Proof. Notice that
\[ \mu_i(x_i) = \sum_{x^{(i)}} \mu(x_i | x^{(i)}) \mu(x^{(i)}) \]
variables at distance \( t \) from \( i \) on \( G \).

and that \( \mu_i(x_i | x^{(i)}) = \mu_t(x_i | x^{(t)}) \).

Proceed as in the previous proof.

Example 1: Ferromagnetic Ising model on regular graphs.

\( G \), graph of degree \( k+1 \)

For \( x_i \in \{ -1, 1 \} \)

\[ \mu(x) = \frac{1}{Z} \exp \left[ \beta \sum_{i} x_i + B \sum_{i,j} x_i x_j \right] \]

\( T(G) \)
BP equations

\[ V_{i \rightarrow j}^{(t+1)}(x_i) \propto e^{B_{x_i}} \prod_{e \in \text{leaves}} \sum_{x_e} e^{B_{x_e} x_e} V_{e \rightarrow i}^{(t)}(x_i) \]

It is convenient to parametrize these in terms of log-likelihoods

\[ h_{i \rightarrow j}^{(t)} \equiv \frac{1}{2} \log \frac{V_{i \rightarrow j}^{(t)}(1)}{V_{i \rightarrow j}^{(t)}(-1)} \]

\[ h_{i \rightarrow j}^{(t+1)} = B + \sum_{e \in \text{leaves}} f_{B_e}(h_{e \rightarrow i}^{(t)}) \]

\[ f_B(x) = \operatorname{atanh} [\tanh B + \tanh x] \]
On the tree $t_i^{(t)}(G_i)$ it is the same story.

\[
\sup_{x \in \mathbb{R}} |\mu(x_i | x_b^{(t)}) - \mu(x_i | x_b^{(t)}_{\text{boundary}})| \leq \frac{1}{2} \sup_{h^{(t)}_+ \neq h^{(t)}_-} |\text{tgh} \ h_i^{(t)} - \text{tgh} \ h_i^{(t+1)}| \
\]

\[
\leq \frac{1}{2} \sup_{h_{\pm}^{(t)} \neq h_{\pm}^{(t+1)}} |h_{\pm}^{(t)} - h_{\pm}^{(t+1)}|
\]

Let $h_{+}^{(t)}$, $h_{-}^{(t)}$ be the log-likelihoods after $t$ levels if the leaves are initialized (resp. to $+\infty$ and $-\infty$.

Then, since $f_\beta$ is monotone,

\[
h_{-}^{(t)} \leq h_i^{(t)} \leq h_{+}^{(t)} \quad \text{for any other bc.}
\]
both for $h^+_t$ and $h^+_t$ the following recursion holds

$$h^{(t+1)} = B + kf^+_t(h^+_t)$$

$B + f^+_t(x)k$

$B + B - \beta k$

In particular

If $x = B + f^+_t(x)k$ have unique solution then $s(t) < A e^{-kt}$ for some $A > 0$.
A careful study show that this happens in the region

\[ \beta \]

\[ \beta_c \]

\[ \text{BP works} \]

\[ \beta_c \text{ can be obtained through a simple argument:} \]

If \( f_\beta(\cdot) \) has Lipschitz constant \( L_\beta \), then

\[ 8(t) \leq |h_+^{(t)} - h_-^{(t)}| \leq (k L_\beta)^{t-1} \leq |h_+^{(1)} - h_-^{(1)}| \]

\[ L_\beta = \tgh \beta \Rightarrow 8(t) \to 0 \text{ if } k \tgh \beta < 1 \]

In fact \( \beta_c = \text{atanh}(\frac{1}{k}) \)

\[ \square \]
**Dobrushin Condition (1968)**

\[ C_i = (V, E, E) \]

\[ \mu(x) = \frac{1}{Z} \prod_{a \in F} \phi_a(x_a) \]

For \( i, j \in V \), define the influence of \( j \) on \( i \) as

\[ C_{ij} = \sup_{x, x_i} \left\{ \| \mu(x_i = 0 | x \_i) - \mu(x_i = 0 | x' \_i) \|_{TV} \right\} \]

\[ x_e = x'_e \quad \forall \ e \in V \setminus \{ i,j \} \]

(Here I use the notation \( x_{\_i} \) for the set of variables \( \{ x_e : e \neq i, j \} \))

Notice that \( C_{ij} \neq 0 \) only if \( d(i,j) = 1 \)
Theorem  If
\[ r = \sup_{i,j} (\Sigma C_{ij}) \leq 1 \]
then
\[ \sup_{x_t, x'_t} \left\| \mu(x_i = 0 | x_t) - \mu(x_i = 0 | x'_t) \right\|_{TV} \leq r^{t/2} - r \]

This can be the measure on the computation tree as well as the measure on the original graph.

Example: Ising model on \((k+1)\)-regular graph

\[ \mu(x_i | x_{\bar{i}}) = \frac{1}{Z_i} \exp\left\{ \beta \sum_{j \in \bar{i}} x_i (x_j + B x_i) \right\} \]

\[ \mu_t = \frac{1}{2} \left\{ 1 + x_i \tanh h(x_{\bar{i}}) \right\} \]

\[ h(x_{\bar{i}}) = \beta \sum_{j \in \bar{i}} x_j + B \]
\[ \| \mu(x_i = 1 | x_{ai}) - \mu(x_i = 0 | x_{ai}) \|_{TV} \]
\[ = \frac{1}{2} \left| \tanh h(x_i) - \tanh h(x'_{ai}) \right| \leq \frac{1}{2} \tanh \left| h(x) - h(x') \right| \leq \theta \]

\[ C_{ij} = \sup_{x \neq x'} \{ \| \mu(1 | x) - \mu(1 | x') \|_{TV} : x = x' + \epsilon \} \]
\[ \leq \frac{1}{2} \tanh 2\beta \]

\[ r \leq (k+1) \frac{1}{2} \tanh 2\beta < 1 \text{ for } \]
\[ \left[ \beta \cdot \frac{1}{2} \text{atanh} \left( \frac{2}{k+1} \right) \right] \]

\( (\text{to be compared with } \beta_c(k) = \text{atanh} \left( \frac{1}{k} \right) ) \)
Proof

Consider for the sake of simplicity, pairwise compatibility functions

\[ \mu(x) = \frac{1}{Z} \prod_{ij \in G} \psi_{ij}(x_i, x_j) \]

We shall identify in the usual manner G with an ordinary graph.

[Diagram of a graph with labeled nodes and edges]

Given \( G \), we shall construct.

Let's simplify the notations

\[ \mu(x) \leftarrow \mu(x \mid x_t) \]

\[ \mu'(x) \leftarrow \mu(x \mid \bar{x}_t) \]
$\mu(s), \mu(\cdot)$ are graphical models on the subgraph induced by nodes whose distance from $i$ is smaller than $t$.

$B_i(t+1)$

**Example $B_i(3)$**

We shall construct a coupling between $\mu$ and $\mu'$ st

$$P\left\{ x_i = x'_i \mid y \leq \delta^t \right\} \leq \frac{\delta^t}{1-\delta} \quad \Rightarrow \quad \|\mu(x)-\mu'(x')\|_1 < \text{THESIS}$$

The coupling is constructed recursively.

@: 2020
1. Start from any coupling (e.g. $x, x'$ independent).

2. Repeat the following.
   * For any $j \in B_i(t-1)$:
     A) Sample $x, x'$ from the current coupling.
     B) Re-sample $x_j, x'_j$ conditional to $x \neq x'$ at $j$ according to the coupling that achieves the TV distance between
     $$\mu(x_j = 1 | x_j), \mu(x_j = 0 | x_j)$$

Consider the operation 2B. Given a coupling $\hat{\mu}$ between $\mu$ and $\mu'$, it returns a new coupling $T_j \hat{\mu}$. 
Suppose that $\hat{\mu}(x_e + x'_e) \leq a \cdot e$

Then

$T_j \hat{\mu}(x_e + x'_e) = a \cdot e \quad \forall \ e \neq j$

$T_j \hat{\mu}(x_j + x'_j) = \text{max} \left\{ \mathbb{E}_{\hat{\mu}} \| \mu(x_j = \cdot | x_j) - \mu(x_j = \cdot | x'_j) \| \right\}$

$\leq \mathbb{E}_{\hat{\mu}} \| \mu - \mu \| = (\ast)$

For simplicity of notation, assume

$\lambda_k \in \mathbb{Z}$

And let $x^{(0)} = (x_1^1, x_2^1, x_3^1)$

Then

$\sum_{e = 1}^k \mathbb{E}_{\hat{\mu}} \| \mu(x_j = \cdot | x^{(0)}) - \mu(x_j = \cdot | x^{(1)}) \| e \leq \sum_{e = 1}^k \mathbb{E}_{\hat{\mu}} \| \mu(x_e = x_e') - \mu(x_e = x_e') \| e$

$\leq \sum_{e \in \mathcal{E}} \sum_{e \in \mathcal{E}} C_{ie} \hat{\mu}(x_e + x_e') \leq \sum_{e \in \mathcal{E}} C_{ie} \lambda_k e$
This relation does not hold if \( j \) has a neighbor outside \( B_{i(t-1)} \). In this case

\[ (\varepsilon) \leq \sum_{e \in \partial B} C_{ie} a_e + \sum_{e \in \partial B \cap \partial \hat{B}} C_{ie} \varepsilon \]

\[ b_i, b_j \leq r \]

After one loop over \( j \), we obtain a new coupling \( \mu' \) such that

\[ \mu'(x_j + x_j') \leq a'_j = (Ca)_j + b_j \]

We thus get a sequence \( \mu^{(n)} \) with

\[ a^{(n+1)} \leq Ca^{(n)} + b \]

\[ \leq \sum_{j} C^{(n+1)} a^{(0)} + (1 + C + \ldots + C^n) b \]

\[ (Ca^{(0)})_j \leq \sum_j (C^n)_{jk} \leq \gamma^n \rightarrow 0 \text{ as } n \rightarrow \infty \]
\((C^n b)_i = 0 \text{ unless } n \geq t-1\)

\[
\lim_{n \to \infty} a_i^{(n)} = \left[ \frac{c^{t-1}}{4-C}(-1)^{-1} b \right]_i
\]

\[
\leq \frac{\sqrt[\nu]{\frac{c^{t-1}}{4-C}(-1)}}{\epsilon} < \frac{x^t}{1-\delta}
\]