1 Characteristic functions

The characteristic function of a real-valued R.V. $X$ is defined as

$$\Phi_X(\theta) = \int \limits_{\mathbb{R}} e^{i\theta x} dP_X(x) = \mathbb{E}[e^{i\theta X}] = \mathbb{E}[\cos(\theta X)] + i\mathbb{E}[\sin(\theta X)].$$

As sines and cosines are bounded, the above expectations exist and are finite, so $\Phi_X(\theta)$ is well defined for all $\theta \in \mathbb{R}$. The following result summarizes some basic properties of the characteristic function.

**Proposition 1.** We have

(a) $\Phi_X(0) = 1$.

(b) $\Phi_X(-\theta) = \overline{\Phi_X(\theta)}$.

(c) $|\Phi_X(\theta)| \leq 1$.

(d) $\theta \mapsto \Phi_X(\theta)$ is a uniformly continuous function on $\mathbb{R}$.

(e) $\Phi_{aX+b}(\theta) = e^{ib\theta} \Phi_X(a\theta)$.

As characteristic functions offer a way to represent a distribution on $\mathbb{R}$, one naturally wonders if such a representation is one-to-one, i.e. does $\Phi_X(\cdot)$ uniquely determine the law of $X$? The answer is yes, which is stated in the following result.

**Theorem 1** (Levy’s inversion formula, Thm 3.3.12 in Dembo’s Notes). Let $X$ have distribution function $F_X$ and characteristic function $\Phi_X$. For any real numbers $a < b$ and $\theta$, let

$$\psi_{a,b}(\theta) = \frac{1}{2\pi} \int_a^b e^{-i\theta u} du = \frac{e^{-i\theta a} - e^{-i\theta b}}{i2\pi\theta}.$$ 

Then,

$$\lim_{T \to \infty} \int_{-T}^{T} \psi_{a,b}(\theta) \Phi_X(\theta) d\theta = \frac{1}{2} [F_X(b) + F_X(b-)] - \frac{1}{2} [F_X(a) + F_X(a-)].$$

(1)

Furthermore, if $\int_{\mathbb{R}} |\Phi_X(\theta)| d\theta < \infty$, then $X$ has the bounded continuous probability density function

$$f_X(x) = \frac{1}{2\pi} \int \limits_{\mathbb{R}} e^{-i\theta x} \Phi_X(\theta) d\theta.$$ 

(2)
Proof of (1) The proof follows by carefully computing and swapping the integrals in the inversion formula. Let \( J_T(a, b) = \int_{-T}^{T} \psi_{a,b}(\theta) \Phi_X(\theta) d\theta \), we aim to compute the limit of \( J_T(a, b) \) as \( T \to \infty \). For this end, we define

\[
S(x, \theta) = \psi_{a,b}(\theta) e^{i\theta x} = \frac{e^{i\theta(x-a) - e^{i\theta(b)}}}{i2\pi \theta}.
\]

We have \(|h_{a,b}(x, \theta)| = |\psi_{a,b}(\theta)| \leq \frac{b-a}{2\pi} \). So on the space \( \mathbb{R} \times [-T, T] \) with the product measure of \( P_X \) and the Lebesgue measure on \([-T, T]\), the bounded function \( h_{a,b}(x, \theta) \) is integrable. So we can apply Fubini theorem to get

\[
J_T(a, b) = \int_{-T}^{T} \psi_{a,b}(\theta) \Phi_X(\theta) d\theta = \int_{-T}^{T} \psi_{a,b}(\theta) \left[ \int_{R} e^{i\theta x} dP_X(x) \right] d\theta = \int_{R \times [-T, T]} h_{a,b}(x, \theta) dP_X(x) d\theta
\]

where

\[
R(u, T) = \int_{-T}^{T} \frac{e^{i\theta u}}{i2\pi \theta} d\theta = \int_{-T}^{T} \cos(\theta u) + i \sin(\theta u) d\theta = \frac{\int_{0}^{\pi} \sin(\theta u) d\theta}{\pi} = \frac{\int_{0}^{\pi} \sin(\theta u) d\theta}{\pi} = \frac{\int_{0}^{\pi} \sin(\theta u) d\theta}{\pi} S(|u| T),
\]

and \( S(t) = \int_{0}^{t} \frac{\sin(\theta u)}{\theta} d\theta \). Applying the fact that \( \lim_{t \to \infty} S(t) = \pi/2 \), we can deduce that

\[
\lim_{T \to \infty} R(x - a, T) - R(x - b, T) = g_{a,b}(x) := \begin{cases} 
0, & x < a \text{ or } x > b \\
1/2, & x = a \text{ or } x = b \\
1, & a < x < b.
\end{cases}
\]

Further, the quantities \( S(t) \) are uniformly bounded: \( \sup_{t \in \mathbb{R}} |S(t)| \leq C < \infty \). By bounded convergence, we get that

\[
\lim_{T \to \infty} J_T(a, b) = \lim_{T \to \infty} \int_{R} [R(x - a, T) - R(x - b, T)] dP_X(x) = \int_{R} \lim_{T \to \infty} [R(x - a, T) - R(x - b, T)] dP_X(x)
\]

\[
= \int_{R} g_{a,b}(x) dP_X(x) = \frac{1}{2} P_X(\{a\}) + P_X(\{a, b\}) + \frac{1}{2} P_X(\{b\})
\]

\[
= \frac{1}{2} (F_X(a) - F_X(a-) + F_X(b) - F_X(b-)) + F_X(b-) - F_X(a)
\]

\[
= \frac{1}{2} (F_X(b) + F_X(b-)) - \frac{1}{2} (F_X(a) + F_X(a-)).
\]

Proof of the density formula As \( \int_{R} |\Phi_X(\theta)| d\theta < \infty \), the integrand \( e^{-i\theta x} \Phi_X(\theta) \) is upper bounded by \( |\Phi_X(\theta)| \) and so also integrable, therefore \( f_X(x) \) is well-defined and finite valued. Taking any \( x \) and a point \( x + h \) close to \( x \), we have

\[
\limsup_{h \to 0} |f_X(x + h) - f_X(x)| \leq \limsup_{h \to 0} \frac{1}{2\pi} \int_{R} |e^{-i\theta h} - 1||\Phi_X(\theta)| d\theta = 0,
\]

equality following from the dominated convergence theorem. This shows that \( f_X(x) \) is continuous in \( x \). Now, as \( \psi_{a,b}(\theta) \) is bounded and \( \Phi_X(\theta) \) is integrable, applying dominated convergence to
\( \psi_{a,b}(\theta) \Phi_X(\theta) \mathbf{1}_{\{\theta \leq T\}} \) gives that

\[
\lim_{T \to \infty} J_T(a, b) = J_\infty(a, b) = \int_{\mathbb{R}} \psi_{a,b}(\theta) \Phi_X(\theta) d\theta = \frac{1}{2\pi} \int_{\mathbb{R}} \left[ \int_{a}^{b} e^{-i\theta u} du \right] \Phi_X(\theta) d\theta
\]

\[
= \int_{a}^{b} \int_{\mathbb{R}} \left[ \frac{1}{2\pi} e^{-i\theta u} \Phi_X(\theta) d\theta \right] du
\]

\[
= \int_{a}^{b} f_X(u) du.
\]

In particular, this shows that \( J_\infty(a, b) \) is continuous in \( a, b \). On the other hand, the result (1) gives

\[
J_\infty(a, b) = \frac{1}{2} (F_X(b) + F_X(b-)) - \frac{1}{2} (F_X(a) + F_X(a-)),
\]

therefore the RHS has to be continuous in \( a, b \). This implies \( F_X \) is itself continuous (check this!), and thus \( F_X(a) = F_X(a-) \), \( F_X(b) = F_X(b-) \) and so

\[
\int_{a}^{b} f_X(u) du = J_\infty(a, b) = F_X(b) - F_X(a).
\]

Hence \( f_X \) is the density of \( X \). \( \square \)

References