1 Density Evolution from a generic message passing algorithm

Density Evolution is also known as Recursive Distributional Equations, because of the recursive nature of the distribution of the messages. For further analysis you are referred to Richardson and Urbanke’s ”Modern Coding Theory” and Aldous and Steele’s ”The Objective Method”. Let’s start with a generic message passing algorithm, described by following equations.

**Generic Message Passing Equations**

\[
\begin{align*}
\hat{\nu}^{(t)}_{a \rightarrow i} &= \Phi_{a \rightarrow i}(\{\nu^{(t)}_{j \rightarrow a} : j \in \partial a \setminus i\}) \\
\nu^{(t+1)}_{i \rightarrow a} &= \Psi_{i \rightarrow a}(\{\hat{\nu}^{(t)}_{b \rightarrow i} : b \in \partial i \setminus a\}) \\
\nu^{(t)}_{i}(x_i) &= \Psi_{i}(\{\hat{\nu}^{(t-1)}_{a \rightarrow i} : a \in \partial i\})
\end{align*}
\]

We first need some assumptions.

- **Random Graph** Assume \(G=(V,F,E)\) to be a random factor graph from \(\mathbb{C}_n(k, \alpha)\) or \(\mathbb{D}_n(\Lambda, \rho)\).

- **Messages** Assume \(\Phi_{a \rightarrow i}\) to depend on \(a \rightarrow i\) only through \(|\partial a|\) the degree of \(a\) and through \(J_a\), a random variable assigned to factor node \(a\).

- **Messages** Assume \(\Psi_{i \rightarrow a}\) to depend on \(i \rightarrow a\) only through \(|\partial i|\) the degree of \(i\) and through \(\hat{J}_i\), a random variable assigned to variable node \(i\).

- **Continuity** \(\Phi_{a \rightarrow i}\) and \(\Psi_{i \rightarrow a}\) have to be continuous wrt \(\{J\}, \{\nu\}, \text{and} \{\hat{\nu}\}\).

**Example 1 Weighted Independent Set**

\(G = (V_G, E_G), S \subseteq V_G\)

\(\mu(S) = \frac{1}{Z} I(S \text{ is an Independent Set}) \prod_{i \in S} \Lambda_i\)

**Assumption:**

\(G\) is random graph \(G_n(2, \alpha)\)

\(\Lambda_i\) are iid random : \(\Lambda_i\) plays the role of \(\hat{J}_i\)

**Proposition 1.** If \(\{\nu^{(0)}_{i \rightarrow a}\}\) are iid random variables distributed as \(\nu^{(0)}\), then for any \(t \geq 0\) and a uniformly random edge \(i-a\), then \(\nu^{(t)}_{i \rightarrow a} \rightarrow \nu^{(t)}_{i \rightarrow a}, \hat{\nu}^{(t)}_{a \rightarrow i} \rightarrow \hat{\nu}^{(t)}_{a \rightarrow i}, \tilde{\nu}^{(t)}_{i} \rightarrow \tilde{\nu}^{(t)}_{i}\), as \(n \rightarrow \infty\), where the random variables \(\nu^{(t)}, \hat{\nu}^{(t)}, \tilde{\nu}^{(t)}\) are defined through the distributional recursion.

**Recursive Distributional Equations**

\[
\begin{align*}
\hat{\nu}^{(t)} &\approx \Phi_k(\{\nu^{(t)}_1, \ldots, \nu^{(t)}_{k-1}; J\}), k \approx \rho \\
\nu^{(t+1)} &\approx \Psi_l(\{\hat{\nu}^{(t)}_1, \ldots, \hat{\nu}^{(t)}_{l-1}; \hat{J}\}), l \approx \lambda \\
\tilde{\nu}^{(t)} &\approx \tilde{\Psi}_l(\{\nu^{(t)}_1, \ldots, \nu^{(t)}_{l}; J\}), l \approx \Lambda
\end{align*}
\]
Figure 1: \( B_{i \rightarrow a}(t) \): Tree with depth \( t \) and root \( a \)

Figure 2: 2-core of \( G \)

Proof

Given a uniformly random edge \( i \rightarrow a \), the message \( \nu_{i \rightarrow a}(t) \) depends uniquely on the depth \( t \) neighborhood of \( i \rightarrow a \), \( B_{i \rightarrow a}(t) \).

As \( n \to \infty \), \( B_{i \rightarrow a}(t) \to T_t^{(s)}(\Lambda, \rho) \). This follows form the proposition in lecture 13. As \( n \) grows large, we know that local structure of graph converges in distribution to a known random tree, and messages are function of local messages. Therefore \( \nu_{i \rightarrow a}(t) \to \nu(t) \) where \( \nu(t) \) is the message through the root edge of \( B_{i \rightarrow a}(t) \) after \( t \) message passing iterations.

Analogously, \( \nu_{a \rightarrow i}(t) \to \nu(t) \). The recursive distribution relationship obviously holds for these quantities.

In statistical physics and coding theory, people often write the recursions through the densities of \( \nu(t) \), \( \nu(t) \).

\( a_t(\nu) \triangleq \text{density of } \nu(t) \)

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Density Evolution: \( a_{t+1}(\nu) = E_t E_f \left[ f(\delta(\nu - \Phi_1(\nu_1, \ldots, \nu_{t-1}, ; J))) \prod_{i=1}^{t-1} d\hat{a}_t(\nu_i) \right] \)

Example 2 k-core percolation

Given an ordinary graph \( G = (V_G, E_G) \), its k-core is the largest subset \( U \subseteq V_G \) such that the induced subgraph has degree \( k \) or more. Notice k-core of a graph \( G \) is unique, since k-core is the union of all subsets
that have minimum degree of k. k-core of G ≡ C_k(G).
The question we’re interested in is, for uniformly random graph G with n vertices and α \binom{n}{2} edges, what can we say about |C_k(G)| or the distribution of |C_k(G)|.
Let’s think about a message passing algorithm,
- \nu_t^{(i)} \in \{0, 1\}
- \nu_t^{(0)} \in \{0, 1\}, \forall i \rightarrow j
- \nu_t^{(t+1)} = 1 if at least k-1 vertices l \in \partial i \\backslash j have \nu_t^{(l)} = 1, 0 otherwise
- \hat{\nu}_t^{(t)} = 1 if at least k vertices l \in \partial i have \nu_t^{(l)} = 1, 0 otherwise

Notice that messages only change from 0 to 1 once as t goes from 0 to \infty. Further C_k(G) = \{i \in V : \nu_t^{(i)} = 1, \forall t\}. Also the algorithm converges after a finite number of iterations, which is at most equal to the number of edges.

Analysis (1) Random graph G \mathbb{C}_n(2, \alpha)
Distribution of the message : \nu^{(t)} = 1 with probability a_t, and 0 w.p 1-a_t
Density Evolution :

\[ a_t = \sum_{l=0}^{\infty} e^{-2\alpha} \frac{(2\alpha)^l}{l!} \sum_{b=0}^{l} \binom{l}{b} a_{t-1}^b (1-a_{t-1})^{l-b} \]
\[ = \sum_{l_0,l_1 \geq k-1} e^{-2\alpha} \frac{(2\alpha)^{l_0+l_1}}{(l_0+l_1)!} \frac{(l_0+l_1)!}{l_0!l_1!} a_{t-1}^{l_1} (1-a_{t-1})^{l_0} \]
\[ = e^{-2\alpha} e^{2\alpha(1-a_{t-1})} \sum_{l=k-1}^{\infty} \frac{(2\alpha)^{l}}{l!} a_{t-1}^{l} \]
\[ = \sum_{l=k-1}^{\infty} e^{-2\alpha a_{t-1}} \frac{(2\alpha a_{t-1})^{l}}{l!} \]
\[ = f(2\alpha a_{t-1}) \]

\[ f(\lambda) = P\{X \geq k - 1\}, X \text{ Poisson(\lambda)} \]