad (k \to \infty).$$

As a result, $f(\theta) = \lim_{k\to\infty} \Phi_{n_k}(\theta) = \exp(ia\theta - (1/2)b\theta^2)$. The limit $X \sim N(a, b)$.

4. Problem 4

$\{X_i : i \ge 1\}$ are a sequence of i.i.d. random variables with uniform distribution on $[0, 1]$. For each $n \ge 1$, we define $Y_1^{(n)} < \cdots < Y_n^{(n)}$ to be the ordered values of $\{X_1, \cdots, X_n\}$.

4.1. **Part a.** In problem statement, it should be $Y_l^{(n)}$.

We wish to show that for $n \ge 1$ and $l \in \{1, \cdots, n\}$, $Y_l^{(n)}$ is a valid random variable. Setting $\mathcal{S} = \{S : S \subset [n], |S| = l\}$, we note that

$$\{Y_l^{(n)} \le b\} = \cup_{S\in\mathcal{S}}\{X_i \le b, i \in S\}.$$

We note that this implies that $Y_l^{(n)}$ is a random variable.

4.2. **Part b.** We define $Z_n = \sqrt{4n}(Y_{\lfloor n/2\rfloor}^{(n)} - 1/2)$. Also, we define the empirical distribution function

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{(X_i \le x)}.$$

Using the Central Limit Theorem, we have, for each $x \in [0, 1]$,

$$\sqrt{n}\frac{(\hat{F}_n(x) - F(x))}{\sqrt{F(x)(1 - F(x))}} \xrightarrow{D} \mathcal{N}(0, 1). \tag{4.1}$$

Now, we have, noting that $F(x) = x$ for the uniform distribution,

$$\begin{aligned}\mathbb{P}[Z_n \le x] &= \mathbb{P}\Big[Y^{(n)}_{\lfloor n/2\rfloor} \le \frac{1}{2} + \frac{x}{\sqrt{4n}}\Big] = \mathbb{P}\Big[\hat{F}_n(\frac{1}{2} + \frac{x}{\sqrt{4n}}) \ge \frac{1}{2}\Big] \\ &= \mathbb{P}\Big[\sqrt{n}\frac{\hat{F}_n(1/2 + x/\sqrt{4n}) - (1/2 + x/\sqrt{4n})}{\frac{1}{2}\sqrt{1 - x^2/n}} \ge -x\Big] \\ &\to \Phi(x),\end{aligned}$$

using the Central Limit Theorem, where $\Phi(\cdot)$ is the cdf of the standard normal distribution. This completes the argument.

4.3. **Part c.** Typo: "... solution of $f(x) = 1/2$" should be $F(x) = 1/2$. Also F' should be continuous.

We proceed similarly as in Part b. Using equation (4.1) and $F(x_0) = 1/2$, we have,

$$\mathbb{P}[Z_n \le x] = \mathbb{P}\Big[\sqrt{n}\frac{\hat{F}_n(x_0 + x/\sqrt{4n}) - F(x_0 + x/\sqrt{4n})}{\sqrt{F(x_0 + x/\sqrt{4n})(1 - F(x_0 + x/\sqrt{4n}))}} \ge \sqrt{n}\frac{F(x_0) - F(x_0 + x/\sqrt{4n})}{\sqrt{F(x_0 + x/\sqrt{4n})(1 - F(x_0 + x/\sqrt{4n}))}}\Big].$$

Using Taylor expansion, $F(x_0 + x/\sqrt{4n}) = F(x_0) + F'(\psi)x/\sqrt{4n}$ for some $\psi \in [x_0, x_0 + x/\sqrt{4n}]$. Finally, if $n \to \infty$, $\psi \to x_0$. Thus, letting $n \to \infty$, we get the desired conclusion, with $\sigma^2 = 1/F'(x_0)^2$. This completes the proof.

5. Problem 5

$Z : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ is a measurable mapping. We note that the mapping $x \to \langle x, v\rangle$ is continuous and thus $\langle Z, v\rangle$ is a random variable for each $v \in \mathcal{H}$.

5.1. **Part a.** Assume $\mathbb{E}[\|Z\|] < \infty$. We wish to show that there exists a unique $u \in \mathcal{H}$ such that for each $v \in \mathcal{H}$, $\langle u, v\rangle = \mathbb{E}[\langle Z, v\rangle]$. We note that $\mathbb{E}[\|Z\|] < \infty$ implies that $\mathbb{E}[|\langle Z, v\rangle|] < \infty$. We will first prove uniqueness. Suppose there exists u_1, u_2 satisfying

$$\langle u_1, v\rangle = \langle u_2, v\rangle = \mathbb{E}[\langle Z, v\rangle].$$

This implies in particular that $\langle u_1, e_j\rangle = \langle u_2, e_j\rangle$ for all $j \ge 1$ which in turn implies that $u_1 = u_2$. This implies the uniqueness. Next, we establish the existence of at least one such $u \in \mathcal{H}$.

We define the sequence $u_n = \sum_{j=1}^n \mathbb{E}[\langle Z, e_j\rangle]e_j$. We will show that this sequence is Cauchy and define $\mathbb{E}[Z] := \lim_{n\to\infty} u_n$. Finally, we will establish that this definition satisfies the required criteria.

To this end, define the sequence of random variables $Z_n = \sum_{j=1}^n \langle Z, e_j\rangle e_j$. It is easy to see by direct calculation that $\mathbb{E}[Z_n] = u_n$, $\|Z_n - Z\| \to 0$ a.s. and that $\|Z_n\| \le \|Z\|$. Thus we have, $\|Z_n - Z\| \le 2\|Z\|$, which has finite expectation. Using DCT, we conclude, $\mathbb{E}[\|Z_n - Z\|] \to 0$ as $n \to \infty$. This further implies that for any $\varepsilon > 0$ and $n > m$ sufficiently large , $\mathbb{E}[\|Z_n - Z_m\|] \le \varepsilon$. Using Parseval identity, we note that $\mathbb{E}[\|Z_n - Z_m\|] = \mathbb{E}\|\phi_{n-m}\|_2$, for the R^{n-m} valued random variable $\phi_{n-m} = (\langle Z, e_{m+1}\rangle, \cdots, \langle Z, e_n\rangle)$. Using Jensen's inequality, we have, $\|\mathbb{E}[\phi_{n-m}]\|_2 \le \mathbb{E}\|\phi_{n-m}\|_2$. Another application of Parseval identity implies that $\|\mathbb{E}[Z_n] - \mathbb{E}[Z_m]\| = \|\mathbb{E}[\phi_{n-m}]\|_2$. This implies that the sequence u_n is Cauchy and therefore convergent. We set $\mathbb{E}[Z] := \lim_{n\to\infty} u_n$. Thus we have, for any $v \in \mathcal{H}$,

$$\langle \mathbb{E}[Z], v\rangle = \lim_{n\to\infty} \langle u_n, v\rangle = \lim_{n\to\infty} \mathbb{E}[\langle Z_n, v\rangle].$$

Finally, we note that $\langle Z_n, v\rangle \to \langle Z, v\rangle$ as $n \to \infty$ a.s. and $|\langle Z_n, v\rangle| \le \|Z\|\|v\|$. Thus by DCT, $\lim_{n\to\infty} \mathbb{E}[\langle Z_n, v\rangle] = \mathbb{E}[\langle Z, v\rangle]$. This completes the proof.

5.2. **Part b.** Z, Y are independent $\mathcal{H}$ valued random variables with $\mathbb{E}[\|Z\|^2] < \infty, \mathbb{E}[\|Y\|^2] < \infty$. This implies that $(\Omega, \mathcal{F}) \to (\mathcal{H}^2, \mathcal{B}_{\mathcal{H}^2})$, $\omega \to (Y(\omega), Z(\omega))$ is measurable. As the inner product $\langle \cdot, \cdot \rangle$ is bicontinuous, this implies that $\langle Z, Y \rangle$ is a valid random variable. Further, $|\langle Z, Y \rangle| \leq \|Z\|\|Y\| \leq (\|Z\|^2 + \|Y\|^2)/2$, which implies $\mathbb{E}[|\langle Z, Y \rangle|] < \infty$.

We define $Z_n = \sum_{j=1}^n \langle Z, e_j \rangle e_j$ and $Y_n = \sum_{j=1}^n \langle Y, e_j \rangle e_j$ as in Part a. Then we have,

$$\mathbb{E}[\langle Z_n, Y_n \rangle] = \mathbb{E}[\sum_{j=1}^n \langle Z, e_j \rangle \langle Y, e_j \rangle] = \sum_{j=1}^n \mathbb{E}[\langle Z, e_j \rangle] \mathbb{E}[\langle Y, e_j \rangle] = \langle \mathbb{E}[Z_n], \mathbb{E}[Y_n] \rangle,$$

where we use that $\langle Z, e_j \rangle$ and $\langle Y, e_j \rangle$ are independent. We have,

$$|\langle Z_n, Y_n \rangle - \langle Z, Y \rangle| \leq |\langle Z_n - Z, Y_n \rangle| + |\langle Z, Y_n - Y \rangle| \leq \|Y_n - Y\|\|Z\| + \|Z_n - Z\|\|Y\|, \tag{5.1}$$

which implies $|\langle Z_n, Y_n \rangle - \langle Z, Y \rangle| \to 0$ a.s. as $n \to \infty$. Further, the expression on the RHS of (5.1) is dominated by $C\|Y\|\|Z\|$ for a sufficiently large constant C and therefore has finite expectation. This implies $\mathbb{E}[\langle Y_n, Z_n \rangle] \to \mathbb{E}[\langle Y, Z \rangle]$ as $n \to \infty$. By definition, $\mathbb{E}[Z_n] \to \mathbb{E}[Z]$ and $\mathbb{E}[Y_n] \to \mathbb{E}[Y]$ which implies, by the bicontinuity of the inner product that $\langle \mathbb{E}[Z_n], \mathbb{E}[Y_n] \rangle \to \langle \mathbb{E}[Z], \mathbb{E}[Y] \rangle$. Combining all these observations yields the desired result.

5.3. **Part c and Part d.** Parts (c) and (d) can be directly adapted from the proofs of Kolmogorov Maximal inequality (Proposition 2.3.15) and the strong law (Theorem 2.3.16).