XORSAT (continuing from Tue)

Before going on, let me explain what I am trying to prove:

\[ \forall \alpha \in \{\alpha_d, \alpha_c\} \quad \text{the following happens w.h.p.} \]

2-core percolation

\[ \exists \text{ a partition of the hypercubic } [0,1]^n \]

\[ \text{into sets } \{\Omega_1, \Omega_e, \Omega_{\bar{e}}\} \quad \text{such that} \]

\[ \rightarrow \quad N = e^{n(\Sigma \varepsilon \tau e)} \quad \text{for some } \Sigma > 0 \]

\[ \rightarrow \quad \mu(x) = \sum_{e=1}^{N} w_e \mu_e(x) \quad \text{with} \]

\[ \mu_e(.) \quad \text{supported on } \Omega_e \]

\[ \rightarrow \quad w_e = e^{-n(\Sigma \varepsilon \tau e)} \]

\[ \rightarrow \quad \frac{\mu(\Omega_{\bar{e}})}{\mu(\Omega_e)[1-\mu(\Omega_{\bar{e}})]} \leq e^{-n C(\varepsilon)} \quad \text{for } \varepsilon \text{ small enough} \]

\[ C > 0 \]
\[ \Omega e \subseteq \{ x : 0 < d(x, \Omega_e) \leq \epsilon \} \]

\[ \Rightarrow \text{Each of the } \mu_e(x) \text{ cannot be further decomposed} \]

["Pure States Decomposition"]

- Why is this important?
  - \( x > x_d \) : no correlation decay  
    \[ \Rightarrow \text{difficulties for } BP \]
  - Origin of correlations:
    far apart variables "know" that the whole system is in state \( e \)
  - If we restrict to a single state
Back to work

We need a second ingredient to be used together with the core size computation.

Consider a factor graph from $G_{\text{in}}(\Lambda, P)$:

$$k \leftarrow \Lambda \quad k \leftarrow P$$

$$e \leftarrow Y \quad e \leftarrow \Lambda.$$

Let $\hat{\mathbf{A}}$ denote the adjacency matrix of $G$:

$$\hat{\mathbf{A}} = \left[ \begin{array}{c|c} \sum & \mathbf{1} \end{array} \right]$$

$$\hat{\mathbf{m}} \leftrightarrow \text{function nodes}$$

$$\hat{\mathbf{n}} \leftrightarrow \text{variable nodes}$$
\[ Z_n(w) = \# \{ x : w(x) = w, \hat{w}(x) = 0 \} \]

"weight of \( x \)"

Then
\[ q_k(z) = \frac{1}{z} \left[ (1+z)^k + (1-z)^k \right] \]

\[ \mathcal{E} Z_n(w) = \sum_{E=0}^{E!} \frac{e!}{(E-e)!} \prod_{l=1}^{\ell_{\text{max}}} \cdot \text{coeff} \left[ \prod_{i=1}^{\hat{w}_{\ell_{\text{max}}}} (1 + x y^i) \cdot x^w y^e \right] \]

\[ \cdot \text{coeff} \left[ \prod_{k=2}^{\hat{w}_k} \hat{w}_k \right] \]

where
\[ E = \hat{w} \sum e P e = \hat{w} \sum P_k k = \# \text{ edges} \]

Proof: The probability of any given factor graph is \( \frac{1}{E!} \). Therefore

\[ \mathcal{E} Z_n(w) = \frac{1}{E!} \cdot \# \{ \text{graph/solution pairs} \} \]

A graph/solution pair can be identified with a "colored graph".
In a colored graph each edge can be either RED or BLUE.

A colored graph is valid if

→ all the edges incoming in the same v-node have the same color.
→ Each f-node has an even number of red edges
→ There are \( w \) red v-nodes

\[ e = \# \text{ red edges} \]

\[ \text{coeff} \left[ \prod_{e=1}^{e_{\text{max}}} (1 + y^e x)^{\tilde{w}_e} x^w y^e \right] = \# \text{ ways of coloring half edges on v-node side} \]

\[ \text{coeff} \left[ \prod_{k=2}^{k_{\text{max}}} q_k(z) z^e \right] = \# \text{ ways of coloring half edges on c-node side} \]
After a (boring) calculation one can find the asymptotic behavior

\[ \mathbb{E} Z_n(\bar{\omega}) = \exp \left\{ \tilde{\phi}(\omega) \right\} \]

weight \( O(n) \)

If \( \Lambda_0 = \Lambda_1 = 0 \) (min variable node degree \( \geq 2 \))

\[ \phi(\omega) = A\omega + O(\omega^2) \]

where \( A = A(\Lambda, \Phi) \)

\[ \begin{array}{c}
A > 0 \\
\text{typical situation}
\end{array} \]

\[ \begin{array}{c}
A < 0 \\
\Rightarrow \text{Any two solutions offset have Hamming distance} \geq \tilde{\phi}(\omega, -\epsilon)
\end{array} \]
Let us apply this to XOR-SAT:

we let \( \tilde{H} \) be the restriction of \( H \) to the core \( K_2(G) \); \( \tilde{n} = |K_2(G)| = \max_{n \in \mathbb{R}} 2^n \cdot \sum_{c(\omega) = c} Q_\omega > 0 \) for \( \alpha > \alpha_d \)

\[ \alpha = \alpha_d \]

\[ \alpha_c > \alpha > \alpha_d \]

\[ \alpha = \alpha_c \]
(* NB: conditional on its degree profile, \( k_2(G) \) is distributed essentially as \( C_1h(\Lambda, P) \). Further \( \Lambda, P \) are exponentially concentrated around typical degree profiles \( \Lambda^*, P^* \) where

\[
P_k^* = 1
\]

\[
\Lambda_e^* = (e^{\lambda} - 1 - \lambda)^{-1} \frac{\lambda^e}{e!} \text{ for } e \geq 2
\]

\[
\Lambda_e^* = 0 \text{ for } e < 1
\]

\[
\lambda = k d Q_{\ast}^{k-1}
\]

Thus we could apply the previous calculation.

\[\Rightarrow \text{ for } \alpha_d < \alpha < \alpha_e \text{ any two sols of the subsystem have dist } \geq \tilde{h}_k(\alpha_0 - \epsilon).\]
What does this mean for the solutions of the original system?

$$\Lambda \approx K_2(q) \text{ and } \nu \approx H_2(q)$$

... clusters, pure states...

$$Q \Rightarrow \text{this gives the desired decomposition.}$$

What does this mean for BP?

\[ \psi_{i \rightarrow o}(x_i) \propto \prod_{b \in i} \Psi_{b \rightarrow i}(x_i) \]

I'll neglect noting time:

\[ \tilde{\psi}_{a \rightarrow i}(x_i) \propto \sum_{\tilde{x} \in \{0,1\}^j} \Pi_{j \in \{0,1\}^j} x_i \Theta \Theta x_{j(k-1)} = 0 \]

\[ Y_{j(k-1)} \rightarrow (x_j) \sim \nu_{j(k-1)} \]
notice that \( S \) be the following subset of measures over \( \{0,1\} \):

\[
S = \{ (1,0); (\frac{1}{2}, \frac{1}{2}); (0,1) \}
\]

If \( k \mapsto \tilde{v}_k \mapsto i \in S \) at some time then the same is true at subsequent times. Let us assume that this is the case and restrict to such messages \( \Rightarrow \) look for a fixed \( \cdots \)

We shall identify \( S \) with

\[
S' = \{ 0, *, 1 \}
\]

Notice:

\[
\tilde{v}_{k \rightarrow i} = \begin{cases} * & \text{if } \tilde{v}_{o \rightarrow i} = * \ 0 \rightarrow i & \text{otherwise} \end{cases}
\]

\[
\tilde{v}_{a \rightarrow i} = \begin{cases} * & \text{if } \tilde{v}_{j \rightarrow a} = * \text{ for some } j \geq 0 \ 0 \rightarrow i \oplus \cdots \oplus \tilde{v}_{j(k-1) \rightarrow a} & \text{otherwise} \end{cases}
\]
(Notice that we can construct a fixed point for each cluster state \( \Omega_e \). Let \( x_i \in \Omega_e \)

\[
\forall i = \begin{cases} \times_i & \text{if } i \in B(G) \\ \ast & \text{otherwise} \end{cases}
\]

where \( B(G) \) is the "backbone of \( G \), ie the set of variables that take the same value in all the sols in \( \Omega_e \).