

Hansung Science High School Talk

Physics: Light

Gary Oas

1.a Introduction, Ancient History of theories of light

To understand modern physics you need to understand light

- *Light is necessarily relativistic.* Since it travels at the speed of light, $c = 3 \times 10^8 \frac{m}{s}$.
- *Light is a quantum phenomena.* Since it classically is described by a wave but necessarily has particle-like properties.

Ancient Greek conceptions of light.

Three main theories of light and/or how human's vision works.

- Empedocles rays of light emanate from eyes and return info..
- Leucippus, emanating veils of atoms (material), *eidola*.
- Aristotle's 'light is the activity of what is transparent'. (Light is a form, not a substance).

1.b From 1500 through 1860s

In the middle ages, Islamic scholars advanced the study of optics. It had become clear that light was no longer a phenomena only tied to human vision but was a physical entity itself. In the later parts of the 1600s two competing of views of what light is.

- *Light is comprised of particles.* This was the notion put forth by Isaac Newton in his treatise 'Opticks'. He thought that light was made of a large number of small particles. On the whole it behaved like a wave.
- *Light is a wave phenomenon.* This view was first put forward by Christian Huygens at roughly the same time as Newtons. Some aspects of light were explained by this theory, though a rigorous mathematical theory was lacking.

In the period from 1700 to about 1860 a debate raged about which view was correct. Many experiments were performed to try to settle the dispute. However, most of these could be explained by either view (though sometimes with some rather strange consequences). Of the key results were the following.

1704 Newton's rings. Interference of light by reflection of light of two surfaces that are close together.

1704 Breaking light into its component colors with a prism.

1801 The [double slit experiment](#) first performed by M. Young. In this experiment light is allowed to pass through two very closely spaced slits. The resulting pattern of alternating light and dark bands results from the interference of light. This experiment is extremely difficult (if not impossible) to explain with a particle view of light. However, Young was not an influential scientist and thus his experiment did not have the immediate impact that it should. This experiment is now easily performed with a laser pointer and appropriate slits. *This is the key experiment to understand in quantum mechanics.*

1818 Poisson's spot. If light is a wave it will diffract around a circular obstacle and constructively interfere at the center of the shadow. Poisson, believing light was a particle, thought this was ludicrous but was soon observed by Arago.

1849 Measuring the speed of light. First terrestrial measurement by H. Fizeau. (Previously determined in 1676 by O. Romer by examining the orbit of Jupiter's moons).

1850 Measuring the speed of light in water as compared to air. Wave theory predicts slower travel through water while particle theory predicts faster travel. The result demonstrated that light traveled slower in water.

1.c Maxwell's theory of electromagnetism

In the mid 1800s Michael Faraday performed many ground breaking experiments in the area of electricity and magnetism. He believed there was an intimate connection between the two and his experiments bore that out. He also believed that light was related as well. His experiment showing that the polarization of light passing through a medium (quartz) was changed by the presence of a strong magnetic field (now called the [Faraday rotation](#)) convinced him he was on the right track. However, Faraday was not an experienced theorist and could develop a theory to further his conviction.

In the 1860s another British scientist James Clerk Maxwell took up Faraday's cause. Maxwell was an experienced theorist (probably one of the top physicists ever) and developed the mathematical theory into one of the pinnacles of modern physics. Maxwell's theory of electromagnetism was quickly hailed as a major advancement in human knowledge. In his theory, he formally unified electric and magnetic phenomena and in the process posited a new phenomena, electromagnetic waves. The mathematics stipulated that these waves travel at a speed of approximately $3 \times 10^8 \frac{m}{s}$ which he knew was close to the speed of light. He hypothesized that this new entity was actually light. His idea was experimentally verified about 20 years later by Heinrich Hertz.

Maxwell's theory unified not only electrical and magnetic phenomena but also optics. It was understood later that other phenomena, radio waves, X rays, gamma rays, microwaves, infrared and ultraviolet radiation all were electromagnetic waves. Maxwell's work was immediately hailed as a success and has withstood the test of time by not being modified in the 150 years of its existence. For within his theory the budding elements of special relativity were built in, i.e. Maxwell's theory is a special relativistic theory of electromagnetic radiation. Even though the later quantum theory would change the view of light is, Maxwell's equations are still valid within it.

Now it is time to actually discuss in as simple manner as possible what Maxwell theory says.

2.a Brief discussion of Maxwell's electromagnetic wave theory

The core relations in Maxwell's electromagnetic theory are his four *Maxwell's equations*. We will not spell out their mathematical form but simply describe in words what they mean. These equations were mostly known prior to Maxwell and their names are given, but Maxwell combined them into a single theory. We are concerned with the last two here.

Gauss law. This formula essentially tells you how to find the electric field in a region where there is a net electric charge. (Electric charges produce electric fields).

No magnetic monopoles. This is identically the same as the previous but with magnetic fields *and* the relation equals zero. (There are no point sources of magnetic fields, i.e. no magnetic charges).

Faraday's law Faraday's earlier work had found that a time changing magnetic field can create an electric field. (Electromagnetic induction, you do not necessarily need point electric charges to create an electric field).

Ampere's law There are two parts to this equation. The first part is an analog of Gauss law and states that a magnetic field is created by electrical currents (moving electric charges). What Maxwell added, based on symmetry, is another term that stated that a changing electric field can create a magnetic field (just like Faraday's law).

The important discovery that Maxwell made involved combining the last two equations when no sources (no electric charges or currents) are present. He saw that there was a self supporting phenomena possible that stems from the basic process:

Changing electric field \longrightarrow **Changing magnetic field** \longrightarrow **Changing electric field** \longrightarrow **Changing magnetic field** \longrightarrow **Changing electric field** \longrightarrow **Changing magnetic field** \longrightarrow **Changing electric field** \longrightarrow ...

This process could go on forever. Also, the mathematical relation that arises from combining Faraday's and Ampere's law takes the form of a wave phenomena. Maxwell recognized this and called this entity an electromagnetic wave. From the mathematical wave equation he could determine just how fast such a wave would travel. It turned out to be very close to the known speed of light.

Maxwell hypothesized that this was the theory of what light *is*; it is an oscillating electric and magnetic field -an electromagnetic wave. Though his discovery did not have immediate experimental verification, the beauty of the idea led to its rapid acceptance by the scientific community. It took another twenty years for experimental confirmation by Heinrich Hertz.

You can demonstrate electromagnetic waves your self by using a battery, a wire, and an AM radio. Connect one end of the wire to one end of the battery. While the radio is tuned to static at a low frequency, tap the other side of the wire to the other terminal. A rapid change in the electric field occurs when contact is made and it sends out EM waves. This can be picked up by the AM radio.

Now, not only is light an electromagnetic wave but so are: microwaves, radio waves, ultraviolet radiation, Xrays, and gamma rays. These are all the same entity at the core. In order to see what makes them different necessitates a short foray into waves.

2.b A little bit about transverse waves

Let's examine the features of a simple harmonic wave. Such a wave can be mathematically described by a simple sine or cosine function, $y(x, t) = A \sin 2\pi(\frac{x}{\lambda} + ft)$. The basic properties of such a wave are:

- Amplitude, A .
- Wavelength, λ .
- frequency, f .
- Period, $T = \frac{1}{f}$.
- Wave speed, $v = f\lambda$. For light we use the letter c , thus $c = f\lambda$.

The difference between types of electromagnetic waves, say X rays and radio waves is simply the wavelength, here X rays having a much smaller ($\sim 10^{-10}m$) wavelength as radio waves (from about a meter up). But at the core, they are essentially the same thing.

2.c Two or three important things that we will need to know about EM waves

(1) The power delivered by a wave in Maxwell's theory is proportional to the square of the amplitude. This fact is easily understood by thinking about water waves. Clearly a wave that is twice as high has much more potential to do damage. It may not seem obvious that it should be proportional to the square of the wave but to convince yourself, think of a mass on the end of a spring. The motion of an oscillating spring is nearly identical to a wave. A slinky that oscillates will have waves propagating down it. Since a slinky is essentially just a spring, we can use the result for the energy in the motion of a spring. Recall that the potential energy of a spring is $PE = \frac{1}{2}kx^2$, where x is the distance from equilibrium. At the maximum displacement, when it has reached the amplitude, A , the total energy is equal to $E = \frac{1}{2}ka^2$. From conservation of energy, this is the energy contained in the motion. The same goes for waves.

(2) The speed of an electromagnetic wave is given by the theory: all electromagnetic waves travel at c . We won't use this fact in our discussion today but just note that this property is the key ingredient leading to the theory of special relativity.

(3) The polarization of light is defined as the direction of oscillation of the electric field component of the EM wave.

This is another property of light that we want to explore a little bit. Polarization is the key property of light that is used in quantum optics experiments, quantum computers, etc. What we want to explore is *linearly polarized light*. This is light such that the electric field component oscillates along a single line. There are various ways to create polarized light (for example using a calcite crystal, or viewing reflected light) but we will just look at a modern, convenient, way - linear polarizers.

2.d Polarization

A linear polarizer only allows light to pass that has light oscillating in one direction -the transmission axis. The key properties of polarized light are really seen when you use two (or three) polarizers together.

Malus law

We will need one mathematical relation that tells us how much linearly polarized light passes through a polarizer. Take one polarizer that produces linearly polarized light. Place another polarizer over it such that its axis of transmission is at an angle θ to the polarization direction of the incoming light. We know that if $\theta = 0$ all the light that came out of the first passes through the second. We also know that if $\theta = 90^\circ$ that no light passes. We can derive a relation for how much light passes, as a function of θ , by resolving the electric field into two perpendicular directions. Then, the component that is parallel to the second polarizer will pass through. Simple trigonometry shows that the magnitude of the passed field is $E_0 \cos \theta$, where E_0 is the amplitude of the original polarized light. Since the intensity of light is simply the amount of power in the light, and since it is proportional to the square of the amplitude, we see that the intensity that emanates from the polarizer is

$$I = I_0 \cos^2 \theta$$

where I_0 is the intensity of the incoming polarized light and θ is the angle between the polarization direction of the incoming light and the transmission axis of the polarizer. This relation is known as Malus' Law and will be important as we proceed.

[That completes our discussion of Maxwell's theory of electromagnetism](#)

3.a Problem: Photoelectric effect

In the later part of the 1800's, Thomas Edison noted in his lab notebook an intriguing phenomena that was later explored more carefully by others. Basically a particular type of metal (Cesium or Zinc), when exposed to ultraviolet light will emit electrons. The basic demonstration is provided by charging up a zinc plate, placing it under a UV lamp (or sunlight), and then noticing that the charge is driven off the plate by the light. (Or, but harder to demonstrate, is placing a plate in UV light will make it become positively charged).

In 1902 this effect was carefully examined and it was noted that Maxwell's theory could not explain some of its key features. Among these are two important ones,

- If electrons are emitted, they will do so immediately after the light is turned on.
- The energy of the electrons emanating are directly proportional to the *wavelength* of the light. In addition, if a long wavelength light is used, no electrons are emitted *no matter how intense the source is*.

The problem with the first is that if a feeble light source is used, the light wave spreads out in all directions, i.e. the energy of the light is distributed over an ever growing large sphere. If the metal plate is rather small, the amount of light hitting will be a very small portion of the entire spherical wave. Thus little energy will be delivered to the plate. However, it can be seen at times that almost immediately after turning on the light an electron is emitted. Maxwell's theory can not deliver a sufficient amount of energy to the plate to kick to the electron in such a case.

The problem with the second is even more glaring. In Maxwell's theory the power delivered by the light wave is proportional to the amplitude (or intensity) of the wave. Thus in order to deliver sufficient energy to the electron, all one would do is to increase the intensity (amplitude). However, the experiment reveals that the intensity does not factor into the problem -it is the wavelength (or frequency) that matters. If you used an infrared terawatt laser on a zinc plate, no electrons will come off (ok this is a bit ridiculous the laser might melt the plate first). However a very feeble X ray source, with low amplitude, will kick out electrons. *Maxwell's theory just can't explain these results.*

3.b Solution: Einstein says light is made of particles

Well in 1905 Einstein proposed that to explain these results you need to think of light as comprised of particles. Each particle (in 1926 called photons) has an energy $E = hf$, where h is Planck's constant and f is the frequency of the light. Then all of the properties of the photoelectric effect can be explained. For this result Einstein won the Nobel Prize in 1922.

3.c How can this be?

But wait! All along I've been pointing to the overwhelming evidence that light is a wave. Now Einstein wants to go back to it being particles? How can this be? We then return to have to explain the interference results discussed earlier. Specifically, we need to explain how the double slit experiment would work, since we said before that it can not be explained by a particle view of light.

We need to be able to encompass both views to explain this experiment. For example consider the double slit experiment performed with ultraviolet light and where a zinc plate is placed behind the slits. One would find that there would be locations where electrons are emitted and others where they are not. The old interference pattern. In this experiment one necessarily needs a wave view to get the interference pattern and also a particle view to explain the appearance of electrons. How can this be?

An experiment in 1909 by G.I. Taylor made this bizarre nature apparent. Essentially he performed a double slit experiment but sent *only one photon through at a time*. What did he see? The familiar interference pattern!

3.d Wait, it gets even more bizarre

In this lecture we have been concentrating on light. However this property we just found, that it needs to have both wave-like and particle-like properties is not limited to photons. It turns out that all objects have this duality. An electron for example can be sent through a double slit one at a time and the overall pattern that emerges after many have been sent through is an interference pattern. This experiment has now been carried out many many times with objects up to a Bose-Einstein condensate of tens of thousands of atoms.

How can a single particle 'know' about the location of the two slits? A particle can only go through one or the other. What if we changed one of the slits right as the particle passes by, does it know about this change?

4.a Answer: Quantum Mechanics

We could sit and ponder this dilemma all day and probably not get too far. Let's jump to the answer – the theory of quantum mechanics.

Here's how we do it. We introduce an entity, *the wavefunction*, Ψ , that behaves like a wave - just like the old electromagnetic wave. However this entity is no longer a real radio wave or whatever but an entity that will tell us the probability of where to find the particle. Why is this so?

Consider the intensity pattern in the original double slit pattern. Recall the intensity pattern is given by the square of the amplitude of the wave. The amplitude in this case being the electric field component (it could also be the magnetic field component but they are related so we only need to talk about one).

$$I \sim |\vec{E}|^2$$

However what does it mean in the case of a single photon? A single photon lands at only a single point on the screen. Only when we send through many many photons does it build up to the full interference pattern. Thus for a single photon, it will probably not land in a dark region of the pattern. *Probably* is the key word. The pattern, for a single particle, represents the probability of where it will land. Thus for photons we no longer talk about intensity patterns but of probability densities -where the photon will tend to land. The actual location of where it lands is random but in accord with the probability distribution.

Now we said that the wavefunction behaves just like the electric field component \vec{E} . Thus since $I \rightarrow P$ and $\vec{E} \rightarrow \Psi$ we have the relation telling us how the wavefunction gives the probability.

$$P = |\Psi|^2$$

This is called the Born probability rule and is the central rule in quantum mechanics.

4.b Measurement in QM

We see that the wavefunction is an entity that only provides a probability as to where the photon (or whatever quantum object) will be found. The key property of quantum mechanics is that it is *impossible* to calculate or predict with certainty where it will be found. In the double slit experiment there is no way to know through which slit the photon went through while seeing an interference pattern. If you do determine, or have the ability to determine, which slit it passed you no longer observe an interference pattern. The reason is the following, if you measure which slit the photon passed, then it is a fact it is in one slit and not the other. In terms of the wavefunction, the act of measurement changed the wavefunction to be nonzero in one slit and zero in the other. The act of obtaining information, or measuring, changes the wavefunction dramatically. This is sometimes called “the collapse of the wavefunction.” It should be clear that since the wavefunction gives the probability that when the photon hits the screen the information on its whereabouts changes instantaneously. If the information changes instantaneously the probability changes instantaneously and thus the wavefunction changes instantaneously.

That the change in the wavefunction must occur instantaneously leads to some truly bizarre results if one thinks the wavefunction is a real entity. Consider the following thought experiment. A photon is emitted in the Andromeda galaxy in an unknown direction. The wavefunction in this case is a spherical shell expanding at the speed of light. Let's say this photon is measured in your eye after nearly 2.5 million years of travel for the photon. The wavefunction just prior to your measurement is a spherical shell 2.5 million light years in radius. The moment after you observe it, it is gone. The information on its location (in your eye) changes the probability, it has zero probability to be anywhere else if it is your eye. Thus this enormous entity disappears in a blink of an eye. This may seem to have issues with relativity (which says nothing can travel faster than light) but it actually turns out to be consistent.

This notion also leads to the metaphysical question as to what is a photon *really* before it is observed. If it can not be described to be in any location, is it a particle before it is measured? Most would say no but that begs the question ‘well what is it then?’ This question is not answered by quantum mechanics. It only says that you need to describe it with a wavefunction. Further experiments lead to a more dramatic conclusion that you can not think of a photon as a particle before it is measured.

5.a Entanglement

There are certain systems that lead to a somewhat magical property called entanglement. Consider a quantum object that decays into two separate objects that fly off in opposite directions. If the direction is unknown, the wavefunction will again be a spherical wave that spreads out. However, the law of conservation of momentum states that if the original object was at rest then the two objects will have equal and opposite momenta. If we know the momentum of one of the objects then we instantaneously know the momentum of the other. However, before the first measurement the two objects do not have a definite value of momentum. Thus measuring one of the objects determines the momentum of the

other far away object, even though no one measured the other object. This property goes under the term *entanglement*. The two objects are correlated, or entangled in momentum.

What we will be interested in later is a particular form of entanglement involving photons. A nonlinear crystal can convert a photon into two. The two outgoing photons are correlated since their energies must add up to the original, the momenta are related as discussed before, and conservation of angular momentum states that the polarization of the two photons are perpendicular. But, as before, the actual values are not determined before the first measurement. We will be concerned with the polarizations, if one is passed through a polarizer then it has a polarization defined by the polarizer's orientation and the other photon will have a perpendicular polarization.

5.b EPR

You might be objecting to all of this thinking the objects have definite position and momentum but we just don't know the values. In 1935, Einstein, Podolsky, and Rosen submitted a paper to prove that objects do have definite values of momentum before a measurement. Einstein, as you may have heard, was not a fan of what quantum mechanics was saying about real objects. He believed the results of quantum mechanics but not the notion that these objects 'are not real' before a measurement. This paper posited that two entangled objects are so far apart that one can not effect the other. Their conclusion was that quantum mechanics is incomplete and that there exists an, as yet unknown, theory that explains where everything is. At the time it was thought no one would ever be able to determine which was the correct view of the universe. Most thought it was a question best left to philosophers.

5.c Bell inequality

However in 1964 the Irish physicist, John Bell came up with a method to experimentally determine whether Einstein's view was correct or whether we can not give objects definite states prior to a measurement. We will give a simplified version of his theorem and show what was found.

Let's quantify the view of Einstein a little bit. The notion that our two photons are in definite states means that each photon 'knows' exactly what will happen when a particular measurement is made. The idea that these are two separate entities is embodied in the fact that one photon knows how it will behave *completely independent of the other photon*. For example, if a detector is set up to measure the polarization along some direction, the photon knows exactly which of the two perpendicular directions it will have. The photon has this information on board and is called a (local) *hidden variable*. What we want to derive is a relation that must be satisfied if each photon has its own hidden variable.

Consider the following setup. A series of individual photons are sent through a nonlinear crystal. About one in ten thousand will form an entangled pair (the process is called parametric down conversion) which fly off in opposite directions. All we care about measuring is the polarization of each photon. Recall that if the polarization is measured along the same direction for each photon they will always have perpendicular values. If the two measurements are not in the same direction then they do not have to be observed to have opposite values.

To construct a hidden variable for this system all we need to do is construct an *instruction set* that rides on board each photon that says what the outcome will be if measured along some direction. To simplify these sets we will only consider three possible orientations of polarizers called A , B , and C . These are essentially angles at which the polarizers are orientated. If the photon passes through we will say that the photon has a + value along that direction. If it does not it will have a - value. We will also specify that the angle between A and B will be the same as between B and C . We can now construct our sets with the constraint that if the two sides choose the same setting, the photons will have opposite values (one + and the other -) of polarization.

There are 8 possible types of sets that particle 1 can have and are listed below. For settings of detector 1 [A, B, C] we can have,

- 1) [+++]
- 2) [++-]
- 3) [+ - +]
- 4) [-++]
- 5) [+ -]
- 6) [- + -]
- 7) [- +]
- 8) [- -]

Particle 1 will be emitted with one of these sets at random. Particle 2 will be given the corresponding set (labeled [ABC]?) that enforces the constraint.

- 1) [+++] [- - -]'
- 2) [++-] [- - +]'
- 3) [+ - +] [- + -]'
- 4) [-++] [+ - -]'
- 5) [+ - -] [-++]'

- 6) [-+-] [++-]
- 7) [- -+] [++-]
- 8) [- - -] [+++]

There will be probabilities for each type of entangled pair coming out, P_1, P_2, \dots, P_8 . Being probabilities, these are all positive numbers between 0 and 1. For each pair, only one setting can be chosen at a time for each detector.

Now consider the following probabilities for the photon to be measured in the + state.

1 chooses A and 2 chooses B = $P(A+, B+)$ Looking at our table we see that $P(A+, B+) = P_3 + P_4$.

1 chooses A and 2 chooses C = $P(A+, C+) = P_2 + P_4$.

1 chooses C and 2 chooses B = $P(C+, B+) = P_3 + P_7$.

Noting that all probabilities are positive we can form the following inequality.

$$P(A+, B+) = P_3 + P_4 \leq P_2 + P_4 + P_3 + P_7 = P(A+, C+) + P(C+, B+)$$

$$P(A+, B+) \leq P(A+, C+) + P(C+, B+)$$

This inequality must be satisfied if the photons have local hidden variables and is called a Bell Inequality, first derived in 1964. Later, other inequalities were developed but this is the simplest and we want to explore it. If we see that quantum mechanics violates this inequality then no local hidden variable model exists and the two photons need to be described as entangled.

To see what quantum mechanics says let's say that particle 1 is measured first in one direction, say $A+$ for example. Then the other photon is in the perpendicular state with probability 1, $A-$ in this case. To find what the probability is to be measured in the + state we use Malus law. Recall that we said that it now corresponds to the probability of passing if the polarizer is at an angle θ to the polarization of the incoming photon.

$$P(\theta) = \cos^2 \theta$$

We will say that the angle between A and B is θ which is the same as between B and C . Then if particle 1 is measured to be + when the detector is set to A the probability for particle 2 to be + if measured in the direction B is,

$$P(A+, C+) = 1 - P(A+, C-) = 1 - \cos^2 \theta = \sin^2 \theta$$

Likewise we have,

$$P(C+, B+) = 1 - P(C+, B-) = 1 - \cos^2 \theta = \sin^2 \theta$$

$$P(A+, B+) = 1 - P(A+, B-) = 1 - \cos^2 2\theta = \sin^2 2\theta$$

Now we can plug into the inequality and see if it is always satisfied,

$$\sin^2 2\theta \stackrel{?}{\leq} 2 \sin^2 \theta$$

You can quickly check that this inequality is violated for a range of angles, check $\frac{\pi}{8} = 22.5^\circ$.

This is the key result [quantum mechanics violates Bell's inequality](#) which means it says the photons are not in definite states before the first measurement!

The last thing to check is whether nature obeys Bell's inequality or not. In 1982 and many times since experiments have shown violations of bell-like inequalities to many many standard deviations. It is no longer possible to have a few of reality that Einstein had that objects have an isolated, independent existence. These experiments are sometimes called metaphysical experimentation and Bell's result is now seen as one of the most significant in the 20th Century even though it is relatively unknown to the general public.

Mathematical Appendix

M.1 Simple trigonometry

Consider the right triangle with edge lengths A , O , and H .

$$\begin{aligned}\cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H} \\ \sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}\end{aligned}$$

The Pythagorean theorem

$$O^2 + A^2 = H^2$$

leads to a relation between the trigonometric functions,

$$\begin{aligned}H^2 &= H^2 \cos^2 \theta + H^2 \sin^2 \theta \\ 1 &= \cos^2 \theta + \sin^2 \theta\end{aligned}$$

M.2 Resolving vectors

We will assume basic familiarity with vectors. Here we just want to remind you about resolving vectors into a coordinate system. A vector is a mathematical entity that has a magnitude and a direction. The magnitude tells of the size of the vector (length of the arrow) and the direction is given by an arrow. In order to relate vectors and give quantitative details of them we need to establish a coordinate system. Once established, the vector can be referenced to the coordinate system by *resolving* the vector into that coordinate system.

We give a simple and relevant example of an electric field vector, \vec{E} , pointing in the two dimensional plane. Define a coordinate system by choosing a location for the origin and define two perpendicular directions x and y . We place the tip of the arrow at the origin and the arrow will point at some angle, θ , to the x axis. By drawing the component of the vector pointing in the x direction and in the y direction (labeled \vec{E}_x and \vec{E}_y) the addition of these two gives the desired vector, $\vec{E}_x + \vec{E}_y = \vec{E}$. The lengths of the components can be found from trigonometry since we have formed a right triangle with \vec{E} being the hypotenuse. This, with the angle θ formed between the vectors \vec{E}_x and \vec{E} the lengths of the components are,

$$\begin{aligned}E_x &= E \cos \theta \\ E_y &= E \sin \theta\end{aligned}$$

(Notice that $E \neq E_x + E_y$ but we do have that $E^2 = E_x^2 + E_y^2$.)

M.3 Wave addition

Here we only discuss the method of adding two one-dimensional waves together. A one dimensional transverse wave can be thought of as a plot of displacement from equilibrium, y , versus the direction of travel, x , that varies in time, t . This can be thought of as the function $y(x, t)$. We will concentrate on simple harmonic waves that can be described by a sine or cosine function. Thus a possible mathematical form is,

$$y(x, t) = A \cos\left(2\pi\left(\frac{x}{\lambda} + ft\right)\right)$$

where A is the maximum displacement that occurs (amplitude); λ is the wavelength (the distance through which the cosine function goes from 0 to 360° (or 2π radians)); and f is the frequency (how many wavelengths pass by one point as cosine function goes from 0 to 2π radians). (Note the wavelength is measured at fixed time and gives the distance after which the cosine repeats itself while the period, $T = \frac{1}{f}$, is measured at fixed position and is the time after which the cosine function repeats itself.)

If two waves are present along the same one-dimensional line, the net wave is given by the addition of these two waves. As functions, this idea is trivial. Consider two waves defined by functions $y_1(x, t)$ and $y_2(x, t)$. The net wave is simply the addition of these two functions,

$$y(x, t) = y_1(x, t) + y_2(x, t).$$

The graphical method of adding these two waves is also easy to perform. Plot each function one above the other or on the same plot. Add each length in the y direction at each point. If one points up and one points down, then the difference of the two should be taken.

M.4 Basic probability

The probability of an occurrence tells us the chance or likelihood of that occurrence. Here we only state the most basic, hopefully obvious, facts about simple probabilities.

- 1) Probabilities are always positive real numbers and belong to the range $[0,1]$ (or 0% to 100%).
- 2) The sum of all the probabilities for a particular outcome is 1 (or 100%).
- 3) If the probability for an outcome to occur is P , the probability for it not to occur is $1 - P$.

Consider a set of outcomes (A, B, C, D, E) that are mutually exclusive. (In any one sample only one of the set is returned). If there are varying probabilities for each of these to occur the facts above can be expressed as

- 1) Each P_i (where $i = A, B, C, D,$ or E) is in the range: $0 \leq P_A \leq 1$
- 2) $P_A + P_B + P_C + P_D + P_E = 1$.
- 3) For example. P_B is the outcome for B to be realized then $1 - P_B$ is the outcome for B not to be realized.

The probabilities can also be thought of in terms of the frequency of each outcome over a large number of samples. Say we ran the same sampling $N = 100$ times and the number of outcomes for each were,

$$A = 28, B = 7, C = 20, D = 24, E = 21$$

Then we could say the probabilities for each outcome are approximately $\frac{\# \text{ of outcomes}}{N}$, or

$$P_A = 0.28, P_B = 0.07, P_C = 0.20, P_D = 0.24, P_E = 0.21$$

Since these are mutually exclusive we can ask questions such as ‘what is the probability for A or D to result?’ This is simply the sum of the two probabilities $P(A \text{ or } D) = P_A + P_D$ (in the case just discussed that would be 0.52).

There many other interesting and important relations in probability theory however this is all we will really need for our discussion.