This course is oriented towards research in applied probability, and requires active participation from all students attending it. Students are asked to get involved in a (small) research project by:

1. Forming a small team (ideally 2-3 students).
2. Reviewing the relevant literature on a research problem chosen from a list proposed by us, and submitting a one-page proposal by October 22.
3. Thinking (!) independently to the problem. By ‘think’ we mean try a few approaches to solving, or make progress on the problem. Some of these will be suggested by us, and some (hopefully) will come from you.

The conclusions will be presented at the end of the course: you are encouraged to report both negative (‘we tried this and did not work’) and positive ones (‘we solved the problem’).

Here is a list of possible possible topics. More details are available from us.

**Spin glasses with correlated couplings.** A generic spin glass model is given by the (random) Boltzmann distribution

$$
\mu(x) = \frac{1}{Z_n(\beta, h)} \exp\left\{ \beta \sum_{i,j} J_{ij} x_i x_j + h \sum_{i=1}^n x_i \right\}.
$$

(1)

over $x \in \{+1, -1\}^n$, where the $J_{ij}$ are random variables and $Z_n(\beta, h)$ is a normalization constant. Of interest is the free entropy density

$$
\phi(\beta, h) \equiv \lim_{n \to \infty} \frac{1}{n} \log Z_n(\beta, h).
$$

(2)

Mathematically rigorous studies have been confined mostly to the case of iid couplings $J_{ij}$. On the other hand, physicists have considered various cases of correlated couplings. We propose to apply mathematical techniques to examples of this type. In particular we are interested in the case $J = O D O^T$ for $D$ diagonal of i.i.d. entries, taking the values $\pm 1$ with equal probability and $O$ a random orthogonal matrix distributed according to the Haar measure.

The objectives would be:

1. Consider the case $h = 0$ and compute the expected partition function $\mathbb{E}\{Z_n(\beta, h)\}$ to leading exponential order.
2. Use the second moment method, plus the above calculation. Does this allow to compute $\phi(\beta, 0)$ for $\beta$ small enough?
3. Conjecture the value of $\phi(\beta, h)$ and investigate the possibility of proving interpolation formulae a la Guerra.
Spin glasses with correlated couplings: TAP equations. Within the context of the previous project, can you conjecture some form of mean field (TAP) equations and maybe also prove them? (Assume, for instance, concentration of the overlap.)

If this is too hard, consider the case of $J_{ij} = n^{-1/2}Z_{ij} + n^{-1}Y$, with $Z_{ij}$ and $Y$ standard i.i.d. normal variables.

The Hopfield model. The Hopfield model is a simple model of associative memory, introduced by John Hopfield in 1982. It stores patterns $\{\xi_1, \ldots, \xi_p\}$, where $\xi^k = (\xi^k_1, \ldots, \xi^k_N) \in \{+1, -1\}^n$ is a vector, in the matrix

$$J_{ij} = \frac{1}{\sqrt{pn}} \sum_{k=1}^p \xi^k_i \xi^k_j.$$  

(3) The idea is that patterns can be retrieved by looking for local minima of the energy function $E(x) \equiv -x^T J x$, where $x \in \{+1, -1\}^n$. For instance if $p = 1$, $E(x) = -(\xi^T x)^2$ has its unique minima at $x = ±\xi$.

In studying the properties of this energy function, it is convenient to consider the Boltzmann measure (1) with the $J_{ij}$ defined as above in terms of patterns. The case of iid patterns, whereby each entry $\xi^k_i$ is an independent Bernoulli($1/2$) random variable has been intensively studied.

We propose students to consider the sparse graph case, namely

$$J_{ij} = \frac{K_{ij}}{\sqrt{p}} \sum_{k=1}^p \xi^k_i \xi^k_j,$$  

(4) where $\gamma > 0$ is a fixed number and $(K_{ij})_{i<j}$ is a collection of i.i.d. random variables $K_{ij} \sim$ Bernoulli($\gamma/n$). In other words, we restrict the interactions of the Hopfield model to an Erdős-Rényi random graph with average degree (approximately) $\gamma$.

Ising models on random hypergraphs. Hypergraphs are generalization of ordinary graphs, whereby each edge (a ‘hyperedge’) may contains $k \geq 3$ vertices. For our purposes, it is actually convenient to use the equivalent formulation of ‘factor graphs.’ These are bipartite graphs $G = (V, F, E)$, whereby each vertex $i \in V = [n]$ corresponds to a variable, and each vertex $a \in F$ to a ‘factor’ in the energy function. We assume factor node to have uniform degree $k \geq 3$. 


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Given such a factor graph, and parameters \( \beta, h \geq 0 \), we can consider the energy function \( E_G : \{+1,-1\}^n \to \mathbb{R} \) by

\[
U_G(x) \equiv -\beta \sum_{a \in F} \prod_{i \in \partial a} x_i - h \sum_{i \in V} x_i,
\]

and the corresponding Gibbs measure over \( \{+1,-1\}^n \):

\[
\mu_G(x) = \frac{1}{Z_G(\beta,h)} e^{-U_G(x)}.
\]

For sequences of factor graphs \( \{G_n = (V_n = [n], F_n, E_n)\}_{n \geq 1} \) of diverging size, we also consider the free energy density (whenever it exists)

\[
\phi(\beta,h) = \lim_{n \to \infty} \frac{1}{n} \log Z_{G_n}(\beta,h).
\]

The project aims at investigating: (i) Existence of the limit free energy density; (ii) Characterization of the limit value; (iii) Local limit of the measures \( \mu_{G_n} \). We suggest to consider the following random models for the graphs \( G_n \)

- Random regular factor graphs. These are uniformly random factor graphs, with degree \( k \) at the factor nodes, and degree \( l \) at variable nodes. If \( |F| = m \), we have, of course, \( mk = nl \).
  If convenient, you should consider the corresponding configuration model.

- Uniformly random factor graphs, with \( m = \lceil n \gamma \rceil \) factors of degree \( k \), and \( n \) variables, with \( \gamma > 0 \) a constant.
  If convenient, you should consider the corresponding ‘independent hyper-edge’ model, whereby for each of the \( \binom{n}{k} \) subset of \( k \) variable nodes, the corresponding degree-\( k \) factor node is present, independently, with probability \( n \gamma / \binom{n}{k} \).

Hint 1: There is an obvious connection between the \( \beta \to \infty \) limit of this model and XORSAT. Hint 2: The appendix of Ref. [8] contains some simple moment calculations for this model.


**Glauber dynamics for Ising models on random hypergraphs.** Consider the model in the previous problem, and assume the case of a random regular hypergraph. Further assume that the degree \( \ell \) is smaller than the hyper-edge size \( k \) (take for instance \( k = 5 \) and \( l = 3 \)). Further assume \( h = 0 \) (no magnetic field). It can then be proved that the model has no theromodynamic phase transition. Namely, for each \( \beta \geq 0 \)

\[
\lim_{n \to \infty} \frac{1}{n} \log Z_G(\beta,0) = \log 2 + \frac{l}{k} \log \cosh \beta.
\]
In this project we consider Glauber dynamics for the measure $\mu(x)$. This is the reversible Markov chain that, at each time step, chooses a uniformly random vertex $i$ and resamples $x_i$ from its conditional distribution given all the other spins.

It is conjectured that there exists $\beta_d \in \mathbb{R}$, such that the mixing time of this Markov chain is $\exp \{ \Theta(n) \}$ if $\beta > \beta_d$ and $O(n \log n)$ if $\beta < \beta_d$.

Can you bring evidence in favor, or against this conjecture?

Random Unique Games. We consider an integer $k$, a graph $G = (V = [n], E)$, and for each directed edge $i \to j$ with $(i,j) \in E$, a permutation $\pi_{i \to j} : [q] \to [q]$ with $\pi_{i \to j} = \pi_{j \to i}^{-1}$. We then consider the energy function $E_{G,\pi} : [q]^n \to \mathbb{R}$ defined by

$$U_{G,\pi}(x) \equiv \sum_{(i,j) \in E, i < j} \mathbb{I}(x_j = \pi_{i \to j}(x_i)).$$

We suggest to consider the following random model for $G$ and $\pi$:

1. The permutations $(\pi_{i \to j})_{(i,j) \in E}$ (assume here that a direction is selected arbitrarily for each edge) are i.i.d. and uniformly random.
2. The graph is a uniformly random $k$-regular graph over $n$, vertices.
3. Alternatively, you can consider the Erdős-Renyi random graph with edge probability $\gamma/n$.

For such graphs, we consider the maximum energy density $u^*(G_n, \pi) \equiv \frac{1}{|E|} \max_{x \in [q]^n} U_{G_n,\pi}(x)$, and its limit $\lim_{n \to \infty} u^*(G_n, \pi)$ (whenever it exists). The objective of the project is (i) Establish upper/lower bounds on the limit; (ii) Prove that the limit exists.

Spin glass models in large dimension. Let $G_{L,d}$ a $d$-dimensional discrete torus of size $L$ (this is the graph with vertex set $\{1, \ldots, L\}^d$ and edges $(x, x + e_i)$ where $e_i$ is the canonical basis, and sums are modulo $L$). Consider the corresponding Ising partition function

$$Z_L(\beta; d) = \sum_{\sigma \in \{+1,-1\}^d} \exp \left\{ \frac{\beta}{\sqrt{d}} \sum_{(x,y) \in E(G_{L,d})} J_{xy} \sigma_x \sigma_y \right\},$$

where $\{J_{xy}\}$ is a collection of standard normal random variables. Define the free energy density

$$\phi(\beta, d) \equiv \lim_{L \to \infty} \frac{1}{L^d} \mathbb{E} \log Z_L(\beta; d).$$

(Why does this limit exist?) A natural conjecture is that, as the dimension diverges, this quantity converges to the free energy of the Sherrington-Kirkpatrick model:

$$\lim_{d \to \infty} \phi(\beta; d) \equiv \phi_{\text{SK}}(\beta).$$

Can you prove or disprove this conjecture? (A related question was studied in [9].)