We discuss transience for reversible MC with countable state space $X = Y$, transition prob $p(x, y)$. irreducible.

We say that $p$ is reversible wrt measure, $\pi : X \rightarrow \mathbb{R}_{\geq 0}$

s.t. $\forall (x, y)$ $\pi(x)p(x, y) = \pi(y)p(y, x)$

We let $c(x, y) := \pi(x)p(x, y)$

$c(x, y) = c(y, x)$

Note that $\sum_y c(x, y) = \pi(x)$

$p(x, y) = \frac{1}{\pi(x)} \sum_y c(x, y)$

In other words any rev MC Markov Chain is a RW on a weighted network $\mathcal{G}(\mathbb{R}_+, \text{V}, \text{E}, c)$
Given $A, B \in V$ disjoint, define $\forall x$

$$
\varphi(x) = \mathbb{P}_x(\tau_A < \tau_B) \quad \text{if } \tau_A \land \tau_b < \infty
$$

$$
= 0 \quad \text{otherwise}
$$

(equivalently, $\varphi(x) = \lim_{n \to \infty} \mathbb{P}_x(\tau_A < (\tau_B \land n))$)

Note that $\varphi$ is harmonic on $V \setminus (A \cup B)$. Indeed,

$$
\varphi(x) = 1 \quad \forall x \in A, \quad = 0 \quad \forall x \in B
$$

$$
\forall x \in V \setminus (A \cup B)
$$

$$
\varphi(x) = \mathbb{E}_x \mathbb{P}_{x_1}(\tau_A < (\tau_B \land n))
$$

$$
\lim_{n \to \infty}
$$

$$
= \lim_{n \to \infty} \sum_y p(x, y) \mathbb{P}_y(\tau_A < (\tau_B \land n))
$$

$$
= \sum_y p(x, y) \varphi(y)
$$
Further if \( V \setminus (A \cup B) =: W \) is finite then \( \phi \) is unique because of the following Max. Principle

Lemma (Max principle) If \( W \) is unique then the max. is at

Lemma If \( \phi \) is harmonic on the finite set \( W \subseteq V \), then

\[
\max_{x \in W} \phi(x) = \max_{x \in \mathcal{W}} \phi(x) \quad \text{in } W^c
\]

where \( \mathcal{W} \) is the set of nodes with at least one neighbor in \( W \).

Proof by contradiction if max "\( \phi \)"

is achieved on \( U \subseteq W \)

then \( \exists x \in U \)

\[
\bar{\phi} = \phi(x) = \sum_y \xi(xy) \phi(y)
\]

hence \( U \) must contain all neighbors of
Corollary: If \( \varphi_1, \varphi_2 \) are harmonic on \( W \) and coincide on \( \partial W \), they coincide on \( W \).

Proof: Consider \( \varphi_1 - \varphi_2 \).

Connection with electrical networks. (These are also physical objects but we give here rigorous defns.)

An electrical network is a weighted graph \( G = (V, E, c) \). Given \( W \subset V \)

We say that \( i : E \to \mathbb{R}, i(x,y) = -i(y,x) \)

\( i \) is a current on \( W \), \( \mathbf{v} : W \to \mathbb{R} \)

\( \mathbf{v} \) are voltages on \( W \).

If \( \forall (x,y) \in E \)

\[ \sum_{(y,z) \in E} i(y,z) = 0 \]

and \( (x,y) \in E \Rightarrow i(x,y) = \frac{\mathbf{v}(x) - \mathbf{v}(y)}{r(x,y)} = c(x,y) (\mathbf{v}(x) - \mathbf{v}(y)) \)
Lemma If \((v, i)\) are current/voltages on \(W\), then \(v\) is harmonic on \(W\) and vice versa (where \(i(x, y) = c(x, y)(v(k) - v(y))\))

Rmk Typically we have \(W = V \setminus (A \cup B)\) and set \(v(x) = v_+\) for \(x \in A\), \(v_+\), \(v(x) = 0\) for \(x \in B\).

The effective \(G_{A, B}\), graph obtained from \(G\) collapsing \(A\) to a single vertex and \(B\) to "".

- If \(\phi\) is harmonic on \(W\) in \(G\), it is harmonic on \(W\) in \(G_{A, B}\).
- Effective conductance

\[C(A \leftrightarrow B) = \text{current flowing across any cut } t \text{ between } A \text{ and } B\]
when \( e \in \mathcal{E} \) we have \( v(x) = 1 \) \( \forall x \in A \)
\( v(x) = 0 \) \( \forall x \in B \)

If \( A = \{ e \} \) we write \( \mathcal{L} (e \leftrightarrow B) \).

**Def** \( S \subseteq \mathcal{E} \) separates \( A \) and \( B \)
in \( G \) if any path btw \( A \) and \( B \) contains at least one edge in \( S \).

**Lemma** Assume \( \tau_B < \infty \) a.s. Then
\[
P_e (\tau_e^+ > \tau_B) = \frac{1}{\pi(e)} \mathcal{L} (e \leftrightarrow B) - 
\]

**Proof**
\[
P_e (\tau_e^+ > \tau_B) = \sum_x p(e,x) P_x (\tau_e > \tau_B)
\]
\[
= \sum_x p(e,x) \left[ 1 - P_x (\tau_B > \tau_e) \right]
\]
\[
= \sum_x p(e,x) \left[ 1 - \frac{q(x)}{q(e)} \right]
\]

where \( q \) is harmonic on \( V \setminus (A \cup B) \)

unique

\( A = \{ e \} \)
\[ P_a(\tau^+_e \rightarrow \tau^+_B) = \frac{1}{\varphi(a) \pi(a)} \sum_x \xi(a,x) \left[ \varphi(x) - \varphi(x') \right] \]

\[ = \frac{1}{\pi(a)} \sum_x \xi(a,x) = \frac{1}{\pi(a)} \hat{\varphi}(a \rightarrow B) \]

\[ = \frac{1}{\pi(a)} \mathcal{E}(a \leftrightarrow B) \]

**Def** Given \( \theta : \hat{E} \rightarrow \mathbb{R} \) antisymmm. (a flow)

its energy is

\[ \mathcal{E}(\theta) := \sum_{(x,y) \in E} \theta^{ij \Delta^2} \]

\[ \mathcal{E}(\theta) := \sum_{(x,y) \in E} \rho(x,y) \theta^{(x,y)^2} \]

[Here \( \hat{E} \subset V \times V \):]

\[ \hat{E} = \{(x,y), x,y \in V, \{x,y\} \in E^2 \} \]

directed edges]
Given \( \varphi : V \to \mathbb{R} \), its energy is (abuse of notation)

\[
\mathcal{E}(\varphi) = \frac{1}{2} \sum_{(x,y) \in E} c(x,y)(\varphi(x) - \varphi(y))^2
\]

**Lemma:** If \( v \) are the voltages st \( v|_A = 1, \ v|_B = 0 \), then \( \forall \varphi \) with same b.c. \( \mathcal{E}(\varphi) \geq \mathcal{E}(v) = \mathcal{E}(\mathcal{A} \leftrightarrow \mathcal{B}) \)

**Proof:** Call \( \mathcal{E}(\varphi, v) = \sum_{(x,y) \in E} (\varphi(x) - v(x))(\varphi(y) - v(y)) \)

Sufficient to show \( \mathcal{E}(\varphi, v) = 0 \) \( \forall \varphi \) such that \( \varphi|_A = 1, \ \varphi|_B = 0 \), \( \mathcal{W} = V \setminus (A \cup B) \)

\[
\mathcal{E}(\varphi, v) = \sum_{(x,y) \in E} (\varphi(x) - v(x))(\varphi(y) - v(y)) = \sum_{x \in \mathcal{W}} \frac{1}{2} \sum_{y \in \mathcal{W}} c(x,y)(v(x) - v(y)) = 0
\]
\[ E_{\text{all}} = \sum_{x \in A} \sum_{y : (x,y) \in E} C(x,y) (v(x) - v(y)) \]

\[ E(v,v) = \sum_{x \in A} \sum_{y : (x,y) \in E} C(x,y) (v(x) - v(y)) = \sum_{x \in A} \sum_{y \in A} \epsilon(x,y) i(x,y) = I(A \leftrightarrow B) \]

**Corollary** If \( c_1 \leq c_2 \) are two sets of conductances, then

\[ E_{c_1}(A \leftrightarrow B) \leq E_{c_2}(A \leftrightarrow B) \]

Assume \( G = (V,E,c) \) is infinite network and let \( V_n \uparrow V \), \( |V_n| = n \)

Define \( G_n \) by “contracting all vertices in \( V \setminus V_n \) in a single one \( b_n \)
Remark: $C_{G_n}(a \leftrightarrow b_n)$ is monotone non-decreasing/increasing.

and

$$C(a \leftrightarrow \infty) = \lim_{n \to \infty} C_{G_n}(a \leftrightarrow b_n)$$

Then $C(a \leftrightarrow \infty)$ does not depend on the sequence $G_n$.

Proof: Via hitting times interpretation or monotonicity principle.

Theorem: $G = (V,E,c)$ is recurrent/transient iff $C(a \leftrightarrow \infty) > 0$.
Lemma. Let $i$ be the current flow from $a \leftrightarrow B$ for $v(a) = 1$, $v_B = 0$.
Then $\forall \theta$ for any other flow $\theta$
$s.t. \ d \theta = d \ i$

$$d \theta(x) := \sum_{(x,y) \in E} \theta(x,y)$$

we have $E(\theta) \geq E(i) = C(a \leftrightarrow B)$

vice versa if $\theta$, $i$ are unit flows

$$E(\theta) \theta \geq E(i) = \frac{1}{C(a \leftrightarrow B)}$$

Proof. Exercise.

Thm. $(V,E,C)$ is transient iff
there exists a unit flow $i \Rightarrow \infty$
with finite energy.
Thm. For $d \geq 3$, SRW on $\mathbb{Z}^d$ is transient.

Proof. By monotonicity, sufficient to consider $d = 3$, $\alpha = 0$.

Construct

$$\theta(x,y) = \mathbb{P}(\theta(x,y) \in R)$$

where $R$ is a random simple path on $\mathbb{Z}^d$ constructed as follows:

Draw $v \sim \text{Unif}(S^2)$. Let $r_0 = R_{\geq 0}$ be the ray along direction $v$.

and $R$ a path from $r_0$ in $\mathbb{Z}^3$, measured on $v$ s.t.

$$d(R, r_0) \leq C$$

for a constant $C$.

$$(d(R, r_0) = \max_{x \in R} \min_{y \in R_0} ||x - y||_2)$$