"Coupling and contraction"

Let us reconsider the Ising spin glass.

\[ G = (V, E) \text{ random regular graph of degree } (k+1) \]

\[ \mu(x) = \frac{1}{Z} \exp \left\{ \beta \sum_{\langle ij \rangle \in E} J_{ij} x_i x_j + B \sum_i x_i \right\} \]

\( J_{ij} \in \{ \pm 1 \} \) uniformly random

\( B \) added magnetic field term.

For \( \beta = 0 \) and \( k(\tanh \beta)^2 < 1 \), we can prove that \( \delta_0 \) is a locally stable fixed point of density evolution.
What about density evolution for $B > 0$?

The local stability calculation does not apply easily because we do not know the fixed point.

\[
\begin{align*}
\mathbb{E}_{\mathcal{H}} [ h_{t+1} ] &= B + \sum_{i=1}^{k} J_i (h_{t,i}) \\
\end{align*}
\]

Idea: Consider two distinct initial conditions $h_0, h'_0$ and ask whether show that $\| h_t - h'_t \| \to 0$?

How? by coupling:

\[
\begin{align*}
h_{t+1} &= B + \sum_{i=1}^{k} J_i (h_{t,i}) \\
h'_{t+1} &= B + \sum_{i=1}^{k} J_i (h'_{t,i}) \\
\end{align*}
\]

\[\text{same } J_i \text{'s} \]
$$|h_{t+1} - h_t'|^2 = \sum_{i=1}^{k} J_i \left[ f(h_{t+1}^i) - f(h_t^i) \right]^2$$

$$E\{ |h_{t+1} - h_t'|^2 \} = k E\{ (f(h_{t+1}) - f(h_t))^2 \} \leq k (f|_{0}) E\{ |h_t - h_t'|^2 \}$$

Therefore, for $$\lambda^2 = k (t gh \beta)^2 < 1$$, (using $$E|X|^2 \leq (E|X|^1)^{\frac{1}{2}}$$)

$$E|h_t - h_t'| \leq \lambda^t (E|h_0 - h_0'|^2)^{\frac{1}{2}}$$

This implies, for $$t < s$$ and some $$C > 0$$ (coupling) $$E|h_t - h_s|^1 \leq Ct$$ initial condition $$h_0 = 0$$

In fact $$h_s^0 = h_t^0$$

And $$E(h_t^0)^2 \leq C^2$$ for some $$C$$

[This requires a separate calculation]
Therefore, for any Lipschitz function \( F \), \( EF(h^0_t) \) is a Cauchy sequence and converges:

\[
EF(h^0_t) - F(h^0_s) \leq L_F \| h^0_t - h^0_s \| \leq L_F \alpha^t
\]

Together with some general measure theory (tightness) this implies

\( h^0_t \rightarrow h^0 \)

Armed with coupling, we can revisit local stability:

Consider \( h_0 \sim h^* \)

More precisely assume \( Eh_0 - h^*, \text{ i.e. } \)

We want to check whether \( h_t \rightarrow h^* \)
or not.
\[ \mathbb{E}[\hat{h}_{t+1} - h_*] = k \mathbb{E}[f(h_t) - \hat{f}(h_t)] \]

\[ \approx k \mathbb{E}[f'(h_*^2) (h_t - h_*^2)] \]

\[ \approx k \mathbb{E}[f'(h_*^2)] \mathbb{E}[h_t - h_*^2] \]

Nota: no completamente justificado

**Local stability condition**

\[ k \mathbb{E}[f'(h_*^2)] < 1 \]
LAST CHAPTER: STATISTICAL PHYSICS IDEAS

- **Warning #1**: Most of this is based on 'very precise' heuristics [and yet has been confirmed in a number of highly non-trivial cases] of Talagrand.
- **Warning #2**: Intuition (and solutions) for random ensembles.

**Basic "dogma"**: For sparse random graphs BP should be asymptotically exact. If not, then something interesting is going on and there exists an "higher order" version of BP that is asymptotically exact.
Higher order versions: 1RSB, 2RSB, 3RSB, \ldots, 0RSB

→ "replica symmetry breaking",

ORSB = RS = BP

→ "should be asymptotically exact, means that with a proper initialization it returns good approximations of the marginals.

→ "Something Interesting is Going On"

Consider the example of XORSAT

\[ \mu(x) = \frac{1}{Z} \sum_{\{H \mid x = 0 \pmod{2}\}} \]

\[ H = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \cdots \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \]

\[ m \rightarrow 0-1 \text{ matrix} \]

\[ n \rightarrow \text{random cisoid vector} \]
H uniformly random among the
n \times m 0-1 matrices with \( k \ (\geq 3) \)
one per row.

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0
\end{bmatrix}
\]

\[
\ell_1(\omega) \quad \ell_2(\omega) \quad \ell_k(\omega)
\]

\[
\mu(\omega) = \frac{1}{Z} \prod_{\ell=1}^{m} \prod_{i=1}^{n} (x_{\ell_1(\omega)} \oplus x_{\ell_2(\omega)} \oplus \ldots \oplus x_{\ell_k(\omega)} = 0)
\]

Factor graph

\[ G \cong G_n(K, \alpha) \quad \alpha = \frac{m}{n} \]
We want to understand the "structure of the set of solutions" of \( \mu \).

Define the \( 2 \)-core of \( G \) as the maximal subset \( F \subseteq F \) such that the induced subgraph \( G[F] \) has variable node degree at least 2.

The typical core size can be done analogously to the \( k \)-core of a random graph. If we denote by \( k_2(G) \) the \( 2 \)-core subgraph, we get

\[
\lim_{n \to \infty} \frac{1}{n} |E(k_2(G))| = Q, \quad \text{where } Q \text{ solves }\]

\[
Q = 1 - e^{-k_* \lambda^{k_* - 1}} \quad \text{number of vertices in the } 2\text{-core}
\]
Further, $|K_2| = E[|K_2|] + o(n)$ w.h.p.

$$\kappa_d(3) = 0.818$$

A simple argument to derive $\kappa_d$:

Consider the model $T(k\alpha)$ for finite neighborhoods in $G_n(k\alpha)$:

Imagine freezing $X_t$ to some value (e.g., $X_t = 0$, does not matter which one)
Let

\[ Q_t = P_t \] 
where \( x_0 \) takes value 0 in all the states which have \( \xi_t = 0 \).

\[ Q_t = P_t \] 
where \( x_0 \) can take value 0 or 1.

Then

\[ Q_{t+1} = 1 - \exp(-k\alpha Q_t) \] 

\[ Q_0 = 1 \]

\[ \alpha < \lambda_d(k) \]

\[ Q_t \to 0 \text{ as } t \to \infty \]

\( x_0 \) is not determined by far away vars.

\[ Q_t \to Q_* \text{ as } t \to \infty \]

\( x_0 \) is determined by far away vars.