Exercises on the law of large numbers and Borel-Cantelli

Exercise [2.1.5]

Let $\epsilon > 0$ and pick $K = K(\epsilon)$ finite such that if $k \geq K$ then $r(k) \leq \epsilon$. Applying the Cauchy-Schwarz inequality for $X_i - \mathbb{E}X_i$ and $X_j - \mathbb{E}X_j$ we have that

$$\text{Cov}(X_i, X_j) \leq \frac{\text{Var}(X_i) \text{Var}(X_j)}{2} \leq r(0) < \infty$$

for all $i, j$. Thus, breaking the double sum in $\text{Var}(S_n) = \sum_{i,j=1}^n \text{Cov}(X_i, X_j)$ into $\{(i, j) : |i - j| < K\}$ and $\{(i, j) : |i - j| \geq K\}$ gives the bound

$$\text{Var}(S_n) \leq 2Kn\epsilon(0) + n^2\epsilon.$$

Dividing by $n^2$ we see that $\limsup_n \text{Var}(n^{-1}S_n) \leq \epsilon$. Since $\epsilon > 0$ is arbitrary and $\mathbb{E}S_n = n\mu$, we have that $n^{-1}S_n \xrightarrow{L^2} \mu$ (with convergence in probability as well).

Exercise [2.1.13]

We have $\mathbb{E}|X_1| = \sum_{k=2}^{\infty} 1/(ck \log k) = \infty$. On the other hand, for $n \in \mathbb{N}$

$$n \mathbb{P}(|X_1| \geq n) = \frac{n}{c} \sum_{k=n}^{\infty} \frac{1}{k \log k} \leq \frac{n}{c} \int_{n-1}^{\infty} \frac{1}{x \log x} \, dx = \frac{n}{c} \int_{\log(n-1)}^{\infty} \frac{1}{z} \, dz \leq \frac{n}{c \log(n-1)} \int_{\log(n-1)}^{\infty} e^{-z} \, dz = \frac{n}{c(n-1) \log(n-1)}.$$

In particular $n\mathbb{P}(|X_1| \geq n) \to 0$ as $n \to \infty$, which implies $\lim_{x \to \infty} x\mathbb{P}(|X_1| \geq x) = 0$. We can therefore apply Proposition 2.1.12, which yields $(S_n/n - \mu_n) \xrightarrow{p} 0$.

It is therefore sufficient to show that $\mu_n$ has a finite limit. We have, for $n$ even

$$\mu_n = \mathbb{E}\{X_1 I_{|X_1| \leq n}\} = \frac{1}{c} \sum_{k=n}^{\infty} (-1)^k \frac{1}{k \log k}$$

$$= \frac{1}{c} \sum_{i=1}^{n/2} \left\{ \frac{1}{2i \log(2i)} - \frac{1}{(2i+1) \log(2i+1)} \right\},$$

and this series is convergent. Further, for $n$ odd, $|\mu_n - \mu_{n-1}| = 1/(cn \log n) \to 0$. Therefore $\mu_n$ has a limit.
Exercise [2.2.9]

Fixing $1 > \lambda > 0$, define $Y_n := \sum_{k \leq n} I_{A_k}$ and set $a_n = \lambda E Y_n$. Since $a_n \to \infty$, we have that,

$$P(A_n \text{ i.o. } ) \geq P(Y_n > a_n \text{ i.o. } ) \geq \limsup_{n \to \infty} P(Y_n > a_n)$$

where the last inequality is due to Fatou’s lemma (c.f. (1.3.10), or Exercise 2.2.2). Applying Exercise 1.3.20, we have that $P(Y_n > a_n) \geq (1 - \lambda)^2 c_n$ for $c_n := (E Y_n^2)^2/E(Y_n^2)$. By the definition of $Y_n$, the assumption of the exercise is precisely that $\alpha = \limsup_n c_n$. Thus, taking first $n \to \infty$ then $\lambda \downarrow 0$ completes the proof of the Kochen-Stone lemma.

Exercise [2.2.26]

1. First note that

$$\text{Var}(S_n) = \sum_{i=1}^{n} P(A_i)(1 - P(A_i)) \leq \sum_{i=1}^{n} P(A_i) = ES_n.$$  

By Markov’s inequality, then,

$$P\left(\frac{|S_n - ES_n|}{ES_n} > \epsilon \right) \leq \frac{\text{Var}(S_n)}{\epsilon^2(ES_n)^2} \leq \frac{1}{\epsilon^2ES_n},$$

and since we assumed that $ES_n = \sum_{i \leq n} P(A_i) \to \infty$, we are done.

2. Since $E(S_{nk}) \geq k^2$, we have from part (a) that

$$P(|S_{nk} - ES_{nk}| > \epsilon ES_{nk}) \leq 1/(\epsilon^2 k^2).$$

Since the series $\sum_k k^{-2}$ is finite, the first Borel-Cantelli lemma implies that $P(|S_{nk} - ES_{nk}| > \epsilon ES_{nk} \text{ i.o. } ) = 0$. Since $\epsilon > 0$ is arbitrary, it follows that $S_{nk}/ES_{nk} \overset{a.s.}{\to} 1$.

3. Since $k^2 \leq ES_{nk} \leq k^2 + 1$ and $(k + 1)^2 \leq ES_{nk+1} \leq (k + 1)^2 + 1$

$$\frac{k^2}{(k + 1)^2 + 1} \leq \frac{E(S_{nk})}{E(S_{nk+1})} \leq \frac{k^2 + 1}{(k + 1)^2},$$

so $E(S_{nk})/E(S_{nk+1}) \to 1$ when $k \to \infty$. Then, for $n_k \leq n \leq n_{k+1},$

$$\frac{S_{nk}}{E(S_{nk})} \frac{E(S_{nk})}{E(S_{nk+1})} \leq \frac{S_n}{E(S_n)} \leq \frac{S_{nk+1}}{E(S_{nk+1})} \frac{E(S_{nk+1})}{E(S_{nk})}.$$  

Hence, by part (b) and the fact that $E(S_{nk})/E(S_{nk+1}) \to 1$, we conclude that $S_n/E(S_n) \overset{a.s.}{\to} 1$.

Exercise [2.3.14]

1. By induction, $\log W_n = \sum_{i=1}^{n} X_i$ for the i.i.d. random variables $X_i = \log(qr + (1 - q)V_1)$. As $\{X_i\}$ are bounded below by $\log(qr) > -\infty$, it follows that $E[(X_1)_{-}]$ is finite, so the strong law of large numbers implies that $n^{-1} \log W_n \overset{a.s.}{\to} w(q)$, as stated.

2. Since $q \mapsto (qr + (1 - q)V_1(\omega))$ is linear and $\log x$ is concave, it follows that $q \mapsto \log(qr + (1 - q)V_1)$ is concave on $(0, 1]$, per $\omega \in \Omega$. The expectation preserves the concavity, hence $q \mapsto w(q)$ is concave on $(0, 1]$. 

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3. By Jensen’s inequality for the concave function \( g(x) = \log x, \ x > 0 \), we have that
\[
w(q) = \mathbb{E} \log qr + (1 - q)V_1 \leq \log(qr + (1 - q)\mathbb{E}V_1).
\]
Hence, if \( \mathbb{E}V_1 \leq r \) then \( w(q) \leq \log(qr + (1 - q)r) = \log r = w(1) \).
Recall that \( (\log x)_- \leq 1/(ex) \) for all \( x \geq 0 \). Hence, if \( \mathbb{E}V_1^{-1} \) is finite, then so is \( \mathbb{E}[(\log V_1)_-]. \) Consequently, the strong law of large numbers of part (a) also applies for \( n^{-1}\log W_n \) in case \( q = 0 \) (i.e., for \( X_i = \log V_i \)). Further, when \( \mathbb{E}[(\log V_1)_-] \) is finite, \( w(q) = w(0) + \mathbb{E} \log(qrV_1^{-1} + 1 - q) \) and by Jensen’s inequality
\[
\mathbb{E} \log(qrV_1^{-1} + 1 - q) \leq \log(qr \mathbb{E}V_1^{-1} + 1 - q) \leq 0
\]
if \( \mathbb{E}V_1^{-1} \leq r^{-1} \), implying that then \( w(q) \leq w(0) \).

4. Our assumption that \( \mathbb{E}V_1^2 < \infty \) and \( \mathbb{E}V_1^{-2} < \infty \) implies that \( \mathbb{E}V_1 < \infty \) and \( \mathbb{E}V_1^{-1} < \infty \). Further, \( w(0) = \mathbb{E} \log V_1 \leq \mathbb{E}V_1 \) is then also finite. We have shown in part (c) that \( w(q) \leq w(1) = \log r \) in case \( \mathbb{E}V_1 \leq r \) and that \( w(q) \leq w(0) \) in case \( \mathbb{E}V_1^{-1} \leq r^{-1} \). Consequently, if \( w(q) \) suffices to show that if \( \mathbb{E}V_1 > r > 1/\mathbb{E}V_1^{-1} \), then there exists \( q^* \in (0,1) \) where \( w(\cdot) \) reaches its supremum (which is hence finite). The former condition is equivalent to \( \mathbb{E}Y > 0 \) and \( \mathbb{E}Z > 0 \) for \( Y = rV_1^{-1} - 1 \geq -1 \) and \( Z = r^{-1}V_1 - 1 \geq -1 \), both of which are in \( L^2 \). Further, since \( q \mapsto w(q) : [0,1] \to \mathbb{R} \) is a concave function, the existence of such \( q^* \in (0,1) \) follows as soon as we check that \( \mathbb{E}(\log(1 + \epsilon Y)) > 0 \) and \( \mathbb{E}(1 - \epsilon - w(1) = \mathbb{E}(\log(1 + \epsilon Z)) > 0 \) when \( \epsilon > 0 \) is small enough. To this end, note that \( \log(1+x) \geq x - x^2 \) for all \( x \geq -1/2 \). Hence, \( \mathbb{E}\log(1 + \epsilon Y) \geq \epsilon \mathbb{E}Y - \epsilon^2 \mathbb{E}Y^2 > 0 \) for \( \epsilon \in (0,1/2) \) small enough. As the same applies for \( \mathbb{E}\log(1 + \epsilon Z) \), we are done.

We see that one should invest only in risky assets whose expected annual growth factor \( \mathbb{E}V_1 \) exceeds that of the risk-less asset, and that if in addition \( \mathbb{E}V_1^{-1} \) is finite, then a unique optimal fraction \( q^* \in (0,1) \) should be re-invested each year in the risky asset.

Exercise [2.3.9]

1. Fix \( \delta > 0 \) such that \( p := \mathbb{P}(\tau_1 > \delta) > \delta \). Note that \( \bar{N}_t + 1 - r \) follows the negative Binomial distribution of parameters \( p \) and \( r = [t/\delta] + 1 \). That is, for \( \ell = 0,1,2,\ldots \),
\[
\mathbb{P}(\bar{N}_t + 1 - r = \ell) = \mathbb{P}(\bar{T}_{t+r-1} \leq t < \bar{T}_{t+r}).
\]
It is easy to check that \( \mathbb{E}[\bar{N}_t] = r/p - 1 \) and \( \text{Var}(\bar{N}_t) = r(1-p)/p^2 \). Consequently, \( \mathbb{E}[\bar{N}_t^2] = (r^2 + r - 3rp + p^2)/p^2 \), and with \( p > 0 \) fixed and \( r \leq t/\delta + 1 \) it follows that \( \sup_{t \geq 1} t^{-2}\mathbb{E}\bar{N}_t^2 < \infty \).

2. Since \( \bar{T}_t \leq \tau_t \), clearly \( N_t \leq \bar{N}_t \). Hence, by part (a), \( \sup_{t \geq 1} t^{-2}\mathbb{E}N_t^2 < \infty \). In view of the criterion of Exercise ?? (for \( f(x) = x^2 \)), this implies that \( \{t^{-1}N_t : t \geq 1 \} \) is a uniformly integrable collection of R.V. As we have seen in Exercise ?? that \( t^{-1}N_t \overset{a.s.}{\to} 1/\mathbb{E}\tau_1 \), it thus follows that also \( t^{-1}N_t \overset{L^1}{\to} 1/\mathbb{E}\tau_1 \) (c.f. Theorem ??), and in particular, \( t^{-1}\mathbb{E}N_t \to 1/\mathbb{E}\tau_1 \) as stated.

Exercise [2.2.24]

1. Substituting \( y = x + z \) and using the bound \( \exp(-z^2/2) \leq 1 \) yields
\[
\int_x^\infty e^{-y^2/2}dy \leq e^{-x^2/2} \int_0^\infty e^{-(y-z)^2}dz = x^{-1}e^{-x^2/2}.
\]
For the other direction, observe that for \( x > 0 \),
\[
(x^{-1} - x^{-3})e^{-x^2/2} = \int_x^\infty (1 - 3y^{-4})e^{-y^2/2}dy \geq \int_x^\infty e^{-y^2/2}dy.
\]
2. Since the probability density function for a standard normal random variable \( G_n \) is \((2\pi)^{-1/2}e^{-x^2/2}\), we get from the bounds of part (a) that
\[
c_\gamma = \lim_{n \to \infty} n^\gamma \sqrt{\log n} P \left( G_n > \sqrt{2\gamma \log n} \right),
\]
exists, is finite and positive. Consequently, fixing \( \epsilon > 0 \) by the first Borel-Cantelli lemma we have that \( P(G_n/\sqrt{2\log n} > 1 + \epsilon \text{ i.o.}) = 0 \). Further, since \( G_n \) are mutually independent, it follows from the second Borel-Cantelli lemma that \( P(G_n/\sqrt{2\log n} > 1 - \epsilon \text{ i.o.}) = 1 \). We see that with probability one, the sequence \( n \mapsto G_n(\omega)/\sqrt{2\log n} \) is infinitely often above \( 1 - \epsilon \) but only finitely often above \( 1 + \epsilon \), in which case \( L(\omega) = \limsup_n G_n(\omega)/\sqrt{2\log n} \) must be in the interval \( (1 - \epsilon, 1 + \epsilon) \). Considering the intersection of the relevant events for \( \epsilon_k \downarrow 0 \), we conclude that \( P(L = 1) = 1 \), as stated.

3. Since \( S_n/\sqrt{n} \) has the same law as \( G_1 \), the upper bound of part (a) implies that \( P(|S_n| \geq 2\sqrt{n \log n}) \leq Cn^{-2} \) for some \( C < \infty \) and all \( n \) large enough. Since the series \( \sum n^{-2} \) is finite, applying the first Borel-Cantelli lemma we get that \( P(|S_n| \geq 2\sqrt{n \log n} \text{ i.o.}) = 0 \), or equivalently, that \( P(|S_n| < 2\sqrt{n \log n} \text{ ev.}) = 1 \).