Exercise [5.2.10]

It is easy to check that \( \{S^n_2 - s^n_2\} \) is a martingale (but you should do it). Let \( A = \{\max_{k=1}^n |S_k| > x\} \) and 
\[
\tau = \inf \{ k : |S_k| > x \} \cap n.
\]
Then
\[
(x + K)^2 P(A) \geq E(S^n_2 I_A) \geq E((S^n_2 - s^n_2)I_A) = E((S^n_2 - s^n_2)I_A),
\]
since \( \tau \) is bounded by \( n \) implies \( E(S^n_2 - s^n_2) = E(S^n_2 - s^n_2) \) (see Corollary 5.1.33), and \( \tau = n \) on \( A^c \). Now add \( E(S^n_2 I_{A^c}) \) to both sides to get
\[
(x + K)^2 P(A) + E(S^n_2 I_{A^c}) \geq E(S^n_2) - s^n_2 P(A).
\]

But \( E(S^n_2 I_{A^c}) \leq (x + K)^2 P(A) \) so
\[
(x + K)^2 = (x + K)^2 P(A) + (x + K)^2 P(A^c) \geq E(S^n_2) - s^n_2 P(A) = s^n_2 P(A).
\]

Exercise [5.2.14]

1. Note that
\[
\log M_n = S_n = \sum_{i=1}^n X_i
\]
for the i.i.d. variables \( X_i = \log Y_i \). Let \( X^{(m)}_1 = \max(X_1, -m) \). Since
\[
\exp(X^{(m)}_1) = \max(Y_1, e^{-m}) \leq Y_1 + e^{-m}
\]
and \( EY_1 = 1 \), it follows from Jensen’s inequality that
\[
E X^{(m)}_1 \leq \log E(\exp(X^{(m)}_1)) \leq \log(1 + e^{-m}).
\]

Since \( (X_1)_+ = X^{(0)}_1 \) we thus deduce that \( E((X_1)_+) \) is finite, which suffices for the strong law of large numbers to apply (c.f. Theorem 2.3.3). That is, \( n^{-1} \log M_n \) converges almost sure to
\[
\mu = EX_1 \leq \lim \limits_{m \to \infty} E X^{(m)}_1 \leq 0.
\]
Further, if \( \mu = 0 \) then yet another application of Jensen’s inequality results with
\[
1 = \exp(\mu/2) \leq E \exp(X_1/2) = E \sqrt{\overline{Y}_1} \leq \sqrt{E \overline{Y}_1} = 1
\]
i.e. with \( \text{Var}(\sqrt{Y}_1) = 0 \), which is ruled out by our assumption that \( Y_1 \) is a non-constant random variable.

2. Since a.s. \( n^{-1} \log M_n \to \mu < 0 \), we have that with probability one \( n^{-1} \log M_n \leq \mu/2 \) for all \( n \) large enough. That is, \( M_n \leq \exp(\mu n/2) \to 0 \) as \( n \to \infty \). For \( \{M_n\} \) uniformly integrable this would imply that \( M_n \to 0 \) in \( L^1 \) and in particular that \( EM_n \to 0 \) as \( n \to \infty \). However, clearly \( EM_n = 1 \) for all \( n \), so necessarily \( \{M_n\} \) is not U.I.
3. If Doob’s $L^p$ maximal inequality applies for $p = 1$, then for any non-negative martingale $X_n$
\[
E[\max_{k \leq n} X_k] \leq qEX_n = qEX_0 < \infty.
\]

Taking $n \to \infty$ it then follows that $\sup_k X_k$ is integrable, hence that $\{X_n\}$ is uniformly integrable.

As we have seen in part (b) a counter example to the latter statement, we conclude that Doob’s $L^p$ maximal inequality cannot extend as is to $p = 1$.

**Exercise [5.3.9]**

For part (a) let $X_n = -1/n$ (non-random), so $X_n^2 = 1/n^2$. Then, obviously $\{X_n\}$ is a submartingale whereas $\{X_n^2\}$ is a super-martingale. No contradiction with Proposition 5.1.22 since $\Phi(x)$ is a decreasing function on the support $(-\infty, 0)$ of the sequence $\{X_n\}$. For part (b) consider $S_n = \sum_{k=1}^n \xi_k$ and independent $\{\xi_k\}$ such that $\xi_k = k^2 - 1$ with probability $1/k^2$ and otherwise $\xi_k = -1$. Clearly, $E\xi_k = 0$ for all $k$ so $\{S_n\}$ is a martingale. However, by Borel-Cantelli II we have that $P(\xi_k = -1, \text{e.v.}) = 1$ hence $S_n \to -\infty$ almost surely. Theorem 5.3.2 does not apply for this example as $E(S_n)_-$ is unbounded. Indeed, take $\ell$ finite and large enough so $\sum_{k>\ell} k^{-2} \leq 1/2$ and set $b = \sum_{k=1}^\ell k^2$. Then, with probability at least half, $\xi_k = -1$ for all $k > \ell$ hence $S_n \leq b - n$. Consequently, $E[(S_n)_-] \geq \frac{1}{2}(n-b)_+ \to \infty$ as $n \to \infty$.

**Exercise [5.3.10]**

Consider the adapted processes $Q_n = \prod_{i=1}^n (1 + Y_i) \geq Q_0 = 1$ and $W_n = (1 + X_n)/Q_{n-1}$. Since $\sum_i Y_i < \infty$ a.s. and $Y_i$ are non-negative, $Q_n$ converges a.s. to a finite limit, say $Q_\infty = \sup_n Q_n$. Therefore, if $W_n$ converges a.s. to a finite limit then so does $X_n = W_n/Q_{n-1} - 1$. Note that $W_n$ is integrable and
\[
E(W_{n+1}|\mathcal{F}_n) \leq (1 + Y_n)(1 + X_n)/Q_n = W_n,
\]

implying that $W_n$ is a super-martingale. Further, $X_n$ is non-negative, hence so is $W_n$ which by Doob’s convergence theorem then converges a.s. to a finite limit.

**Exercise [5.2.11]**

The same line of reasoning applies in all three parts of this exercise. Namely, for a certain convex non-negative function $\Phi(\cdot)$ the relevant inequality trivially holds when $E\Phi(Y_n)$ is infinite. In part (a) assuming $E\Phi(Y_n) < \infty$ for the non-decreasing $\Phi(y) = (y)_+^p$, we have from the $L^p$-maximal inequalities that $E\Phi(Y_k)$ is finite for all $k \leq n$, so $\{X_k = \Phi(Y_k), k \leq n\}$ is a sub-MG (by Proposition 5.1.22). In parts (b) and (c) we start with a martingale $\{Y_k\}$ so again $X_k = \Phi(Y_k)$ is a sub-MG although the relevant functions $\Phi(y) = |y|^p$ and $\Phi(y) = (y + c)^2$, for $c \geq 0$, respectively, are non-decreasing only for $y \geq 0$. In all three cases if $Y_k \geq y > 0$ then $X_k \geq x = \Phi(y)$, so we bound $P(\max_k Y_k \geq y)$ by $P(\max_k X_k \geq x)$ which in turn is further bounded via Doob’s inequality. This procedure results with the stated bounds of (a) and (b), whereas in part (c) it provides the bound
\[
P(\max_{k=0}^n Y_k \geq y) \leq (y + c)^{-2}E(Y_n + c)^2.
\]

However, here $EY_n = EY_0 = 0$, so the preceding bound simplifies to $(y + c)^{-2}(EY_n^2 + c^2)$ and setting $c = EY_n^2/y$ yields after a bit of algebra the stated bound of part (c).

**Exercise [5.2.19]**

1. Since $|W_n|$ and $|Y_n|$ are both bounded by $|X_n^1| + |X_n^2|$, the integrability of $W_n$ and $Y_n$ follows from that of $X_n^1$ and $X_n^2$. It is also easy to see that for an $\mathcal{F}_\tau$-stopping time $\tau$ and $\mathcal{F}_\tau$-adapted $\{X^1_\tau\}, \{X^2_\tau\}$ the processes $\{W_n\}$ and $\{Y_n\}$ are also $\mathcal{F}_\tau$-adapted. Our assumption that $X^2_\tau \geq X^2_\tau$ implies that $W_n \leq W_n + (X^1_\tau - X^2_\tau)I_{(\tau = n)} = Y_n$. 

2
Further, \( \tau \) is an \( \mathcal{F}_n \)-stopping time, so the event \( \{ \tau < n \} = \{ \tau \leq n - 1 \} \) and its complement \( \{ \tau \geq n \} = \{ \tau > n - 1 \} \) are both in \( \mathcal{F}_{n-1} \). Hence, taking out the known \( I_{\{ \tau < n \}} \) and \( I_{\{ \tau \geq n \}} \) we deduce from the sup-MG property of \( X^1_n \) and \( X^2_n \) that

\[
E[W_n | \mathcal{F}_{n-1}] \leq E[Y_n | \mathcal{F}_{n-1}]
= I_{\{ \tau \geq n \}} E[X^1_n | \mathcal{F}_{n-1}] + I_{\{ \tau < n \}} E[X^2_n | \mathcal{F}_{n-1}]
\leq X^1_{n-1} I_{\{ \tau > n-1 \}} + X^2_{n-1} I_{\{ \tau \leq n-1 \}} = W_{n-1} \leq Y_{n-1}.
\]

That is, both \( \{ W_n \} \) and \( \{ Y_n \} \) are sup-MGs for \( \mathcal{F}_n \).

2. Fixing a positive integer \( n \), consider the partition of \( \Omega \) to the disjoint events \( \{ A_\ell, B_\ell, \ell \geq 0 \} \) where \( A_\ell = \{ \omega : \tau_\ell(\omega) < n \leq \tau_{\ell+1}(\omega) \} \) and \( B_\ell = \{ \omega : \tau_\ell(\omega) < n \leq \tau_{\ell+1}(\omega) \} \) for \( \ell = 0, 1, \ldots \). As \( \tau_\ell, \ell \geq 0 \) and \( \tau_\ell, \ell \geq 0 \) are stopping time for the filtration \( \{ \mathcal{F}_n \} \) it is easy to check that each of these events is in \( \mathcal{F}^X_{n-1} \). We claim that if the event \( A_\ell \) occurs, then \( Z_n - Z_{n-1} \leq 0 \). Indeed, for \( \omega \in A_\ell \) either \( Z_n - Z_{n-1} = a^{-\ell}b^\ell \) or in case \( n = \tau_\ell \), by definition \( X_n \leq a \) and \( Z_n - Z_{n-1} = a^{-\ell+1}b^\ell(X_n/a - 1) \leq 0 \). We further claim that if the event \( B_\ell \) occurs, then \( Z_n - Z_{n-1} \leq a^{-\ell+1}b^\ell(X_n - X_{n-1}) \). Indeed, for \( \omega \in B_\ell \) the preceding inequality holds with equality except when \( n = \tau_{\ell+1} \) in which case it follows from the fact that \( X_{\tau_{\ell+1}} \geq b \). Thus, decomposing \( Z_n - Z_{n-1} \) according to this partition of \( \Omega \), we deduce that \( Z_n - Z_{n-1} \leq V_n(X_n - X_{n-1}) \) with \( V_n = \sum_{\ell=0}^{\infty} a^{-\ell-1}b^\ell I_{B_\ell} \) non-negative and measurable on \( \mathcal{F}^X_{n-1} \). Further, as \( \theta-1 \leq 2\ell \), the disjoint events \( B_\ell \) are empty for \( \ell \geq n/2 \), hence \( V_n \leq a^{-1}(b/a)^{n/2} \) is bounded. Taking out the known \( V_n \) and recalling that \( (X_n, \mathcal{F}^X_n) \) is a sup-MG, we see that \( V_n(X_n - X_{n-1}) \) is integrable with

\[
E[V_n(X_n - X_{n-1}) | \mathcal{F}^X_{n-1}] = V_nE[X_n - X_{n-1} | \mathcal{F}^X_{n-1}] \leq 0.
\]

Consequently, as \( Z_0 \leq 1 \), the non-negative

\[
Z_n \leq Z_0 + \sum_{k=1}^{n} V_k(X_k - X_{k-1})
\]

are integrable, with

\[
E[Z_n - Z_{n-1} | \mathcal{F}^X_{n-1}] \leq V_nE[X_n - X_{n-1} | \mathcal{F}^X_{n-1}] \leq 0,
\]

establishing that \( (Z_n, \mathcal{F}^X_n) \) is a sup-MG, as claimed.

3. We have that \( \theta_0 = 0 \) if and only if \( X_0 \leq a \) and consequently, \( Z_0 = \min(1, X_0/a) \). Recall that \( Z_n \geq 0 \) and \( Z_{\tau_\ell} = a^{-\ell}b^\ell \) when \( \tau_\ell \) is finite, that is, when \( U_\infty(a, b) \geq \ell \). Further from part (b) we have that \( E(Z_0) \geq E[Z_n \wedge \tau_\ell] \) for any \( n, \ell \geq 1 \). Thus, taking \( n \to \infty \) and applying Fatou’s lemma we see that

\[
E[\min(X_0/a, 1)] \geq E[\liminf_{n \to \infty} Z_n \wedge \tau_\ell] \geq E[Z_{\tau_\ell} I_{\{\tau_\ell < \infty\}}] = \left( \frac{b}{a} \right)^\ell P(U_\infty(a, b) \geq \ell).
\]