Stats 310A Session 7

November 16, 2017

1 Characteristic functions

The characteristic function of a real-valued R.V. X is defined as

$$\Phi_X(\theta) = \int_{\mathbb{R}} e^{i\theta x} dP_X(x) = \mathbb{E}[e^{i\theta X}] = \mathbb{E}[\cos(\theta X)] + i\mathbb{E}[\sin(\theta X)].$$

As sines and cosines are bounded, the above expectations exist and are finite, so $\Phi_X(\theta)$ is well defined for all $\theta \in \mathbb{R}$. The following result summarizes some basic properties of the characteristic function.

Proposition 1. *We have*

(a) $\Phi_X(0) = 1$.

(b) $\Phi_X(-\theta) = \overline{\Phi_X(\theta)}$.

(c) $|\Phi_X(\theta)| \leq 1$.

(d) $\theta \mapsto \Phi_X(\theta)$ *is a uniformly continuous function on* $\mathbb{R}$.

(e) $\Phi_{aX+b}(\theta) = e^{ib\theta}\Phi_X(a\theta)$.

As characteristic functions offer a way to represent a distribution on $\mathbb{R}$, one naturally wonders if such a representation is one-to-one, i.e. does $\Phi_X(\cdot)$ uniquely determine. the law of X? The answer is yes, which is stated in the following result.

Theorem 1 (Levy's inversion formula, Thm 3.3.12 in Dembo's Notes)**.** *Let X have distribution function F_X and characteristic function Φ_X. For any real numbers $a < b$ and θ, let*

$$\psi_{a,b}(\theta) = \frac{1}{2\pi}\int_a^b e^{-i\theta u} du = \frac{e^{-i\theta a} - e^{-i\theta b}}{i2\pi\theta}.$$

Then,

$$\lim_{T\uparrow\infty} \int_{-T}^{T} \psi_{a,b}(\theta)\Phi_X(\theta)d\theta = \frac{1}{2}[F_X(b) + F_X(b-)] - \frac{1}{2}[F_X(a) + F_X(a-)]. \tag{1}$$

Furthermore, if $\int_R |\Phi_X(\theta)|d\theta < \infty$, then X has the bounded continuous probability density function

$$f_X(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\theta x}\Phi_X(\theta)d\theta. \tag{2}$$

x **Proof**

Proof of (1) The proof follows by carefully computing and swapping the integrals in the inversion formula. Let $J_T(a,b) = \int_{-T}^{T} \psi_{a,b}(\theta)\Phi_X(\theta)d\theta$, we aim to compute the limit of $J_T(a,b)$ as $T \to \infty$. For this end, we define

$$h_{a,b}(x,\theta) = \psi_{a,b}(\theta)e^{i\theta x} = \frac{e^{i\theta(x-a)-e^{i\theta(x-b)}}}{i2\pi\theta}.$$

We have $|h_{a,b}(x,\theta)| = |\psi_{a,b}(\theta)| \leq \frac{b-a}{2\pi}$. So on the space $\mathbb{R} \times [-T,T]$ with the product measure of P_X and the Lebsgue measure on $[-T,T]$, the bounded function $h_{a,b}(x,\theta)$ is integrable. So we can apply Fubini theorem to get

$$J_T(a,b) = \int_{-T}^{T} \psi_{a,b}(\theta)\Phi_X(\theta)d\theta = \int_{-T}^{T} \psi_{a,b}(\theta)\left[\int_R e^{i\theta x}dP_X(x)\right]d\theta = \int_{\mathbb{R}\times[-T,T]} h_{a,b}(x,\theta)dP_X(x)d\theta$$
$$= \int_{\mathbb{R}}\left[\int_{-T}^{T} h_{a,b}(x,\theta)d\theta\right]dP_X(x) = \int_{\mathbb{R}}\left[R(x-a,T) - R(x-b,T)\right]dP_X(x),$$

where

$$R(u,T) = \int_{-T}^{T} \frac{e^{i\theta u}}{i2\pi\theta}d\theta = \int_{-T}^{T} \frac{\cos(\theta u) + i\sin(\theta u)}{i2\pi\theta} = \int_0^T \frac{\sin(\theta u)}{\pi\theta}d\theta = \frac{\text{sign}(u)}{\pi}\int_0^{|u|T} \frac{\sin\theta}{\theta}d\theta = \frac{\text{sign}(u)}{\pi}S(|u|T),$$

and $S(t) = \int_0^t \frac{\sin\theta}{\theta}d\theta$. Applying the fact that $\lim_{t\to\infty} S(t) = \pi/2$, we can deduce that

$$\lim_{T\to\infty} R(x-a,T) - R(x-b,T) = g_{a,b}(x) := \begin{cases} 0, & x < a \text{ or } x > b \\ \frac{1}{2}, & x = a \text{ or } x = b \\ 1, & a < x < b. \end{cases}$$

Further, the quantities $S(t)$ are uniformly bounded: $\sup_{t\in\mathbb{R}} |S(t)| \leq C < \infty$. By bounded convergence, we get that

$$\lim_{T\to\infty} J_T(a,b) = \lim_{T\to\infty}\int_{\mathbb{R}}[R(x-a,T) - R(x-b,T)]dP_X(x) = \int_{\mathbb{R}}\lim_{T\to\infty}[R(x-a,T) - R(x-b,T)]dP_X(x)$$
$$= \int_{\mathbb{R}} g_{a,b}(x)dP_X(x) = \frac{1}{2}P_X(\{a\}) + P_X((a,b)) + \frac{1}{2}P_X(\{b\})$$
$$= \frac{1}{2}(F_X(a) - F_X(a-) + F_X(b) - F_X(b-)) + F_X(b-) - F_X(a)$$
$$= \frac{1}{2}(F_X(b) + F_X(b-)) - \frac{1}{2}(F_X(a) + F_X(a-)).$$

Proof of the density formula As $\int_{\mathbb{R}} |\Phi_X(\theta)|d\theta < \infty$, the integrand $e^{-i\theta x}\Phi_X(\theta)$ is upper bounded by $|\Phi_X(\theta)$ and so also integrable, therefore $f_X(x)$ is well-defined and finite valued. Taking any x and a point $x+h$ close to x, we have

$$\limsup_{h\to 0} |f_X(x+h) - f_X(x)| \leq \limsup_{h\to 0} \frac{1}{2\pi}\int_{\mathbb{R}} |e^{-i\theta h} - 1||\Phi_X(\theta)|d\theta = 0,$$

equality following from the dominated convergence theorem. This shows that $f_X(x)$ is continuous in x. Now, as $\psi_{a,b}(\theta)$ is bounded and $\Phi_X(\theta)$ is integrable, applying dominated convergence to

$\psi_{a,b}(\theta)\Phi_X(\theta)\mathbf{1}\{|\theta| \leq T\}$ gives that

$$\lim_{T\to\infty} J_T(a,b) = J_\infty(a,b) = \int_{\mathbb{R}} \psi_{a,b}(\theta)\Phi_X(\theta)d\theta = \frac{1}{2\pi}\int_{\mathbb{R}} \left[\int_a^b e^{-i\theta u}du\right]\Phi_X(\theta)d\theta$$
$$= \int_a^b \int_{\mathbb{R}} \left[\frac{1}{2\pi}e^{-i\theta u}\Phi_X(\theta)d\theta\right]du$$
$$= \int_a^b f_X(u)du.$$

In particular, this shows that $J_\infty(a,b)$ is continuous in a, b. On the other hand, the result (1) gives

$$J_\infty(a,b) = \frac{1}{2}(F_X(b) + F_X(b-)) - \frac{1}{2}(F_X(a) + F_X(a-)),$$

therefore the RHS has to be continuous in a, b. This implies F_X is itself continuous (check this!), and thus $F_X(a) = F_X(a-)$,$F_X(b) = F_X(b-)$ and so

$$\int_a^b f_X(u)du = J_\infty(a,b) = F_X(b) - F_X(a).$$

Hence f_X is the density of X. $\square$

References