Exercise [5.3.20]

1. We claim that

\[ E[h|F_n] = 2^n \sum_{i=1}^{2^n} \int_{A_{i,n}} h(u) \, du \, I_{A_{i,n}}(t) \quad (1) \]

Indeed, integrability and \( F_n \)-measurability of the RHS of (1) are obvious. Further, denoting the RHS of (1) by \( f_n \), clearly \( E[f_i|A_{i,n}] = E[h|A_{i,n}] \) for each value of \( i \) and \( n \) (as \( E(A_{i,n}) = P(A_{i,n}) = 2^{-n} \)), which suffices since \( \{A_{i,n} : i = 1, \ldots , 2^n\} \) is a \( \pi \)-system.

2. Each function \( X_n(t) \) is piecewise constant on the intervals generating \( F_n \), having the value \( h_{i,n} \) on the interval \( A_{i,n} \). Thus, \( X_n(t) \) is \( F_n \) measurable and further integrable with

\[ c_n = E|X_n| = 2^{-n} \sum_{i=1}^{2^n} |h_{i,n}| \]

finite. It is easy to check that \( h_{i,n} = (h_{2i-1,n+1} + h_{2i,n+1})/2 \), namely, the constant value of \( X_n(t) \) on each interval \( A_{i,n} \) is the average of the values that \( X_{n+1}(t) \) take on the two adjacent intervals \( A_{2i-1,n+1} \) and \( A_{2i,n+1} \) into which \( A_{i,n} \) split. By part (a)

\[ E[X_{n+1}|F_n] = \sum_{i=1}^{2^n} g_{i,n} I_{A_{i,n}}(t) \]

where \( g_{i,n} \) is the expected value of \( X_{n+1}(t) \) with respect to the uniform measure on \( A_{i,n} \). Since \( A_{i,n} \) is the disjoint union of the intervals \( A_{2i-1,n+1} \) and \( A_{2i,n+1} \) of same length on each of which \( X_{n+1}(t) \) is constant, it follows that \( g_{i,n} = h_{i,n} \) and consequently, that \( X_n \) is a martingale.

3. Let \( c_n = E|X_n| \) and \( c = \sup_n c_n \). If \( c = \infty \) then there exist \( n_k \geq k \) such that \( c_{n_k} \geq 2^k \). Taking for each \( k \) the collection of intervals \( A_{i,n_k} \) with the \( m = m^{(k)} = 2^{n_k - k} \) largest values of \( |h_{i,n_k}| \) then yields some \( s_1^{(k)} < t_1^{(k)} \leq s_2^{(k)} < t_2^{(k)} \cdots t_m^{(k)} \) for which

\[ \sum_{\ell=1}^{m} |t_\ell^{(k)} - s_\ell^{(k)}| \leq 2^{-k} \xrightarrow{k \to \infty} 0 \quad \text{while} \quad \sum_{\ell=1}^{m} |x(t_\ell^{(k)}) - x(s_\ell^{(k)})| \geq 1 \]

contradicting the absolute continuity of \( x(\cdot) \). Therefore, \( c < \infty \) and hence for all \( \rho > 0 \) by Markov’s inequality

\[ 2^{-n} \sum_{j=1}^{2^n} I_{|X_j| > \rho} = P(|X_n| > \rho) \leq E|X_n|/\rho \leq c/\rho \quad (2) \]

Further, note that

\[ E|X_n|I_{|X_n| > \rho} = \sum_{\{j:|X_j| > \rho\}} 2^{-n}|h_{j,n}|. \quad (3) \]
Let $\epsilon > 0$ be fixed and $\delta = \delta(\epsilon, x) > 0$ be determined as in the definition of absolute continuity. Taking $\rho = c/\delta$ observe that (2) and (3) imply by the absolute continuity of $x(\cdot)$ that

$$
E[|X_n|I_{|X_n|>\rho}] \leq \epsilon \text{ for all } n,
$$

hence $\{X_n\}$ is U.I.

4. Assuming hereafter that $x(\cdot)$ is absolutely continuous, hence $\{X_n\}$ is a U.I. martingale, by Corollary 5.3.14,

$$
X_n = E[h|F_n] \text{ for some } h = X_\infty \in L^1.
$$

Hence, by (1) we see that $x((j+1)2^{-n}) - x((j-1)2^{-n}) = \int_{(j-1)2^{-n}}^{j2^{-n}} h(u)du$ for $n = 0, 1, \cdots$ and $i = 1, \cdots, 2^n$. By linearity of the integral we thus have

$$
x(j2^{-n}) - x(i2^{-n}) = \int_{i2^{-n}}^{j2^{-n}} h(u)du \text{ for all } j \geq i, \text{ and all } n \text{ values.} \quad (4)
$$

Consider now arbitrary $1 > t \geq s \geq 0$ and let $j_m \geq i_m, n_m$ be such that $j_m2^{-n_m} \to t$ and $i_m2^{-n_m} \to s$ as $m \to \infty$. By continuity of $x(\cdot)$, the LHS of (4) for this sequence converges to $x(t) - x(s)$, while the RHS of (4) is $\int_s^t h(u)I_{i_m2^{-n_m} \leq u < j_m2^{-n_m}}du$, with the integrand converging a.e. to $h(u)I_{[s,t]}(u)$ and dominated by the integrable function $|h|$. Thus, by dominated convergence these integrals converge to $\int_s^t h(u)du$.

5. Consider now

$$
\Delta^{-1}[x(s+\Delta) - x(s)] - h(s) = \Delta^{-1}\int_s^{s+\Delta} [h(u) - h(s)]du.
$$

So that

$$
\lim_{\Delta \to 0} |\Delta^{-1}[x(s+\Delta) - x(s)] - h(s)| \leq \lim_{\Delta \to 0} \Delta^{-1}\int_s^{s+\Delta} |h(u) - h(s)|du = 0 \text{ a.e. } [0,1].
$$

Hence $h(t) = \frac{dx}{dt}$ a.e. $[0,1]$ as claimed.

**Exercise [5.3.39]**

Set $\lambda > 0$ and let $\psi = e^{\lambda} - \lambda - 1$.

1. Obviously, $N_n$ is measurable on $\mathcal{F}_n$. By our assumptions about the $L^2$ martingale $(M_n, \mathcal{F}_n)$, part (a) of Exercise 1.4.40 applies for the law of $Y = \lambda(M_{n+1} - M_n)$ conditional on $\mathcal{F}_n$, taking there $\kappa = \lambda$ and

$$
\lambda^{-2}E[Y^2|\mathcal{F}_n] = E[(M_{n+1} - M_n)^2|\mathcal{F}_n] = \langle M \rangle_{n+1} - \langle M \rangle_n.
$$

With $\langle M \rangle_{n+1} \in m\mathcal{F}_n$, we consequently have that

$$
E[N_{n+1}|\mathcal{F}_n] = N_n \exp(-\psi(\langle M \rangle_{n+1} - \langle M \rangle_n))E[e^Y|\mathcal{F}_n] \leq N_n,
$$

implying that $(N_n, \mathcal{F}_n)$ is a non-negative sup-MG.

2. Since $\lambda > 0$ the event $\{M_\tau \geq u, \langle M \rangle_\tau \leq r\}$ implies that $N_\tau \geq a$ for $a = \exp(\lambda u - \psi r)$. Hence, by Markov’s inequality

$$
P(M_\tau \geq u, \langle M \rangle_\tau \leq r) \leq P(N_\tau \geq a) \leq a^{-1}EN_\tau.
$$

Applying Doob’s convergence theorem for the non-negative sup-MG $\{N_{n\wedge \tau}\}$ whose a.s. limit is $N_\tau$ (by the a.s. finiteness of $\tau$), you thus deduce that $EN_\tau \leq EN_0 = 1$ as needed to complete the proof.
3. Recall that the $L^2$ bounded martingale $S_n$ of Example 5.3.23 converge a.s. (and in $L^2$) to a finite limit $S_\infty$. As the martingale $S_n$ has independent increments, we deduce that $\langle S \rangle_n = \mathbb{E}(S_n^2)$ is a non-random sequence which converges to the finite constant $\langle S \rangle_\infty = \sum_k \mathbb{E}\xi_k^2$. By part (a) and our assumption that $|\xi_k| \leq 1$ we further have that $N_n = \exp(\lambda S_n - \psi(S)_n)$ is a non-negative sup-MG for any $\lambda > 0$. Thus, by Doob’s convergence theorem $N_n \rightarrow N_\infty$ almost surely and $\mathbb{E}N_\infty \leq \mathbb{E}N_0 = 1$. Since necessarily $N_\infty = \exp(\lambda S_\infty - \psi(S)_\infty)$, it follows that $\mathbb{E}[\exp(\lambda S_\infty)] \leq \exp(\psi(S)_\infty)$ is finite. This conclusion extends to all $\lambda \in \mathbb{R}$ since the same argument applies also for the martingale $-S_n$.

**Exercise [5.4.10]**

1. Since $\tau$ is a stopping time for $\mathcal{F}_n^\xi$, we know that $I_{k\leq \tau} = 1 - I_{\tau < k-1}$ is measurable on $\mathcal{F}_{k-1}$, and hence independent of $\xi_k$. Consequently, with $\xi_k$ identically distributed,

$$
\mathbb{E}\xi_k I_{k\leq \tau} = \mathbb{E}\xi_k \mathbb{P}(k \leq \tau) = \mathbb{E}\xi_1 \mathbb{P}(\tau \geq k).
$$

The representation $S_\tau = \sum_{k=1}^\infty \xi_k I_{k\leq \tau}$ applies when $\tau < \infty$ a.s. (hence when $\mathbb{E}\tau < \infty$ as assumed). Thus, by Fubini’s theorem with respect to the product of the probability measure $\mathbb{P}$ and the counting measure on $k \in \{1, 2, \ldots\}$, we find that

$$
\mathbb{E}S_\tau = \mathbb{E}\left[\sum_{k=1}^\infty \xi_k I_{k\leq \tau}\right] = \sum_{k=1}^\infty \mathbb{E}[\xi_k] \sum_{k=1}^\infty \mathbb{P}(\tau \geq k) = \mathbb{E}[\xi_1] \mathbb{E}[\tau],
$$

where the integrability condition for Fubini’s theorem is merely that

$$
\sum_{k=1}^\infty \mathbb{E}[|\xi_k| I_{k\leq \tau}] = \sum_{k=1}^\infty \mathbb{E}[|\xi_1|] \mathbb{P}(\tau \geq k) = \mathbb{E}[|\xi_1|] \mathbb{E}[\tau]
$$

is finite. As the latter follows from the assumed finiteness of $\mathbb{E}\tau$, we are done.

2. Without loss of generality assume that $\mathbb{E}\xi_1 = 0$, for otherwise, we can always work with $\{\xi_i - \mathbb{E}\xi_i\}$ which are i.i.d. and have the same variance as $\{\xi_i\}$. Setting $v := \text{Var}(\xi_1)$, recall that $X_n = S_n^2 - vn$ is a martingale with $X_0 = 0$. Since $\mathbb{E}X_{\tau \wedge T} = \mathbb{E}X_0 = 0$ and $\tau < \infty$ a.s., we have by monotone convergence that as $n \rightarrow \infty$

$$
\mathbb{E}S_{\tau \wedge T}^2 = v \mathbb{E}[n \wedge \tau] + v \mathbb{E}\tau < \infty.
$$

This shows that the martingale $\{S_{n \wedge \tau}\}$ is $L^2$-bounded and by Doob’s $L^2$-martingale convergence theorem, $S_{n \wedge \tau} \rightarrow S_\tau$ in $L^2$, resulting with

$$
\mathbb{E}S_{\tau \wedge T}^2 = \lim_{n \rightarrow \infty} \mathbb{E}S_{n \wedge \tau}^2 = v \mathbb{E}\tau.
$$

3. When establishing Wald’s identity in part (a) we used the condition $\mathbb{E}\tau < \infty$ only for justifying the representation $S_\tau = \sum_{k=1}^\infty \xi_k I_{k\leq \tau}$ and for establishing Fubini’s theorem integrability condition when interchanging the order of summation (over $k$) and expectation (with respect to $\mathbb{P}$). For a non-negative sequence $\xi_k$ we have a non-negative integrand, in which case Fubini’s theorem requires no integrability assumption (under the convention that $0 \times \infty = 0$), and the representation for $S_\tau$ is then valid even when $\tau(\omega) = \infty$.

**Exercise [5.4.12]**

This is the strictly positive product martingale $M_n$ of Example 5.1.10, for the positive i.i.d. variables $Y_k = e^{\lambda \xi_k}/M(\lambda)$ of mean one.
1. For $p = 1 - q \geq 1/2$ we know from parts (c) and (d) of Exercise 5.4.11 that $\tau_b$ is finite a.s. Further, by definition $S_{n \wedge \tau_b} \leq b$ for all $n$ and as $M(\lambda) = pe^\lambda + qe^{-\lambda} \geq 1$ whenever $\lambda \geq 0$, in this case

$$M_{n \wedge \tau_b} = \exp(\lambda S_{n \wedge \tau_b} - (n \wedge \tau_b) \log M(\lambda)) \leq \exp(nb).$$

Thus, $\{M_{n \wedge \tau_b}\}$ is a uniformly bounded, hence U.I. martingale. With $S_{\tau_b} = b$, it then follows from Doob’s optional stopping theorem that

$$1 = EM_0 = EM_{\tau_b} = e^{nb}E[M(\lambda)^{-\tau_b}].$$

2. Setting $0 < s < 1$ there exists for $p \geq 1/2$ a unique $\lambda > 0$ such that $M(\lambda) = pe^\lambda + qe^{-\lambda} = 1/s$. Indeed, solving $qsa^2 - x + ps = 0$ for $x = e^{-\lambda}$ in $(0, 1)$, we find from part (a) that $E[s^{\tau}] = x^b$ and

$$E[s^{\tau}] = x = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs},$$

when $q \in (0, 1/2]$, whereas $x = s$ in the trivial case $q = 0$.

3. Since $\tau_{a,b} = \min(\tau_b, \tau_{-a})$ is finite a.s. and $\tau_{-a} \geq a$ we have that

$$P(\tau_b < a) \leq P(\tau_b < \tau_{-a}) \leq P(\tau_b < \infty).$$

Consequently, $P(\tau_b < \tau_{-a}) \rightarrow P(\tau_b < \infty)$ as $a \rightarrow \infty$. To complete the proof recall that from Corollary 5.4.8 we have that

$$P(\tau_b < \tau_{-a}) = 1 - r = \frac{1 - e^{-\lambda_a a}}{e^{\lambda_a}b - e^{-\lambda_a a}} \rightarrow e^{-\lambda_a b}$$

when $a \rightarrow \infty$ (as $\lambda_a > 0$).

4. Clearly, $\{\tau_b < \infty\}$ if and only if $\{Z \geq b + 1\}$. Hence, for any positive integer $b$,

$$P(Z = b) = P(Z \geq b) - P(Z \geq b + 1) = (1 - e^{-\lambda_a})e^{-\lambda_a(b-1)}.$$

**Exercise [5.5.8]**

1. First, we have that

$$L(s) = E(s^N) = \sum_{k=0}^{\infty} s^k p(1-p)^k = p \sum_{k=0}^{\infty} (s(1-p))^k = \frac{p}{1 - s(1-p)}. $$

Then, note that for $s = 1/m$ we have

$$L\left(\frac{1}{m}\right) = \frac{p}{1 - (1-p)/m} = \frac{p}{1 - p} = \frac{1}{m}\]$$

implying that $\rho$, being the unique solution of $s = L(s)$ in $[0, 1)$ must equal $1/m$.

Recall that Proposition 5.5.7 gives the recursive formula $L_n(s) = L[L_{n-1}(s)]$ starting at $L_0(s) = s$. The stated formula we have for the non-critical case is of the form

$$L_n(s) = (pa_n(s) + b(s))/(1 - p)a_n(s) + b(s))$$

with $a_n(s) = m^n(1-s)$. Moreover, $b(s) = (1-p)s - p$ is such that the identity $pa_0(s) + b(s) = s[(1-p)a_0(s) + b(s)]$ holds, or equivalently $L_0(s) = s$. Having established the case $n = 0$, we proceed
to verify this formula by induction over \( n \). To this end, suppressing the argument of \( a_n \) and \( b \), note that by the induction hypothesis

\[
L[L_n(s)] = \frac{p((1-p)a_n + b)}{(1-p)a_n + b - (1-p)(pa_n + b)} = \frac{pa_{n+1} + b}{(1-p)a_{n+1} + b},
\]

where the second identity follows from our choice of \( a_{n+1} = (1-p)a_n/p \) and so we are done.

In the critical case, \( L(s) = 1/(2-s) \) and our formula \( L_n(s) = [na(s) + s]/[na(s) + 1] \) clearly starts at \( L_0(s) = s \). In the induction step, by the induction hypothesis (and suppressing the argument of \( a \)),

\[
L[L_n(s)] = \frac{na + 1}{2(na + 1) - (na + s)} = \frac{(n+1)a + s}{(n+1)a + 1},
\]

where the second identity follows from our choice of \( a = (1-s) \) and we are once more done.

2. Recall Lemma 5.5.4 that \( X_n = m^{-n}Z_n \) is a non-negative MG, so by Doob’s convergence theorem \( X_n \xrightarrow{D} X_\infty \geq 0 \). Further, \( f(x) = \exp(-\lambda x) \) is continuous and bounded on \([0, \infty)\) when \( \lambda \geq 0 \). Thus, \( \widehat{L}_\infty(e^{-\lambda}) = \mathbb{E}[e^{-\lambda X_\infty}] \) is the limit of \( L_n(s_n) \) as \( n \to \infty \), where \( s_n = \exp(-\lambda/m^n) \uparrow 1 \) in the sub-critical case \( m > 1 \). More precisely, \( a_n(s_n) = m^n(1 - s_n) \to \lambda \) and \( b(s_n) = (1-p)s_n - p \to 1 - 2p \).

Hence, from the expression in part (a) we deduce that

\[
L_n(s_n) = \frac{pa_n(s_n) + b(s_n)}{(1-p)a_n(s_n) + b(s_n)} \to \frac{p\lambda + 1 - 2p}{(1-p)\lambda + 1 - 2p} = \frac{\rho \lambda + (1-\rho)}{\lambda + (1-\rho)}. \]

We obtained the right most identity by substituting \( (1-p)(1-\rho) = (1-2p) \) and \( (1-p)\rho = p \) (recall part (a) that \( \rho = 1/m \)), and it is easy to verify the right most expression of the preceding display matches the formula we gave for \( \widehat{L}_\infty(e^{-\lambda}) \).

Since \( \mathbf{P}(X_\infty = 0) = \widehat{L}_\infty(0) = \rho \), clearly

\[
\mathbb{E}[e^{-\lambda X_\infty} | X_\infty > 0] = \frac{\mathbb{E}[e^{-\lambda X_\infty}] - \mathbb{E}[e^{-\lambda X_\infty} I_{\{X_\infty = 0\}}]}{1 - \mathbf{P}(X_\infty = 0)} = \frac{\widehat{L}_\infty(e^{-\lambda}) - \widehat{L}_\infty(0)}{1 - \widehat{L}_\infty(0)} = \frac{1 - \rho}{\lambda + (1-\rho)} = \mathbb{E}[e^{-\lambda T}],
\]

where \( T \) has the exponential distribution of parameter \( 1-\rho \). Recall part (a) of Exercise 3.2.40 that the law of a non-negative random variable is uniquely determined by its Laplace transform, hence the law of \( X_\infty \) conditional on \( X_\infty > 0 \) is an exponential distribution of parameter \( 1-\rho \). Further, the event \( X_\infty > 0 \) implies non-extinction and though the converse is in general not obvious, here \( \mathbf{P}(X_\infty = 0) = \rho \) is also the probability of extinction, hence if there is a difference between these two events, it is of probability zero (and hence can be ignored).

3. In the sub-critical case, \( p > 1-p \) and thus \( m < 1 \). Then, \( a_n(s) \to 0 \) when \( n \to \infty \) and hence \( L_n(s) \uparrow 1 \) for all \( s \in [0, 1] \). More precisely, using the formula (*) for \( L_n(s) \) we find that

\[
m^{-n}(1 - L_n(s)) = \frac{(1-2p)(1-s)}{(1-p)a_n(s) + b(s)} \to (1-2p)(1-s)/b(s)
\]

as \( n \to \infty \). It is easy to check that \( b(s) = b(0)(1-ms) \) and hence

\[
\lim_{n \to \infty} \frac{1 - L_n(s)}{1 - L_n(0)} = \frac{(1-s)b(0)}{b(s)} = \frac{1-s}{1-ms}.
\]
Moreover, following our treatment in part (b) of the conditioning on being positive, we thus get here that
\[ E[s^{Z_n}|Z_n \neq 0] = \frac{L_n(s) - L_n(0)}{1 - L_n(0)} = 1 - \frac{1 - L_n(s)}{1 - L_n(0)} \rightarrow \frac{(1 - m)s}{1 - ms}. \]

Setting \( s = e^{-\lambda} \) we get that
\[ E[e^{-\lambda Z_n}|Z_n \neq 0] \rightarrow \frac{(1 - m)e^{-\lambda}}{1 - me^{-\lambda}} \]
which is precisely the Laplace transform of a Geometric\((1 - m)\) random variable. Recall part (b) of Exercise 3.2.40 that such convergence of Laplace transforms of non-negative random variables implies the corresponding weak convergence, which leads to our thesis.

4. For the critical case, taking \( s_n = e^{-\lambda/n} \uparrow 1 \) yields \( L_n(s_n) \uparrow 1 \). More precisely, from our formula \( L_n(s) = (na(s) + s)/(na(s) + 1) \) we find that \( n(1 - L_n(s_n)) = na(s)/(na(s) + 1) \) since \( a(s) = 1 - s \), and with \( na(s_n) \rightarrow \lambda \) it follows that \( n(1 - L_n(s_n)) \rightarrow \lambda/(1 + \lambda) \). Similarly, \( n(1 - L_n(0)) \rightarrow 1 \). Consequently, similarly to part (c), here
\[
E[e^{-\lambda Z_n/n}|Z_n \neq 0] = \frac{L_n(s_n) - L_n(0)}{1 - L_n(0)} = 1 - \frac{1 - L_n(s_n)}{1 - L_n(0)}
\]
\[ \rightarrow 1 - \frac{\lambda}{1 + \lambda} = \frac{1}{1 + \lambda}. \]

The last expression is the Laplace transform of the standard exponential distribution (with parameter one), from which we deduce that conditional to non-extinction, \( n^{-1}Z_n \) converges weakly to an exponential distribution.

**Exercise [5.3.32]**

1. Since \((S_n, \mathcal{F}_n^\xi)\) is a martingale, the same applies for the stopped martingale \( X_n = S_{n\wedge \tau} \). With \( n\wedge \tau \leq n \), we further have that \( E[S_{n\wedge \tau}^2] \leq E[S_n^2] \) is finite, so \( \{X_n\} \) is an \( L^2 \)-martingale. Clearly, \( X_0 = 0 \) and \( X_k - X_{k-1} = \xi_k I_{k\leq \tau} \). Further, \( \tau \) is a stopping time for \( \mathcal{F}_k^\xi \) so the event \( \{k \leq \tau\} = \{\tau \leq k - 1\}^c \) is in \( \mathcal{F}_{k-1}^\xi \) for each \( k \geq 1 \). With \( \{\xi_k\} \) i.i.d. the predictable compensator of \( \{X_n\} \) is thus
\[
\langle X \rangle_n = \sum_{k=1}^n E[\xi_k^2 I_{k\leq \tau}|\mathcal{F}_{k-1}^\xi] = \sum_{k=1}^n I_{k\leq \tau} E[\xi_k^2|\mathcal{F}_{k-1}^\xi] = (n \wedge \tau) E[\xi_1^2],
\]
with \( \langle X \rangle_\infty = \tau E[\xi_1^2] \).

2. Since \( E[\sqrt{\tau}] \) is finite, we have from part (a) that so is \( E[\langle X \rangle_\infty^{1/2}] \). It then follows from Proposition 5.3.31 that \( \sup_k |X_k| \) is integrable, hence \( \{X_n\} \) is a U.I. martingale. In particular, \( E[X_n] = E[X_0] = 0 \) for all \( n \). Further, with \( \tau \) a.s. finite, \( X_n = S_{n\wedge \tau} \rightarrow S_\tau \) as \( n \rightarrow \infty \) which by uniform integrability of \( \{X_n\} \) implies that \( E[S_\tau] = 0 \).