Technical Addendum to
Market Culture: How Rules Governing Exploding Offers Affect Market Performance[1]

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Proposition 1: In the exploding offer treatment, when firms and applicants are all risk neutral, there is a perfect Bayesian equilibrium with efficient late matching. At this equilibrium, in period 7 firm *j* makes an exploding offer to the applicant of quality *j*+1, who accepts that offer.

Proposition 2: In the Open offer condition, there is a perfect Bayesian equilibrium with efficient late matching in which firm *j* matches to the applicant of quality *j*+1.

Proposition 3: In the Renege condition, there is a perfect Bayesian equilibrium with efficient late matching. At this equilibrium, in period 9, firm *j* makes an exploding offer to the applicant of quality *j*+1, who accepts that offer.

Proofs of Propositions:

To facilitate the presentation of the equilibrium strategies, we first introduce some notation.

Let σ be a permutation on (1,2,3,4,5,6), with $\sigma = (\sigma(1), \sigma(2), .., \sigma(6))$, where $\sigma(i)$ is the quality of applicant *i*, and let *S* be the set of all such permutations σ.

Let $p : S \rightarrow [0,1]$ s.t. $\sum_{\sigma} p(\sigma) = 1$ be a probability distribution over all possible outcomes, and let *P* be the set of all such probability distributions.

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At any period $1 \le t \le 6$ let s_i be the sum of all signals that applicant *i* has received so far, s_{-i} the unknown signals the other applicants received, and let *r* be the remaining signal vector all applicants receive by period 7, so $s+r$ is the sum of the signals, which determines relative quality.

If at period *t* applicant *i* already has signals s_i, the probability distributions on quality that are still feasible are $p^{s_i} \in P$ such that $p^{s_i}(\sigma) > 0$ if and only if there exist s_{-i}, *r* such that $\forall 1 \le k, j \le 6 : \sigma(k) > \sigma(j) \Rightarrow s_k + r_k \ge s_j + r_j$. Let P^{s_i} be the set of all such feasible probability distributions over outcomes p^{s_i}. Let $q \in P^{s_i}$ be the applicants' subjective belief about the distribution over outcomes.

For each player *i* let $p_X(i) = \sum_{\sigma:\sigma(i)=x} p(\sigma)$ be the probability that applicant *i* has quality *x* and let and $q_x(i) = \sum_{\sigma:\sigma(i)=x} q(\sigma)$ be applicant *i*'s beliefs about these probabilities.

Let F^M be the set of firms that are already matched and let $m(j)$ be the quality of the applicant matched to firm *j*.

Proof of Proposition 1 (late equilibrium in the exploding offer condition):

To specify the equilibrium we need to specify not only the strategies of the players, but also their beliefs at all information sets.

Firms and applicants believe that all unobserved offers are exploding offers, that is, they believe that no applicant holds an offer, and that all unmatched firms are able to make an offer.

At all information sets before period 7 in which no firm is matched, and at all nodes in period 7-9, the strategies of firms and applicants call for them to behave as follows:

In period $t<7$, no firm *j* makes an offer.

In periods 7, 8 and 9 the k-highest unmatched firm makes an exploding offer to the k-highest unmatched applicant.

The applicant holds the highest of all open offers he receives, and, in case he did not accept an exploding offer before, accepts the highest available offer in period 9.

In period 7 and 8, the (unmatched) applicant of the k-highest quality among all unmatched applicants, accepts any exploding offer from a firm whose quality is equal to or higher than the k-highest unmatched firm, and in case of receiving multiple offers accepts the highest one as long as he does not hold an open offer from a higher quality firm.

In periods t<7, suppose applicant i has received signals s_i so far and receives an exploding offer from firm j. The applicant believes that he is the only applicant to receive an offer, and that rejecting the offer has no impact on other applicants' behavior, as rejected offers are private information. The applicant forms as "favorable" beliefs as possible over the probability distribution of future qualities, and hence his expected payoff from rejecting the offer, that are consistent with receiving an offer from firm j. If there is no chance that firm j can do better by matching to applicant i than to wait for period 7 and match to the applicant of quality j+1, that is if $\left\{p : p \in P^{s_i} and \sum_{\sigma} p(\sigma)\sigma(i)j > j(j+1)\right\} = \phi$ then the applicant forms beliefs $q^{s_i} = \arg\max_{p}\left\{\sum_{\sigma} p(\sigma)\sigma(i)(\sigma(i)-1-j) \mid p \in P^{s_i}\right\}$, which maximize the difference between the expected profit of waiting and receiving in period 7 an assortative match, that is receiving $\sum_{\sigma} p(\sigma)\sigma(i)(\sigma(i)-1)$ and accepting firm j's offer and receiving $\sum_{\sigma} p(\sigma)\sigma(i)j$. If $\left\{p : p \in P^{s_i} and \sum_{\sigma} p(\sigma)\sigma(i)j > j(j+1)\right\} \neq \phi$ the applicant forms beliefs $q^{s_i} = \arg\max_{p}\left\{\sum_{\sigma} p(\sigma)\sigma(i)(\sigma(i)-1-j) \mid p \in P^{s_i} and \sum_{\sigma} p(\sigma)\sigma(i)j > j(j+1)\right\}$. The applicant accepts the offer when $\sum_{\sigma} q^{s_i}(\sigma)\sigma(i)j > \sum_{\sigma} q^{s_i}(\sigma)\sigma(i)(\sigma(i)-1)$, otherwise he rejects the offer.

Suppose there is a match in period t<7. We assume that all unmatched firms and applicants believe that there are no outstanding open offers they are not aware of. The game consisting of the remaining firms and applicants starting from their current information[2] has at least one perfect Bayesian equilibrium, and we assume that players use the strategies of one of these for every such continuation game. We do not need to specify these strategies exactly (beyond that they are equilibrium strategies), because these strategies do not affect the payoffs of the firms and applicants who have deviated by matching, and hence do not affect whether they find it profitable or not to deviate.

We now verify that these strategies form a perfect Bayesian equilibrium.

Along the equilibrium path, all firms *j* make an offer in period 7 to the respective applicant of quality *j*+1. A firm *j* cannot profitably deviate by making an offer to *j*+2, or *j*, and applicant *i* of quality *j*+1 cannot profitably deviate by rejecting firm *j*'s offer.

Along the equilibrium path: Assume all firms are still unmatched, and workers and firms believe that no applicant holds an open offer from a firm. Can firm *j* profit from deviating, by making an early exploding offer in period *t*<7 to applicant *i* who has probabilities $p_x(i)$ to be of quality *x*? This is profitable for firm *j* only if $\sum_{x=1,..,6} j \cdot p_x(i) > j(j+1)$.

To verify that applicant *i*'s best response is to reject such an offer (and hence assure that in equilibrium no firm finds it profitable to deviate and make an early offer), we need to find beliefs $q_x(i)$ for applicant *i* such that he prefers to reject the offer and wait, i.e. $\sum_x j \cdot q_x(i) < \sum_x x(x-1) \cdot q_x(i)$, subject to $\sum_{x=1,..,6} j \cdot q_x(i) > j(j+1)$ (that is the applicant believes that firm *j* has not made a mistake by making an early exploding offer) and $q \in P^{s_i}$, that is beliefs *q* are feasible given applicant *i*'s signal s_i.

We will show that such beliefs exist by showing at the same time that this perfect Bayesian equilibrium does not hinge on the applicants not knowing the true probabilities of being of various types, that is not observing the vector of signals *s*. Specifically, we show that, along the equilibrium path, when all firms $k \neq j$ play according to their strategies, there is no nontrivial probability distribution over applicants' qualities (i.e.

[2] To be precise, the information sets starting at period t+1 are not singletons, so the continuation of the game is not a subgame. But consider an auxiliary game G(t), derived from the path of play in the original game in which there is a match at period t<7. The players in the auxiliary game G(t) are all those players who are not yet matched by period t. The game G(t) begins at period t, and for that period only, firms may only make open offers. From period t+1 onward, the rules of the auxiliary game are those of the original game (i.e. both open and exploding offers may be made). Then the continuation strategies of the remaining (not yet matched) players in the original game are precisely equal to the strategies of players in the auxiliary game holding the same open offers.

such that $0<p_x(i)<1$ for some $1\le x\le 6$), such that it is profitable for both the firm and applicant to deviate, that is the firm to make an early offer and the applicant to accept it.

► For a risk neutral firm j it is profitable to make an early offer (that is accepted) to applicant i of probabilities $p_x = p_x(i)$ to be of quality x whenever:

$$p_1 j + p_2 2j + p_3 3j + p_4 4j + p_5 5j + p_6 6j \ge j(j+1) \qquad \text{(F)}$$

The (risk neutral) applicant i with probabilities $p_x = p_x(i)$ to be of quality x, prefers to accept firm j's offer whenever

$$\begin{aligned} &p_1 0 + p_2 2\cdot 1 + p_3 3\cdot 2 + p_4 4\cdot 3 + p_5 5\cdot 4 + p_6 6\cdot 5 \le \\ &p_1 j + p_2 2j + p_3 3j + p_4 4j + p_5 5j + p_6 6j. \end{aligned} \qquad \text{(A)}$$

We now show, for each firm j that (F) and (A) can only be fulfilled if $p_{j+1}(i)=1$, as long as no applicant is yet matched.

It is clear that when $\boldsymbol{j} = \mathbf{5}$, (F) cannot be fulfilled unless p_6=1.

For $\boldsymbol{j} = \mathbf{4}$:

Inequality (A) is $\begin{aligned} &p_2 2\cdot 1 + p_3 3\cdot 2 + p_4 4\cdot 3 + p_5 5\cdot 4 + p_6 6\cdot 5 \le \\ &p_1 4 + p_2 2\cdot 4 + p_3 3\cdot 4 + p_4 4\cdot 4 + p_5 5\cdot 4 + p_6 6\cdot 4. \end{aligned}$

$$\Leftrightarrow \quad 4p_1 \ge -6p_2 - 6p_3 - 4p_4 + 4p_6.$$

Inequality (F) is $4p_1 + 8p_2 + 12p_3 + 16p_4 + 20p_5 + 24p_6 \ge 20$

We use that $\sum_i p_i = 1$ and obtain $\quad 4p_1 \le -3p_2 - 2p_3 - p_4 + p_6$.

Combining (A) and (F) implies: $-3p_2 - 4p_3 - 3p_4 + 5p_6 \le 0$

Which implies that $-15p_2 - 10p_3 - 5p_4 + 5p_6 < 0$ if $p_i > 0$ for at least one i of $\{2,3,4\}$.

The last strict inequality, with (F) delivers that $20p_1 < 0$: contradiction.

If $p_i = 0$ for i =2,3,4, then the combination of (A) and (F) imply that $p_6 = 0$, which using (F) implies that $p_1 = 0$ that means $p_5 = 1$.

For $\boldsymbol{j} = \mathbf{3}$:

Inequality (A) is $\begin{aligned} &p_2 2\cdot 1 + p_3 3\cdot 2 + p_4 4\cdot 3 + p_5 5\cdot 4 + p_6 6\cdot 5 \le \\ &3p_1 + 6p_2 + 9p_3 + 12p_4 + 15p_5 + 18p_6. \end{aligned}$

$$\Leftrightarrow \quad 3p_1 \ge -4p_2 - 3p_3 + 5p_5 + 12p_6.$$

Inequality (F) is $3p_1 + 6p_2 + 9p_3 + 12p_4 + 15p_5 + 18p_6 \geq 12$

We use that $\sum_i p_i = 1$ and obtain $3p_1 \leq -2p_2 - p_3 + p_5 + 2p_6$.

Combining (A) and (F) implies: $-p_2 - p_3 + 2p_5 + 5p_6 \leq 0$

Which implies that $-2p_2 - p_3 + p_5 + 2p_6 < 0$ if $p_i > 0$ for at least one *i* of $\{2,5,6\}$. The last strict inequality, with (F) delivers that $3p_1 < 0$: contradiction.

If $p_i = 0$ for i =2,5,6, then (F) implies that $p_3 = 0 = p_1$, which means $p_4 = 1$.

For ***j* = 2**:

Inequality (A) is $\begin{array}{l} p_2 2\cdot 1 + p_3 3\cdot 2 + p_4 4\cdot 3 + p_5 5\cdot 4 + p_6 6\cdot 5 \leq \\ 2p_1 + 4p_2 + 6p_3 + 8p_4 + 10p_5 + 12p_6. \end{array}$

$\Leftrightarrow \quad 2p_1 \geq -2p_2 + 4p_4 + 10p_5 + 18p_6.$

Inequality (F) is $2p_1 + 4p_2 + 6p_3 + 8p_4 + 10p_5 + 12p_6 \geq 6$

We use that $\sum_i p_i = 1$ and obtain $2p_1 \leq -p_2 + p_4 + 2p_5 + 3p_6$.

Combining (A) and (F) implies: $-p_2 + 3p_4 + 8p_5 + 15p_6 \leq 0$

Which implies that $-p_2 + p_4 + 2p_5 + 3p_6 < 0$ if $p_i > 0$ for at least one *i* of $\{4,5,6\}$. The last strict inequality, with (F) delivers that $2p_1 < 0$: contradiction.

If $p_i = 0$ for i =4,5,6, then (F) implies that $p_2 = 0 = p_1$, which means $p_3 = 1$.

For ***j* = 1**:

Inequality (A) is $\begin{array}{l} p_2 2\cdot 1 + p_3 3\cdot 2 + p_4 4\cdot 3 + p_5 5\cdot 4 + p_6 6\cdot 5 \leq \\ p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5 + 6p_6. \end{array}$

$\Leftrightarrow \quad p_1 \geq 3p_3 + 8p_4 + 15p_5 + 24p_6.$

Inequality (F) is $p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5 + 6p_6 \geq 2$

We use that $\sum_i p_i = 1$ and obtain $p_1 \leq p_3 + 2p_4 + 3p_5 + 4p_6$.

Combining (A) and (F) implies: $2p_3 + 6p_4 + 12p_5 + 20p_6 \leq 0$

Which implies that $p_i = 0$ for i =3,4,5,6, then (F) implies that $p_1 = 0$, which means $p_2 = 1$. ◄

Proof of Proposition 2 (late equilibrium in the Open offer only condition):

The strategies and beliefs are as follows:

In period 7, 8 and 9, firm *j,* the k-highest unmatched firm makes an offer to the k-highest unmatched applicant. Firms believe that no other firm makes or made an open offer to that applicant.

- In periods *t*<9 applicants hold the best offer they receive and in period 9 accept the best available offer. (Applicants can accept the offer of the highest remaining firm as soon as they receive it, specifically, they can accept the offer of firm 5, the highest quality firm, in period *t*<9.)

Since offers are private, holding the best offer is always (weakly) better than accepting or rejecting it.

No firm has an incentive to make an early offer, as an applicant will simply hold any early offer, and accept it only if she didn't receive a better offer. This means, the firm will be matched with the applicant only if the applicant is of the quality the firm would make an offer to in period 7 anyway, or, of worse quality, in which case the firm is strictly worse off.

Firm *j,* the j-highest remaining firm in period 7, cannot gain by making an offer in period 7 to an applicant who is of a quality higher than the *j*+1 remaining applicant, as the applicant will receive a better offer in period 7. ◊

Proof of Proposition 3 (late equilibrium in the Renege condition):

We introduce a bit more notation for the equilibrium strategies of proposition 3, to define to which applicant each unmatched firm should make an offer at every information set in period 9. Each unmatched firm should make an offer to the correspondingly ranked unmatched applicant, or the highest ranked matched worker who would renege on his previous acceptance and accept the firm's offer (given the other offers made simultaneously).

Let $of : F \setminus F^M \rightarrow C = \{1,2,...,6\}$ be the offer function defined by the following algorithm:

If firm $5 \in F \setminus F^M$ let $of(5) = 6 = \max\{c : c \in C = \{1,2,3,4,5,6\}\}$ and let $C^4 = C \setminus \{6\}$, if $5 \in F^M$ let $C^4 = C \setminus \{m(5)\}$.

In general, given C^k, for the firm of quality k the algorithm is: If firm $k \in F \setminus F^M$ let $of(k) = \max\{c : c \in C^k\}$ and let $C^{k-1} = C^k \setminus \{of(k)\}$, if $k \in F^M$ let $C^{k-1} = C^k \setminus \{m(k)\}$.

The algorithm stops at firm 1, and each firm in $f \in F \setminus F^M$ is assigned an applicant who is either unmatched or matched to a lower quality firm.

Let g^{F^M} be the generalized inverse offer function extended to matched as well as unmatched firms, that is $g^{F^M} : C = \{1,2,...,6\} \to F \cup \{0\}$ such that, given F^M, $g^{F^M}(c) = of^{-1}(c)$ for $c \in of(F \setminus F^M)$, $g^{F^M}(c) = f$ if $m(f) = c$, $c \notin of(F \setminus F^M)$ and $f \in F^M$, and $g^{F^M}(c) = 0$ for $c \notin \left(of(F \setminus F^M) \cup m(F^M)\right)$.

The beliefs and strategies that constitute this equilibrium are as follows:

Firms and applicants believe that all unobserved offers are exploding offers, that is, they believe that no applicant holds an offer, and that all unmatched firm are able to make an offer.

In period 9, the unmatched firm *j* among all the unmatched firms $F \setminus F^M$ makes an exploding offer to the applicant of quality $of(j)$ as prescribed by the offer function *of*.

In periods *t*<9 firms do not make offers.

The applicant holds the highest of all open offers he receives, and, in case he did not accept an exploding offer before, accepts the highest available offer in period 9.

In period 7 and 8, the applicant of quality c accepts any exploding offer from a firm whose quality is equal or higher than $g^M(c)$ (whether he is already matched or not), in case of receiving multiple offers he accepts the highest one.

In periods *t*<7, suppose the applicant has received signals s_i so far and receives an exploding offer from firm *j*. The applicant believes that he is the only applicant to receive an offer, and that rejecting the offer has no impact on other applicants' behavior, as rejected offers are private information. Given the set of matched firms F^M, the applicant forms beliefs $q^{s_i} \in P^{s_i}$, so as to maximize the difference in expected payoff from accepting the offer and rejecting the offer, that is q^{s_i} maximizes, subject to $p \in P^{s_i}$,

$$Q = \left[\sum_{\sigma} p(\sigma)\sigma(i) \max(j, g^{F^M \cup \{j\}}(\sigma(i)) - 1/\sigma(i)) \right] - \left[\sum_{\sigma} p(\sigma)\sigma(i) g^{F^M}(\sigma(i)) \right].$$ The applicant accepts the offer if $Q \geq 0$ and rejects it otherwise.

Given F^M, *s* and hence *p* we show that firm *j* has no incentive to make an exploding (or open) offer to applicant *i*, which the applicant would accept. (The firm knows whether the applicant would accept, and there are no incentives to make offers that are rejected.) If firm *j* makes an early offer that is accepted, the expected payoff is $\sum_{\sigma} p(\sigma)\sigma(i) jI$ where $I = 1$ if $j = \max\{j, g^{F^M \cup \{j\}}(\sigma(i))\}$ and 0 otherwise and where $g^{F^M \cup \{j\}}$ depends on σ.

If firm *j* does not make an early exploding or open offer (and no other firm does), then the expected payoff is $\sum_{\sigma} p(\sigma) j \cdot of(j)$, where of is the offer function, which depends on σ.

There are 3 possible outcomes:

1. $\sigma(i) = of(j)$. Then firm *j* is simply indifferent between hiring early or late.
2. $\sigma(i) < of(j)$. Then $j = \max\{j, g^{F^M \cup \{j\}}(\sigma(i))\}$ and firm *j* remains matched to $\sigma(i)$ and is strictly worse off from making an early offer.
3. $\sigma(i) > of(j)$. Then there exists a firm *k* such that $\sigma(i) = of(k)$, firm *k* will make an offer to applicant *i* in period 9, the applicant will renege and firm *j* will be unmatched, so, firm *j* is strictly worse off from making an early offer.

Therefore, firm *j* has no incentive to make an early offer.

Furthermore, in periods 7 or 8, firm *j* has no incentive to make an offer to an applicant of quality lower than $of(j)$, but also not to an applicant of quality higher than $of(j)$, as then in period 9 the firm expects the applicant to renege on the acceptance and hence to be strictly worse off.

The strategies of the applicants are also a best response given their beliefs. ◊