Matching: The Theory

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October 4, 2011

Many-to-one matching: The college admissions model

Players: Firms $\{f_1, .., f_n\}$ with number of positions: $q_1, ..., q_n$
Workers: $\{w_1, ..., w_p\}$
Synonyms (sorry:): $F =$ Firms, $C =$ Colleges, $H =$ Hospitals
$W =$ Workers, $S =$ Students.

Preferences over individuals (complete and transitive), as in the marriage model:

$$P(fi) = w3, w2, ...fi... \qquad [w3 \succ_{f_i} w2]$$
$$P(wj) = f2, f4, ...wj...$$

An outcome of the game is a matching: $\mu : F \cup W \longrightarrow F \cup W$ s.t.

- $w \in \mu(f)$ iff $\mu(w) = f$ for all f, w
- $|\mu(f)| \leqslant q_f$
- $\mu(w) \in F \cup \{w\}$.

so f is matched to the set of workers $\mu(f)$.

Firm preferences over groups of workers. The simplest model is

Responsive preferences: $\forall\ S \subseteq W$ with $|S| < q_i$, and w and w' in $W \backslash S$:

- $S \cup w \succ_{f_i} S \cup w'$ if and only if $w \succ_{f_i} w'$, and
- $S \cup w \succ_{f_i} S$ if and only if w is acceptable to f_i.

A matching μ with $\mu(w) = f$ is blocked by an individual if either the worker is unacceptable to the firm or the firm is unacceptable to the worker.
A matching μ is blocked by a pair of agents (f, w) if they each prefer each other to μ:

- $w \succ_f w'$ for some w' in $\mu(f)$ or $w \succ_f f$ if $|\mu(f)| < q_f$
- $f \succ_w \mu(w)$

As in the marriage model, a matching is (pairwise) stable if it isn't blocked by any individual or pair of agents.

Responsive preferences allow us to concentrate on pairwise stability only.

A matching μ is blocked by a coalition $A \subseteq W \cup F$ if there exists another matching μ' s.t. $\forall w, f$ in A

- $\mu'(w) \in A$
- $\mu'(w) \succ_w \mu(w)$
- $\widetilde{w} \in \mu'(f)$ implies $\widetilde{w} \in A \cup \mu(f)$ (i.e. every firm in A is matched at μ' to new students only from A, although it may continue to be matched with some of its "old" students from μ. (this differs from the standard definition of the core...)
- $\mu'(f) \succ_f \mu(f)$

A matching is group stable if it is not blocked by a coalition of any size.

Lemma 5.5: When preferences are responsive, a matching is group stable if and only if it is (pairwise) stable.

Proof: instability clearly implies group instability.
Now suppose μ is blocked via coalition A and outcome μ'. Then there must be a worker w and a firm f such that w is in $\mu'(f)$ but not in $\mu(f)$ such that w and f block μ. (Otherwise it couldn't be that $\mu'(f) \succ_f \mu(f)$, since f has responsive preferences.)

A related marriage market

- Replace college C by q_C positions of C denoted by $c_1, c_2, ..., c_{q_C}$. Each c_i has C's preferences over individuals. Since c_i has a quota of 1, no need for preferences over groups of students.
- Each student's preference list is modified by replacing C, wherever it appears on his list, by the string $c_1, c_2, ..., c_{q_C}$, in that order.
- A matching μ of the college admissions problem, corresponds to a matching μ' in the related marriage market
  - students in $\mu(C)$ are matched in the order given by $P(C)$, with the ordered positions of C. (If preferences are not strict, there will be more than one such matching.)

Lemma 5.6: A matching of the college admissions problem is stable if and only if the corresponding matchings of the related marriage market are stable.

(NB: some results from the marriage model translate, but not those involving both stable and unstable matchings. . .)

Geographic distribution:
Theorem 5.12
When all preferences over individuals are strict, the set of students employed and positions filled is the same at every stable matching.
Proof: immediate via the construction of the corresponding marriage problem (Lemma 5.6) and the result for the marriage problem.

Who gets matched to such undesirable hospitals?

Theorem 5.13 Rural hospitals theorem (Roth '86):
When preferences over individuals are strict, any hospital that does not fill its quota at some stable matching is assigned precisely the same set of students at every stable matching.
(This will be easy to prove after Lemma 5.25)

Comparison of stable matchings in the college admissions model:

Suppose C evaluates students by their scores on an exam, and evaluates entering classes according to their average score on the exam. (So even if college C has strict preferences over students, it may not have strict preferences over entering classes).

Different stable matchings may give college C different entering classes.

- However *no two distinct entering classes of C at stable matchings will have the same average exam score.*
- For any two distinct entering classes of C at stable matchings:
  - Apart from students who are in both entering classes
  - Every student in one of the entering classes will have a higher exam score than any student in the other entering class.

Lemma 5.25 (Roth and Sotomayor)
Suppose colleges and students have strict individual preferences, and let μ and μ' be stable matchings for (S, C, P), such that $\mu(C) \neq \mu'(C)$ for some C.

Let $\widetilde{\mu}$ and $\widetilde{\mu}'$ be the stable matchings corresponding to μ and μ' in the related marriage market.

If $\widetilde{\mu}(c_i) \succ_C \widetilde{\mu}'(c_i)$ for some position c_i of C then $\widetilde{\mu}(c_i) \succeq_C \widetilde{\mu}'(c_i)$ for all positions c_i of C.

Proof: It is enough to show that $\widetilde{\mu}(c_j) \succ_C \widetilde{\mu}'(c_j)$ for all $j > i$. Suppose not. $\exists$ j such that $\widetilde{\mu}(c_j) \succ_C \widetilde{\mu}'(c_j)$, but $\widetilde{\mu}'(c_{j+1}) \succeq_C \widetilde{\mu}(c_{j+1})$. Theorem 5.12 (constant employment) implies $\widetilde{\mu}'(c_j) \in S$. Let $s' = \widetilde{\mu}'(c_j)$. By the decomposition lemma $c_j = \widetilde{\mu}'(s') \succ_{s'} \widetilde{\mu}(s')$.

Furthermore, $\widetilde{\mu}(s') \neq c_{j+1}$, since $s' = \widetilde{\mu}'(c_j) \succ_C \widetilde{\mu}'(c_{j+1}) \succeq_C \widetilde{\mu}(c_{j+1})$ (since for any stable matching $\widetilde{\mu}'$ in the related marriage market, $\widetilde{\mu}'(c_j) \succ_C \widetilde{\mu}'(c_{j+1})$ for all j).

c_{j+1} comes right after c_j in the preferences of s' in the related marriage problem. So $\widetilde{\mu}$ is blocked via s' and c_{j+1}, contradicting (via Lemma 5.6) the stability of μ.

(This proof also establishes the rural hospitals theorem).

Theorem 5.26: (Roth and Sotomayor)
If colleges and students have strict preferences over individuals, then colleges have strict preferences over those groups of students that they may be assigned at stable matchings. Let μ and μ' be stable matchings, then college C is indifferent between $\mu(C)$ and $\mu'(C)$ only if $\mu(C) = \mu'(C)$.

Proof: via the lemma, and repeated application of responsive preferences.

Theorem 5.27: (Roth and Sotomayor)
Let preferences over individuals be strict, and let μ and μ' be stable matchings for (S, C, P).

- If $\mu(C) \succ_C \mu'(C)$ for some college C, then $s \succ_C s'$ for all s in $\mu(C)$ and s' in $\mu'(C) \backslash \mu(C)$.

That is, C prefers every student in its entering class at μ to every student who is in its entering class at μ' but not at μ.

Proof: Consider the related marriage market and the stable matchings $\widetilde{\mu}$ and $\widetilde{\mu}'$ corresponding to μ and μ'. Let $q_C = k$, so that the positions of C are $c_1, .., c_k$.

- C fills its quota under μ and μ', else Theorem 5.13 (Rural hospitals) implies that $\mu(C) = \mu'(C)$.
- So $\varnothing \neq \mu'(C) \backslash \mu(C) \subseteq S$. Let $s' = \widetilde{\mu}'(c_j)$ for some position c_j such that s' is not in $\mu(C)$. Then $\widetilde{\mu}(c_j) \neq \widetilde{\mu}'(c_j)$.

By Lemma 5.25 $\widetilde{\mu}(c_j) \succ_C \widetilde{\mu}'(c_j) = s'$.

The Decomposition Lemma implies $c_j \succ_{s'} \widetilde{\mu}(s')$.
So the construction of the related marriage problem implies $c \succ_{s'} \mu(s')$, since $C \succ_{s'} \mu(s')$, since $\mu(s') \neq C$.
Thus $s \succ_C s'$ for all $s \in \mu(C)$ by the stability of μ, which completes the proof.

So, for considering stable matchings, we have some slack in how carefully we have to model preferences over groups. (This is lucky for design, since it reduces the complication of soliciting preferences from firms with responsive preferences...)

The results also have an unusual mathematical aspect, since they allow us to say quite a bit about stable matchings even without knowing all the preferences of potential blocking pairs.

Consider a College C with quota 2 and preferences over individuals $P(C) = s1, s2, s3, s4$. Suppose that at various matchings $1 - 4$, C is matched to

1. $\{s1, s4\}$,
2. $\{s2, s3\}$,
3. $\{s1, s3\}$, and
4. $\{s2, s4\}$.

Which matchings can be simultaneously stable for some responsive preferences over individuals?

So long as all preferences over groups are responsive, matchings 1 and 2 cannot both be stable (Lemma 5.25), nor can matchings 3 and 4 (Theorem 5.27).

Strategic questions in the College Admissions model:

Theorem 5.16 (Roth)
A stable matching procedure which yields the student-optimal stable matching makes it a dominant strategy for all students to state their true preferences.
Proof: immediate from the related marriage market.

Strategic questions in the College Admissions model:

Theorem 5.14 (Roth)
No stable matching mechanism exists that makes it a dominant strategy for all hospitals to state their true preferences.
Proof: consider a market consisting of 3 hospitals and 4 students. H1 has a quota of 2, and both other hospitals have a quota of 1. The preferences are:

$s1 : H3, H1, H2$ $\quad$ $s3 : H1, H3, H2$ $\quad$ $H1 : s1, s2, s3, s4$

$s2 : H2, H1, H3$ $\quad$ $s4 : H1, H2, H3$ $\quad$ $H2 : s1, s2, s3, s4$

$H3 : s3, s1, s2, s4$

The unique stable matching is $\{[H1, s3, s4], [H2, s2], [H3, s1]\}$
But if H1: instead submitted the preferences $s1, s4$ the unique stable matching is $\{[H1, s1, s4], [H2, s2], [H3, s3]\}$.
Intuition: a college is like a coalition of players in terms of strategies.
Note that in one-to-one matching, DA cannot be manipulated by an agent if and only if there is a unique stable partner. The statement is false in many-to-one matching (see above)

A More Complex Market: Matching with Couples

This model is the same as the college admissions model, except the set of workers is replaced by a set of applicants that includes both individuals and couples.
Denote the set of applicants by $A = A_1 \cup C$, where A_1 is the set of (single) applicants who seek no more than one position, and C is the set of couples $\{a_i, a_j\}$.

- Each couple $c = \{a_i, a_j\}$ in C has preferences over ordered pairs of positions, i.e. an ordered list of elements of $F \times F$.
- Applicants in the set A_1 have preferences over residency programs, and residency programs (firms) have preferences over the individuals in A, just as in the simple model discussed earlier. A matching is a set of pairs in $F \times A$.

Each single applicant, each couple, and each residency program submits to the centralized clearinghouse a Rank Order List (ROL) that is their stated preference ordering of acceptable alternatives.

A matching μ is blocked by a single applicant (in the set A_1), or by a residency program, if μ matches that agent to some individual or residency program not on its ROL. A matching is blocked by an individual couple $\{a_i, a_j\}$ if they are matched to a pair (r_i, r_j) not on their ROL.

A residency program r and a single applicant a in A_1 together block a matching μ precisely as in the college admissions market, if they are not matched to one another and would both prefer to be.

A couple $c = \{a_i, a_j\}$ in A and residency programs r_1 and r_2 in F block a matching μ if the couple prefers (r_1, r_2) to $\mu(c)$, and if either r_1 and r_2 each would prefer to be matched to the corresponding member of the couple, or if one of them would prefer, and the other already is matched to the corresponding couple member. That is, c and (r_1, r_2) block μ if

- $.(r_1, r_2) \succ_c \mu(c)$ and if

either

- $\{a_1 \notin \mu(r_1)$ and $a_1 \succ_{r_1} a_i$ for some $a_i \in \mu(r_1)$ or a_1 is acceptable to r_1 and $|\mu(r_1)| < q_1$ $\}$ *and* either $a_2 \in \mu(r_2)$ or $\{a_2 \notin \mu(r_2)$, $a_2 \succ_{r_2} a_j$ for some $a_j \in \mu(r_1)$ or a_2 is acceptable to r_2 and $|\mu(r_2)| < q_2$ $\}$

or

- $a_1 \in \mu(r_1)$ and $\{a_2 \notin \mu(r_2)$, $a_2 \succ_{r_2} a_j$ for some $a_j \in \mu(r_2)$ or a_2 is acceptable to r_2 and $|\mu(r_2)| < q_2\}$.

A matching is stable if it is not blocked by any individual agent or by a pair of agents consisting of an individual and a residency program, or by a couple together with one or two residency programs

Theorem 5.11 (Roth '84): In the college admissions model with couples, the set of stable matchings may be empty.

Proof: by example.

Furthermore, the following example shows that even when the set of stable matchings is non-empty, it may no longer have the nice properties we've come to expect.

There are $C = \{c1; c2\}$ and one single student s and one couple $(m; w)$

$\succ_s$: $c1; c2;$

$\succ_{(m;w)}$: $(c1; c2);$

$\succ_{c1}$:$m; s$

$\succ_{c2}$: $s; w$:

There is no stable matching (exercise).

Matching with couples (Example of Aldershof and Carducci, '96)

4 hospitals $\{h1, \ldots h4\}$ each with one position;

2 couples $\{s1, s2\}$ and $\{s3, s4\}$

Preferences:

h_1	h_2	h_3	h_4	$\{s_1, s_2\}$	$\{s_3, s_4\}$
s_4	s_2	s_2	s_2	h_3, h_2	h_2, h_1
s_3	s_3	s_4	s_3	h_2, h_3	h_2, h_3
	s_1	s_1		h_2, h_4	h_1, h_3
				h_3, h_4	h_4, h_1
				u, h_3	h_4, h_3
				u, h_2	
				u, h_4	
				h_3, u	
				h_2, u	

There are exactly two stable matchings:

$(h_1, s_4), (h_2, s_2), (h_3, s_1), (h_4, s_3)$ preferred by $h_1, h_2, h_4, \{s_1, s_2)$

$(h_1, s_4), (h_2, s_3), (h_3, s_{12}), (h_4, u)$ preferred by $h_3, \{s_2, s_3)$

So, even when stable matchings exist, there need not be an optimal stable matching for either side, and employment levels may vary.

When thinking about redesigning the NRMP to accomodate couples: All that seems available are counterexamples:

1. Stables matches with couples may not exist
2. There is no stable mechanism that is strategy-proof for hospitals.

On the other hand, empirically, markets that use stable mechanisms seem to function quite well - they continue to be used.

Let's look at real markets and see how theory can (or cannot) guide the design of a market institution.