Online Appendix to
"Strategic Trading in Informationally Complex Environments"

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Abstract

This online appendix contains supplementary material omitted from the main body of the paper.

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1 Zero Aggregate Demand in Equilibrium

In this section, we formally derive equilibria in the special examples in which aggregate demand ends up being zero on the equilibrium path (footnotes 12 in Section 3.2 and 20 in Section 6 in the main body of the paper). In Section 1.1 we work out the example with one informed trader, corresponding to footnote 12, and in Section 1.2 we work out the example with multiple informed traders, corresponding to footnote 20.

1.1 One Informed Trader

Example OA.1 *The value of the security, v, is distributed normally with mean zero and variance one. There is one strategic trader with signal θ_1 who observes the value perfectly: $\theta_1 = v$. The demand of liquidity traders is $u = -v$. The market maker does not observe any signals beyond the aggregate demand.*

This example satisfies Assumptions 1 and 2 in Section 3, and thus we can use the closed form solutions derived in the proof of Theorem 1 and presented in Section 4.2 of the paper (for the special case $k_M = 0$). Because we have only one strategic trader in the example, many of the matrices become scalars, simplifying the calculation.

Specifically, $\Sigma_{\theta\theta} = \Sigma_{diag} = 1$, and therefore $\Lambda = 2$ and $\Lambda^{-1} = 1/2$. Next, $\Sigma_{\theta v} = 1$, while $\Sigma_{\theta u} = -1$. Thus, $A_v = 1/2$ and $A_u = -1/2$. The coefficients of the quadratic equation in γ are: $a = -1/4$, $b = (1/2) \cdot 4 \cdot (-1/2) + 1 = 0$, $c = Var(-\theta/2 - u) = Var(v/2) = 1/4$. Therefore, $\gamma = 1$, $\beta_D = 1/\gamma = 1$, and $\alpha = \gamma A_v - A_u = 1$. Thus, on the equilibrium path, aggregate demand is equal to $D = \alpha\theta + u = v - v = 0$.

1.2 Multiple Informed Traders

Example OA.2 *The value of the security, v, is distributed normally with mean zero and variance one. There are m strategic traders with the same signal $\theta_1 = v$. The demand of liquidity traders is $u = -v$. The market maker does not observe any signals beyond the aggregate demand.*

This example also satisfies Assumptions 1 and 2 in Section 3, and thus we can use the closed form solutions derived in the proof of Theorem 1 and presented in Section 4.2 of the paper (again, for the special case $k_M = 0$). We now have multiple strategic traders, so the calculations involve matrix manipulations.

Specifically, $\Sigma_{\theta\theta}$ is an m-dimensional matrix whose elements are all equal to one, while Σ_{diag} is an m-dimensional identity matrix. We thus have

$$\Lambda = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix} \quad \text{and} \quad \Lambda^{-1} = \frac{1}{m+1} \begin{pmatrix} m & -1 & \cdots & -1 \\ -1 & m & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & m \end{pmatrix}.$$

Next,

$$\Sigma_{\theta v} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \Sigma_{\theta u} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}, \quad A_v = \begin{pmatrix} 1/(m+1) \\ \vdots \\ 1/(m+1) \end{pmatrix}, \quad A_u = \begin{pmatrix} -1/(m+1) \\ \vdots \\ -1/(m+1) \end{pmatrix}.$$

The coefficients of the quadratic equation on γ are therefore $a = -m/(m+1)^2$, $b = -(m-1)/(m+1)^2$, and $c = 1/(m+1)^2$, which in turn gives us $\gamma = 1/m$ and $\beta_D = m$. Thus,

$$\alpha = \gamma A_v - A_u = \frac{1}{m} \begin{pmatrix} 1/(m+1) \\ \vdots \\ 1/(m+1) \end{pmatrix} + \begin{pmatrix} 1/(m+1) \\ \vdots \\ 1/(m+1) \end{pmatrix} = \frac{1}{m} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and so on the equilibrium path, aggregate demand is equal to $D = \alpha^T \theta + u = (1/m) \cdot m \cdot v - v = 0$. Thus, for every m, the market price on the equilibrium path is also always equal to zero, not revealing any information contained in the signals of the strategic traders and in the demand coming from the liquidity traders.

2 Zero Intercepts in Equilibrium

In this section we prove the statement made informally in footnote 13 in Section 3.2 of the main body of the paper that in our setting, linear equilibria with nonzero intercepts do not exist.

Proposition OA.1 *Suppose there exists an equilibrium of the form:*

- $d_i(\theta_i) = \delta_i + \alpha_i^T \theta_i$;
- $P(\theta_M, D) = \beta_0 + \beta_M^T \theta_M + \beta_D D$.

Then $\beta_0 = 0$ and for all i, $\delta_i = 0$.

Proof. Consider a particular realization of θ_i's, θ_M, and u.

Then in this equilibrium, the realized price will be given by

$$\begin{aligned} P = P(\theta_M, D) &= \beta_0 + \beta_M^T \theta_M + \beta_D D \\ &= \beta_0 + \beta_M^T \theta_M + \beta_D u + \beta_D \sum_{i=1}^n \delta_i + \beta_D \sum_{i=1}^n \alpha_i^T \theta_i. \end{aligned} \tag{OA.1}$$

By the definition of equilibrium, for every realization of θ_M and D, the price set by the market maker is equal to the expected value of the security conditional on θ_M and D:

$$P(\theta_M, D) = E[v|\theta_M, D].$$

Integrating over all possible realizations of θ_M and D, we thus get the following for the unconditional expectation of the price:

$$E[P] = E[v].$$

Since by assumption, $E[v] = 0$, and also $E[\theta_M]$, $E[u]$, and $E[\theta_i]$ (for all i) are equal to zero, by taking the unconditional expectation of equation (OA.1), we get

$$0 = \beta_0 + \beta_D \sum_{i=1}^{n} \delta_i. \tag{OA.2}$$

Now, as in Step 2 of the proof of Theorem 1 in the paper, consider the expected payoff of strategic trader i from submitting demand d after observing signal $\widetilde{\theta}_i$. It is equal to

$$E\left[d \left(v - \beta_0 - \beta_M^T \theta_M - \beta_D \left(d + \sum_{j \neq i} \delta_j + \sum_{j \neq i} \alpha_j^T \theta_j + u \right) \right) \middle| \theta_i = \widetilde{\theta}_i \right].$$

Except for the presence of constants β_0 and δ_j, this is the same expression as in Step 2 of the proof of Theorem 1. By the same logic as in that Step, it has to be the case that $\beta_D > 0$ and the unique optimal demand d^* is given by:

$$\begin{aligned} d^* &= \frac{1}{2\beta_D} E\left[v - \beta_0 - \beta_M^T \theta_M - \beta_D \left(\sum_{j \neq i} \delta_j + \sum_{j \neq i} \alpha_j^T \theta_j + u \right) \middle| \theta_i = \widetilde{\theta}_i \right] \\ &= -\frac{1}{2\beta_D} \left(\beta_0 + \beta_D \sum_{j \neq i} \delta_j \right) + \frac{1}{2\beta_D} \left(\Sigma_{iv}^T - \beta_M^T \Sigma_{iM}^T - \beta_D \left(\sum_{j \neq i} \alpha_j^T \Sigma_{ij}^T + \Sigma_{iu}^T \right) \right) \Sigma_{ii}^{-1} \widetilde{\theta}_i, \end{aligned}$$

By assumption, we also have $d^* = \delta_i + \alpha_i^T \widetilde{\theta}_i$. Since the equalities above have to hold for all realizations $\widetilde{\theta}_i$, it has to be the case that

$$\delta_i = -\frac{1}{2\beta_D} \left(\beta_0 + \beta_D \sum_{j \neq i} \delta_i \right),$$

which can be rearranged as

$$\beta_0 + \beta_D \sum_{j \neq i} \delta_j + 2\beta_D \delta_i = 0. \tag{OA.3}$$

Combining equations (OA.2) and (OA.3), we see that for every i, $\beta_D \delta_i = 0$, and thus $\delta_i = 0$, and therefore we also have $\beta_0 = 0$. ■

3 The Impact of Informative and Uninformative Signals on Trading Strategies

In this section, we prove the statement made in footnote 16 in Section 5.1 and also following Example 6 in Section 5.3 that a trader who possesses neither fundamental nor technical information (and who is thus less informed than the market maker) does not trade in equilibrium and thus does not make a profit. We also show that the presence of such a trader does not affect the trading behavior of other agents or the pricing behavior of the market maker. We then present two examples showing that the interaction of different types of information can in general be quite subtle: a trader who possesses only technical information may be able to make a positive profit even if he is the

only strategic trader in the market, and a trader who has some signals less informative than those of the market maker may nevertheless use those signals for trading if he also has some other, more informative signals. We conclude by showing that if a sub-vector of strategic traders' signals is uninformative (rather than simply being less informative than the signal of the market maker), then the traders do not use this sub-vector of signals in equilibrium, and the presence of these signals has no impact on the agents' trading behavior or on the pricing behavior of the market maker.

Formally, we say that trader i is (weakly) *less informed* than the market maker if conditional on the market maker's signal θ_M, trader i's signal θ_i does not contain any additional information about other random variables in the model: v, u, and θ_j for $j \neq i$:

$$Cov(\theta_i, v|\theta_M) = Cov(\theta_i, u|\theta_M) = Cov(\theta_i, \theta_j|\theta_M) = 0.$$

Note that this is equivalent to the trader having neither fundamental nor technical information, as defined in footnote 16 in Section 5.1 in the main body of the paper.

Proposition OA.2 *Suppose trader i is less informed than the market maker. Then in equilibrium, $\alpha_i = 0$, and the profit of trader i is zero. The equilibrium strategies of all other traders and the pricing behavior of the market maker are the same as in the economy in which trader i is not present.*

Proof. For notational simplicity, suppose $i = 1$, and suppose the dimensionality of trader 1's signal is k (i.e., $\theta_1 \in \mathbb{R}^k$). Note that by the maintained Assumption 1 of Section 3, at least one strategic trader receives at least some information about the value of the security that is not contained in the market maker's signal, and thus there are at least two strategic traders in this market: $n \geq 2$.

From the closed-form solution given in Sections 4.1 and 4.2 of the main body of the paper, the vector α that "stacks" all trading strategy vectors α_j on top of each other is given by

$$\begin{aligned}\alpha &= \frac{1}{\beta_D} A_v - A_u \\ &= \Lambda^{-1} \left(\frac{1}{\beta_D} Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M) \right).\end{aligned}$$

Note first that the assumption that trader 1 is less informed than the market maker implies that the top k elements of vector $\left(\frac{1}{\beta_D} Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M) \right)$ are all zero.

Next, consider matrix $\Lambda = \Sigma_{diag} + Var(\theta|\theta_M)$.

$$\Lambda = \begin{pmatrix} \Sigma_{11} & 0 & \cdots & 0 \\ 0 & \Sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{nn} \end{pmatrix} + \begin{pmatrix} Var(\theta_1|\theta_M) & 0 & \cdots & 0 \\ 0 & Var(\theta_2|\theta_M) & \cdots & Cov(\theta_2, \theta_n|\theta_M) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & Cov(\theta_n, \theta_2|\theta_M) & \cdots & Var(\theta_n|\theta_M) \end{pmatrix} \tag{OA.4}$$

$$= \begin{pmatrix} \Sigma_{11} + Var(\theta_1|\theta_M) & 0 \\ 0 & M \end{pmatrix}, \tag{OA.5}$$

where M is an invertible matrix.

Thus,

$$\Lambda^{-1} = \begin{pmatrix} (\Sigma_{11} + Var(\theta_1|\theta_M))^{-1} & 0 \\ 0 & M^{-1} \end{pmatrix}, \tag{OA.6}$$

and therefore in vector

$$\alpha = \Lambda^{-1} \left(\frac{1}{\beta_D} Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M) \right),$$

the first k elements are all zero—and these are precisely the elements that describe the equilibrium trading strategy of trader 1, α_1. The result about the zero profit of trader 1 is then immediate.

Next, the first k elements of vector $A_v = \Lambda^{-1} Cov(\theta, v|\theta_M)$ and vector $A_u = \Lambda^{-1} Cov(\theta, u|\theta_M)$ are all zero, and so it is immediate that the coefficients a, b, and c of the quadratic equation that pins down market depth $\gamma = 1/\beta_D$ are the same as in the corresponding quadratic equation for the market in which trader 1 is not present (see Section 4.2 of the main body of the paper for the closed-form solution formulas). Thus, the presence of trader 1 does not affect β_D.

Finally, from the block-diagonal formulas in equations (OA.4), (OA.5), and (OA.6), it follows that the elements of vectors A_v and A_u *after* the first k are exactly the same as in the corresponding vectors for the market in which trader 1 is not present. This observation, combined with the observations that the first k elements of A_v and A_u are all zero, and that β_D is not affected by the presence of trader 1, imply that β_M is not affected by the presence of trader 1 and that the elements of vector α *after* the first k are exactly the same as in the corresponding vector for the market in which trader 1 is not present. ■

The statement of Proposition OA.2 is intuitive, but it is important to note that the interaction of information and trading can in general be quite subtle. To illustrate the subtlety, we present two examples. The first example shows that a trader who possesses only technical information may still be able to make a positive profit even if he is the only strategic trader in the market.[1] In the example, the reason for the ability of the technical trader to make a profit is that liquidity demand is informative about the value of the security, and so the market maker's pricing rule is sensitive to aggregate demand. The technical trader, in turn, is informed about the "bias" in liquidity demand.

[1]Strictly speaking, the setting of the example violates Assumption 1 of our model, which states that at least one strategic trader in the market must possess some fundamental information. Thus, our general results are not applicable to this setting. Nevertheless, as we show, there still exists a unique linear equilibrium in this example.

Example OA.3 *The value of the security is distributed as $v \sim N(0,1)$. Liquidity demand is distributed as $u = v + \epsilon$, where $\epsilon \sim N(0,1)$. Random variables v and ϵ are independent. There is a single strategic trader in the market, whose signal is given by $\theta_1 = \epsilon$. The market maker does not directly observe any signals ($k_m = 0$).*

To compute a linear equilibrium, suppose the sensitivity of the market maker's pricing rule is $\beta_D > 0$ and suppose the strategic trader's trading rule is given by multiplier α_1. The aggregate demand D is then equal to $v + (1 + \alpha_1)\epsilon$, and the expected value of the security conditional on aggregate demand is then given by

$$E[v|D] = \frac{1}{1 + (1 + \alpha_1)^2} D.$$

Thus, $\beta_D = \frac{1}{1+(1+\alpha_1)^2}$.

On the other hand, suppose the strategic trader observes a realization $\tilde{\theta}$ of his signal. If he submits demand d, his expected profit is equal to

$$E\left[d\left(v - \beta_D(v + \tilde{\theta} + d)\right)\right] = -\beta_D\left(\tilde{\theta}d + d^2\right).$$

Thus, the optimal demand is $d = -\tilde{\theta}/2$, and so $\alpha_1 = -1/2$, which in turn implies $\beta_D = 4/5$ and a positive expected profit of trader 1 (conditional on observing realization $\tilde{\theta}$ of his signal, trader 1's expected profit is equal to $\tilde{\theta}^2/5$).

Our next example shows that for Proposition OA.2 to hold, it is important that a trader's entire vector of signals is less informative than the signal of the market maker. If only some of trader i's signals are less informative, while other ones contain useful information that the market maker does not observe, then the trader may end up putting non-zero weights on the "less informative" signals in equilibrium.

Example OA.4 *The value of the security is distributed as $v \sim N(0,1)$. There is a single strategic trader in the market, who observes a vector of signals $(\theta^1; \theta^2) \in \mathbb{R}^2$. The first component of the vector is given by $\theta^1 = v + \delta$, where $\delta \sim N(0,1)$ is a random variable independent of v. The second component is $\theta^2 = \delta$. The market maker's signal is given by $\theta_M = \delta$. Finally, the demand from liquidity traders is $u \sim N(0,1)$, distributed independently of v and δ.*

Note that the second component of the strategic trader's vector of signals is (weakly) less informative than the market maker's signal: it contains no additional information. By Proposition OA.2, if that were the only information that the trader observed, he would not trade based on it. In Example OA.4, however, that is not the case. In the unique equilibrium, the strategic trader will put negative weight on component θ^2: his demand is given by $d = (\theta^1 - \theta^2) = v$, and so $\alpha_1 = (1; -1)$. The market maker ignores her signal ($\beta_M = 0$), and puts weight $\beta_D = 1/2$ on aggregate demand. (Note that this equilibrium is essentially the same as the equilibrium of the standard Kyle (1985) model, and that trader 1 having access to signal θ^2 is essential for this equivalence: if trader 1 had access only to signal θ^1, the equilibrium would be substantively different, with, e.g., $\beta_M \neq 0$.)

The last result of this section shows that if some elements of traders' vectors of signals are uninformative about *all* other relevant random variables (including the information of the market maker), then traders will indeed not trade on that information—it will have no impact on their trading. We use this result in the proof of the general case of Theorem 3 in Section 6 of this Online Appendix, where in some cases we add auxiliary uninformative signals to avoid having to deal with zero variance matrices.

Formally, suppose each strategic trader i's signal θ_i can be represented as a pair of signals $(\theta_i^{'};\theta_i^{''})$ in such a way that the combined vectors $\theta' = (\theta_1^{'};\ldots;\theta_n^{'})$ and $\theta'' = (\theta_1^{''};\ldots;\theta_n^{''})$ have the property that vector θ'' is *uninformative*:

$$Cov(\theta'', v) = Cov(\theta'', u) = Cov(\theta'', \theta') = Cov(\theta'', \theta_M) = 0.$$

(Note that for some traders i, $\theta_i^{'}$ or $\theta_i^{''}$ may be empty, but we assume that both θ' and θ'' have at least one element.) Let α' and α'' be the vectors of components of equilibrium demand multiplier α corresponding to θ' and θ''.

Proposition OA.3 *Suppose θ'' is uninformative. Then $\alpha'' = 0$. Moreover, the equilibrium is not affected by the presence of signals in θ'': pricing multipliers β_D and β_M are the same as those in the market where only signals in θ' are observed, and vector of strategy multipliers α' is the same as the vector of strategy multipliers in the market where only signals in θ' are observed.*

Proof. The proof is similar to that of Proposition OA.2. From the closed-form solution given in Sections 4.1 and 4.2 of the main body of the paper, the vector α is given by

$$\begin{aligned}\alpha &= \frac{1}{\beta_D}A_v - A_u \\ &= \Lambda^{-1}\left(\frac{1}{\beta_D}Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M)\right).\end{aligned}$$

Since by assumption, θ'' is uninformative, it is independent of u, v, and θ_M, and so the rows corresponding to elements of θ'' in the matrix $\left(\frac{1}{\beta_D}Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M)\right)$ are all zero. Call this "Observation 1".

Next, consider matrix $\Lambda = \Sigma_{diag} + Var(\theta|\theta_M) = \Sigma_{diag} + \Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^{T}$. Take any element i from vector θ' and any element j from vector θ'' (these elements can be parts of the same trader's signal or be observed by different traders). Since by assumption, θ'' is independent of both θ' and θ_M, the entries in matrix Λ in cells (i, j) and (j, i) are both zero (because the corresponding entries are all zero in matrices Σ_{diag}, $\Sigma_{\theta\theta}$, and $\Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^{T}$. This, in turn, implies that the entries in matrix Λ^{-1} in cells (i, j) and (j, i) are also both zero.[2] Call this "Observation 2".

Combining Observation 1 and Observation 2, we find that the entries in vector $\alpha = \Lambda^{-1}\times\left(\frac{1}{\beta_D}Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M)\right)$ corresponding to the elements of θ'' are all equal to zero.

[2]To see this, note that by rearranging the rows and columns of matrix Λ to first list the entries corresponding to θ' and then the entries corresponding to θ'', one gets a block-diagonal matrix

$$\begin{pmatrix} \Sigma_{diag}^{'} + Var(\theta^{'}|\theta_M) & 0 \\ 0 & \Sigma_{diag}^{''} + Var(\theta^{''}) \end{pmatrix},$$

The proof of the statement that the pricing multipliers β_D and β_M are the same as those in the market where only signals in θ' are observed, and that the vector of strategy multipliers α' is the same as the vector of strategy multipliers in the market where only signals in θ' are observed, is completely analogous to the proof of the statement in Proposition OA.2 that the equilibrium strategies of traders $j \neq i$ and the pricing behavior of the market maker are the same as in the economy in which trader i (less informed than the market maker) is not present. ■

4 Rescaling Liquidity Demand

In this section, we formally show that equilibrium prices are not affected by the scale of liquidity demand.

Take an "original" market as defined in Section 3, and consider an "alternative" market, in which the demand from liquidity traders, $u^{(alt)}$, is equal to the demand from liquidity traders in the original market multiplied by a constant factor $\rho > 0$: $u^{(alt)} = \rho u$. For notational convenience, we append the superscript (alt) to variables associated with the alternative market, while variables associated with the original markets have the same notation as before.

The total demand observed by the market maker in the equilibrium of the alternative market is thus

$$D^{(alt)} = \sum_i d_i^{(alt)} + u^{(alt)},$$

where $d_i^{(alt)} = (\alpha^{(alt)})^T \theta_i$ is the demand of trader i. The price in the equilibrium of this alternative market is then a random variable given by

$$p^{(alt)} = \beta_D^{(alt)} \left(\sum_i (\alpha^{(alt)})^T \theta_i + \rho u \right) + (\beta_M^{(alt)})^T \theta_M.$$

Proposition OA.4 *In the unique linear equilibrium of the alternative market,*

$$\begin{aligned} \beta_D^{(alt)} &= \frac{\beta_D}{\rho}, \\ \beta_M^{(alt)} &= \beta_M, \\ \alpha^{(alt)} &= \rho \alpha. \end{aligned}$$

In particular, equilibrium price does not depend on the scale of liquidity demand:

$$p^{(alt)} = p.$$

Proof. The proof follows directly from the equilibrium characterization of Section 4.2. First,

the inverse of which is also block-diagonal:

$$\begin{pmatrix} \left(\Sigma'_{diag} + Var(\theta' | \theta_M)\right)^{-1} & 0 \\ 0 & \left(\Sigma''_{diag} + Var(\theta'')\right)^{-1} \end{pmatrix}.$$

note that $\Sigma_{\theta\theta}^{(alt)} = Cov(\theta,\theta) = \Sigma_{\theta\theta}$ and similarly $\Sigma_{diag}^{(alt)} = \Sigma_{diag}$, $\Sigma_{MM}^{(alt)} = Cov(\theta_M,\theta_M) = \Sigma_{MM}$, $\Sigma_{\theta v}^{(alt)} = Cov(\theta,v) = \Sigma_{\theta v}$, and $\Sigma_{Mv}^{(alt)} = Cov(\theta_M,v) = \Sigma_{Mv}$.

Next, note that $\Sigma_{\theta u}^{(alt)} = Cov(\theta,\rho u) = \rho\Sigma_{\theta u}$, $\Sigma_{Mu}^{(alt)} = Cov(\theta_M,\rho u) = \rho\Sigma_{Mu}$, $\sigma_{uu}^{(alt)} = Var(\rho u) = \rho^2\sigma_{uu}$, and $\sigma_{vu}^{(alt)} = Cov(v,\rho u) = \rho\sigma_{vu}$.

This immediately yields $A_v^{(alt)} = A_v$, $A_u^{(alt)} = \rho A_u^{(alt)}$, $a^{(alt)} = a$, $b^{(alt)} = \rho b$, and $c^{(alt)} = \rho^2 c$. Hence, $\gamma^{(alt)} = \rho\gamma$, $\beta_D^{(alt)} = \beta_D/\rho$, $\beta_M^{(alt)} = \beta_M$, and $\alpha^{(alt)} = \rho\alpha$, and therefore $p^{(alt)} = p$. ■

5 Proof of Theorem 2 (General Case)

The proof of Theorem 2 in the main body of the paper applies to the special case in which the covariance matrix of random vector $(\theta;\theta_M;u)$ is full rank. In this section, we prove Theorem 2 for the general case, imposing only Assumptions 1 and 2L: $Cov(v,\theta|\theta_M) \neq 0$ and $Var(u|\theta,\theta_M) > 0$.

Before proceeding to the proof, we first prove the following lemma.

Lemma OA.1 *Let ϕ be a random vector normally distributed with mean zero and variance-covariance matrix Σ. Let α be a vector of the same dimensionality, and let $\{\alpha_k\}$ be a sequence of vectors such that*

$$\lim_{k\to\infty} (\alpha_k)^T\widetilde{\phi} = \alpha^T\widetilde{\phi}$$

for every realization $\widetilde{\phi}$ of random vector ϕ except possibly on a set of probability zero. Then the limit also holds in the L^2 sense, that is,

$$\lim_{k\to\infty} E\left[\left((\alpha_k)^T\phi - \alpha^T\phi\right)^2\right] = 0.$$

Proof. Let N be the rank of matrix $Var(\phi)$, and take an orthogonal matrix Φ such that

$$\Phi^T Var(\phi)\Phi = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},$$

where M is a symmetric positive definite matrix of size N (if matrix $Var(\phi)$ is itself positive definite, and thus full rank, then we can simply take Φ to be the identity matrix and thus $M = Var(\phi)$.)

Let $\psi = \Phi^T\phi$ (and thus, since Φ is orthogonal, $\phi = \Phi\psi$). If, for almost all realizations $\widetilde{\phi}$ of ϕ, we have $(\alpha_k)^T\widetilde{\phi} \to \alpha^T\widetilde{\phi}$, then for almost all realizations $\widetilde{\psi}$ of ψ, we also have $(\alpha_k)^T\Phi\widetilde{\psi} \to \alpha^T\Phi\widetilde{\psi}$, which can be rewritten as

$$(\Phi^T\alpha_k)^T\widetilde{\psi} \to (\Phi^T\alpha)^T\widetilde{\psi}. \tag{OA.7}$$

Since ψ is distributed normally with mean zero and variance $\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$, where M is a positive definite matrix of rank N, the fact that the convergence in equation (OA.7) holds for almost all realizations $\widetilde{\psi}$ of random vector ψ implies that the first N components of vector $\Phi^T\alpha_k$ converge to

the first N components of vector $\Phi^T\alpha$. Then,

$$\begin{aligned}
E\left[\left((\alpha_k)^T\phi - \alpha^T\phi\right)^2\right] &= E\left[\left((\alpha_k)^T\Phi\psi - \alpha^T\Phi\psi\right)^2\right] \\
&= E\left[(\Phi^T\alpha_k - \Phi^T\alpha)^T\psi\psi^T(\Phi^T\alpha_k - \Phi^T\alpha)\right] \\
&= (\Phi^T\alpha_k - \Phi^T\alpha)^T Var(\psi)(\Phi^T\alpha_k - \Phi^T\alpha) \\
&= (\Phi^T\alpha_k - \Phi^T\alpha)^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} (\Phi^T\alpha_k - \Phi^T\alpha).
\end{aligned}$$

Since the first N components of $\Phi^T\alpha_k$ converge to the first N components of $\Phi^T\alpha$, we get that the first N components of $(\Phi^T\alpha_k - \Phi^T\alpha)$ converge to 0, which in turn implies by the last equation that $E\left[\left((\alpha_k)^T\phi - \alpha^T\phi\right)^2\right] \to 0$. ■

We are now ready to prove Theorem 2. The proof proceeds in five steps.

Step 1. Step 1 is identical to Step 1 in the proof of the special case in Appendix B, and is therefore omitted. The remaining steps are different from those in the proof of the special case.

Step 2. Let us now consider the entire sequence of markets, and restore superscript (m) for the variables. We introduce, for each market m, a random vector $\widehat{\theta}^{(m)}$, which is independent of the other random variables of the model, and is distributed normally with mean 0 and variance-covariance matrix $\widehat{\Sigma}^{(m)}_{diag}$.

Applying the usual projection formulas for jointly normal random variables, we note that the expressions for $\widehat{\Lambda}^{(m)}$, $\widehat{A}^{(m)}_u$, and $\widehat{A}^{(m)}_v$ that were obtained in Step 1 can be written as

$$\begin{aligned}
\widehat{\Lambda}^{(m)} &= Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M), \\
\widehat{A}^{(m)}_u &= [Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1} Cov(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, u|\theta_M), \\
\widehat{A}^{(m)}_v &= [Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1} Cov(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, v|\theta_M).
\end{aligned}$$

Note that, unlike the special case considered in Appendix B, the limit of $\widehat{\Lambda}^{(m)}$ is not guaranteed to be invertible, and so we cannot directly consider the limits of $\widehat{A}^{(m)}_u$ and $\widehat{A}^{(m)}_v$. Instead, we (in essence) will perform a change of basis, and will work in the new system of coordinates.

Formally, let Φ be an orthogonal matrix such that

$$\Phi^T Var(\theta|\theta_M)\Phi = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},$$

where M is a symmetric positive definite matrix whose size is the rank of $Var(\theta|\theta_M)$ (and, if $Var(\theta|\theta_M)$ is itself positive definite, we can simply take Φ to be the identity matrix and $M = Var(\theta|\theta_M)$). Note that Assumption 1 ($Cov(v, \theta|\theta_M) \neq 0$) implies that $Var(\theta|\theta_M) \neq 0$, and thus the size of matrix M is greater than zero.

Let $(\theta'; \theta'')$ be the random vector defined as $\Phi^T\theta$, where the dimensionality of θ' is equal to the rank of M. (Note that $Var(\theta'|\theta_M) = M$ and that $Var(\theta''|\theta_M) = 0$.) In a similar fashion, we let $((\widehat{\theta}^{(m)})'; (\widehat{\theta}^{(m)})'') = \Phi^T\widehat{\theta}^{(m)}$ and $((\xi^{(m)})'; (\xi^{(m)})'') = \Phi^T\xi^{(m)}$.

We will first show that the following limits hold.

$$\begin{aligned}
Var((\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'') &\to 0,\\
Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) &\to 0,\\
Var((\widehat{A}_u^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) &\to 0,\\
Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) &\to 0,\\
Var((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) &\to 0,\\
Var((\widehat{A}_v^{(m)})^T\xi^{(m)}) &\to 0,\\
Var((\widehat{A}_u^{(m)})^T\xi^{(m)}) &\to 0,\\
Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) &\to 0,\\
Cov((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) &\to 0,\\
Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \xi^{(m)}) &\to 0,\\
Cov((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}, \xi^{(m)}) &\to 0,\\
Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) &\to 0.
\end{aligned}$$

Step 2(a). We show that $Var((\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'') \to 0$. Note that

$$Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'') \leq Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i),$$

where $(\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i$ denotes the i-th element of random vector $(\xi^{(m)})' + (\widehat{\theta}^{(m)})'$. We then observe that, using the variance-covariance inequality,

$$\begin{aligned}
&\left|Cov((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i, (\xi^{(m)})'_j + (\widehat{\theta}^{(m)})'_j|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')\right|^2\\
&\quad\leq Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')\\
&\qquad\times Var((\xi^{(m)})'_j + (\widehat{\theta}^{(m)})'_j|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')\\
&\quad\leq Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i)Var((\xi^{(m)})'_j + (\widehat{\theta}^{(m)})'_j).
\end{aligned}$$

Since $Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i) = Var((\xi^{(m)})'_i) + Var((\widehat{\theta}^{(m)})'_i) \to 0$ for all i, we conclude that every element of matrix $Var((\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')$ converges to 0.

Step 2(b). We now show that $Var((\widehat{A}_v^{(m)})^T(\xi^{(m)}+\widehat{\theta}^{(m)}) \to 0$ and $Var((\widehat{A}_u^{(m)})^T(\xi^{(m)}+\widehat{\theta}^{(m)})) \to 0$. We focus on the first limit; the second one can be obtained via a similar derivation. For any vector $\widetilde{\theta}^{(m)}$ of the same dimensionality as $\widehat{\theta}^{(m)}$ and $\xi^{(m)}$, we can rewrite $(\widehat{A}_v^{(m)})^T\widetilde{\theta}^{(m)}$ as

$$\begin{aligned}
&E[v|\theta + \xi^{(m)} + \widehat{\theta}^{(m)} = \widetilde{\theta}^{(m)}, \theta_M = 0]\\
&\quad= E[v|\Phi^T\theta + \Phi^T\xi^{(m)} + \Phi^T\widehat{\theta}^{(m)} = \Phi^T\widetilde{\theta}^{(m)}, \theta_M = 0]\\
&\quad= E[v|\theta' + (\xi^{(m)})' + (\widehat{\theta}^{(m)})' = (\widetilde{\theta}^{(m)})', (\xi^{(m)})'' + (\widehat{\theta}^{(m)})'' = (\widetilde{\theta}^{(m)})'', \theta_M = 0],
\end{aligned}$$

where we define $((\widetilde{\theta}^{(m)})'; (\widetilde{\theta}^{(m)})'') = \Phi^T\widetilde{\theta}^{(m)}$ and the second equality makes use of the fact that, conditionally on $\theta_M = 0$, it is the case that $\theta'' = 0$ almost surely.

Let $\chi^{(m)} = (\xi^{(m)})' + (\widehat{\theta}^{(m)})' - E[(\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'']$, i.e., $\chi^{(m)}$ is the residual of

the projection of $(\xi^{(m)})' + (\widehat{\theta}^{(m)})'$ on $(\xi^{(m)})'' + (\widehat{\theta}^{(m)})''$. As $(\xi^{(m)})$ and $(\widehat{\theta}^{(m)})$ are independent of v, $(\xi^{(m)})'' + (\widehat{\theta}^{(m)})''$ is also independent of v, and noting that $(\xi^{(m)})'' + (\widehat{\theta}^{(m)})''$ is independent of θ', $\chi^{(m)}$, and θ_M, we get

$$\begin{aligned}(\widehat{A}_v^{(m)})^T\widetilde{\theta}^{(m)} &= E[v|\theta' + (\xi^{(m)})' + (\widehat{\theta}^{(m)})' = (\widetilde{\theta}^{(m)})', (\xi^{(m)})'' + (\widehat{\theta}^{(m)})'' = (\widetilde{\theta}^{(m)})'', \theta_M = 0] \\ &= E[v|\theta' + \chi^{(m)} = f(\widetilde{\theta}^{(m)}), \theta_M = 0],\end{aligned}$$

where we define $f(\widetilde{\theta}^{(m)}) = (\widetilde{\theta}^{(m)})' - E[(\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'' = (\widetilde{\theta}^{(m)})'']$. Next,

$$E[v|\theta' + (\chi^{(m)})' = f(\widetilde{\theta}^{(m)}), \theta_M = 0] = Cov(v, \theta' + \chi^{(m)}|\theta_M)[Var(\theta' + \chi^{(m)}|\theta_M)]^{-1} f(\widetilde{\theta}^{(m)}).$$

Hence, since $f(\xi^{(m)} + \widehat{\theta}^{(m)}) = \chi^{(m)}$,

$$Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) = Var(Cov(v, \theta' + \chi^{(m)}|\theta_M)[Var(\theta' + \chi^{(m)}|\theta_M)]^{-1}\chi^{(m)}).$$

Therefore, to show that $Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)}) \to 0$, it is enough to show that

$$Var(\chi^{(m)}) \to 0,$$

and that

$$Cov(v, \theta' + \chi^{(m)}|\theta_M)[Var(\theta' + \chi^{(m)}|\theta_M)]^{-1} \to Cov(v, \theta'|\theta_M)[Var(\theta'|\theta_M)]^{-1}.$$

To get the first limit, note that $Var(\chi^{(m)}) = Var((\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')$, which was shown to converge to 0 in Step 2(a). The second limit follows directly from the first one, along with the observation that $Var(\theta'|\theta_M)$ is, by construction, positive definite, and thus invertible.

Thus, $Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) \to 0$.

Step 2(c). Since $\xi^{(m)}$ and $\widehat{\theta}^{(m)}$ are independent, we have that $Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) = Var((\widehat{A}_v^{(m)})^T\xi^{(m)}) + Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)})$, and as the left-hand side converges to 0 as $m \to \infty$, and the two terms of the right-hand side are nonnegative, we get that $Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) \to 0$ and $Var((\widehat{A}_v^{(m)})^T\xi^{(m)}) \to 0$. We also get $Var((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) \to 0$ and $Var((\widehat{A}_u^{(m)})^T\xi^{(m)}) \to 0$ by the same argument.

Step 2(d). We now show that $Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \to 0$; the proof of the convergence for $Cov((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \to 0$, $Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \xi^{(m)}) \to 0$, and $Cov((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}, \xi^{(m)}) \to 0$ is again completely analogous.

By the covariance inequality,

$$|Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}_i^{(m)})|^2 \le Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)})Var(\widehat{\theta}_i^{(m)})$$

and we know from Step 2(c) that $Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) \to 0$. Besides we also have that $Var(\widehat{\theta}_i^{(m)}) \to 0$. Hence $Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}_i^{(m)}) \to 0$ for all i, and thus $Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \to 0$.

Step 2(e). Again, by the covariance inequality,

$$|Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)})|^2 \le Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)})Var((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}),$$

and we have just shown that both variances converge to 0. Thus,

$$Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) \to 0.$$

Step 3. In this step we will show the following:

$$\begin{aligned} a^{(m)} &\to 0, \\ b^{(m)} &\to -Cov(u, v|\theta, \theta_M), \\ c^{(m)} &\to Var(u|\theta, \theta_M). \end{aligned}$$

By the projection formulas for jointly normal random variables, we observe that the expressions for the coefficients $a^{(m)}$, $b^{(m)}$ and $c^{(m)}$, obtained in Step 1, can be written as follows:

$$\begin{aligned} a^{(m)} &= Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}), \\ b^{(m)} &= 2Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) - Cov(u, v|\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, \theta_M), \\ c^{(m)} &= Var((\widehat{A}_u^{(m)})^T(\theta + \xi^{(m)}) - u|\theta_M). \end{aligned}$$

Step 3(a). We showed in Step 2 that $Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) \to 0$. It implies $a^{(m)} \to 0$.

Step 3(b). Let us show that $b^{(m)} \to -Cov(u, v|\theta, \theta_M)$.

In Step 2, we showed that

$$Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) \to 0$$

so it remains to show $Cov(u, v|\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, \theta_M) \to Cov(u, v|\theta, \theta_M)$.

Using the projection formula, we write $Cov(u, v|\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, \theta_M)$ as

$$Cov(u, v|\theta_M) - Cov(u, \theta|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\theta, v|\theta_M),$$

where we used the fact that $\xi^{(m)} + \widehat{\theta}^{(m)}$ is independent of v and θ. Then, using the orthogonality of Φ, i.e., that $\Phi^T = \Phi^{-1}$, we observe that

$$\begin{aligned} &Cov(u, \theta|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\theta, v|\theta_M) \\ &\qquad = Cov(u, \Phi^T\theta|\theta_M)[Var(\Phi^T\theta + \Phi^T\xi^{(m)} + \Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\Phi^T\theta, v|\theta_M). \end{aligned}$$

Also observe that $Cov(u, \Phi^T\theta|\theta_M) = (Cov(u, \theta'|\theta_M), 0)$ and $Cov(\Phi^T\theta, v|\theta_M) = (Cov(\theta', v|\theta_M); 0)$. From the block matrix inversion formula,[3] and using the facts that $Var(\theta''|\theta_M) = 0$ and that

[3] See, for example, `http://en.wikipedia.org/wiki/Block_matrix_pseudoinverse`.

random vectors θ and $\xi^{(m)} + \widehat{\theta}^{(m)}$ are independent, we get

$$
\begin{aligned}
&Cov(u, \Phi^T\theta|\theta_M)[Var(\Phi^T\theta + \Phi^T\xi^{(m)} + \Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\Phi^T\theta, v|\theta_M)\\
&= Cov(u, \theta'|\theta_M)\\
&\times [Var(\theta' + (\xi^{(m)})' + (\widehat{\theta}^{(m)})'|\theta_M)\\
&- Cov((\xi^{(m)})' + (\widehat{\theta}^{(m)})', (\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')Var((\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')Cov((\xi^{(m)})'' + (\widehat{\theta}^{(m)})'', (\xi^{(m)})' + (\widehat{\theta}^{(m)})')]^{-1}\\
&\times Cov(\theta', v|\theta_M)\\
&= Cov(u, \theta'|\theta_M)[Var(\theta' + (\xi^{(m)})' + (\widehat{\theta}^{(m)})'|\theta_M, (\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')]^{-1}Cov(\theta', v|\theta_M).
\end{aligned}
$$

Now, as

$$
Var(\theta'+(\xi^{(m)})'+(\widehat{\theta}^{(m)})'|\theta_M, (\xi^{(m)})''+(\widehat{\theta}^{(m)})'') = Var(\theta'|\theta_M)+Var((\xi^{(m)})'+(\widehat{\theta}^{(m)})'|(\xi^{(m)})''+(\widehat{\theta}^{(m)})''),
$$

and since we have already proven in Step 2 that $Var((\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'') \to 0$, we have

$$
\begin{aligned}
&Cov(u, \Phi^T\theta|\theta_M)[Var(\Phi^T\theta + \Phi^T\xi^{(m)} + \Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\Phi^T\theta, v|\theta_M) \to\\
&\qquad Cov(u, \theta')[Var(\theta'|\theta_M)]^{-1}Cov(\theta', v|\theta_M).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&Cov(u, v|\theta_M) - Cov(u, \theta|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\theta, v|\theta_M) \to\\
&\qquad Cov(u, v|\theta_M) - Cov(u, \theta'|\theta_M)[Var(\theta'|\theta_M)]^{-1}Cov(\theta', v|\theta_M),
\end{aligned}
$$

and the latter expression is equal to $Cov(u, v|\theta', \theta_M)$, which in turn is equal to $Cov(u, v|\theta, \theta_M)$. Thus, $b^{(m)} \to -Cov(u, v|\theta, \theta_M)$.

Step 3(c). Finally, let us show that $c^{(m)} \to Var(u|\theta, \theta_M)$. We have already shown that

$$
\begin{aligned}
c^{(m)} &= Var(Cov(u, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}(\theta + \xi^{(m)}) - u|\theta_M)\\
&= Var(Cov(u, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}\theta - u|\theta_M) + Var((\widehat{A}_u^{(m)})^T\xi^{(m)}).
\end{aligned}
$$

We have shown in Step 2 that $Var((\widehat{A}_u^{(m)})^T\xi^{(m)}) \to 0$. Now, again using $\Phi^T = \Phi^{-1}$, we note that

$$
\begin{aligned}
&Cov(u, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}\theta\\
&\qquad = Cov(u, \Phi^T\theta|\theta_M)[Var(\Phi^T\theta + \Phi^T\xi^{(m)} + \Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}\Phi^T\theta.
\end{aligned}
$$

Next, using the same block matrix inversion formula as in Step 3(b), and the same facts about vectors θ'', θ, and $\widehat{\theta}^{(m)}$, we get

$$
\begin{aligned}
&Var(Cov(u, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}\theta - u|\theta_M) =\\
&\qquad Var(Cov(u, \Phi^T\theta|\theta_M)[Var(\Phi^T\theta + \Phi^T\xi^{(m)} + \Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}\Phi^T\theta - u|\theta_M) \to\\
&\qquad\qquad Var(Cov(u, \theta'|\theta_M)[Var(\theta'|\theta_M)]^{-1}\theta' - u|\theta_M).
\end{aligned}
$$

Finally,

$$Var(Cov(u,\theta'|\theta_M)[Var(\theta'|\theta_M)]^{-1}\theta' - u|\theta_M) = Var(u|\theta',\theta_M)$$
$$= Var(u|\theta,\theta_M),$$

and thus

$$c^{(m)} \to Var(u|\theta,\theta_M).$$

For the last observation in Step 3, note that by Assumption 2L, $Var(u|\theta,\theta_M) > 0$, and thus the results above on the limits of $a^{(m)}$, $b^{(m)}$, and $c^{(m)}$ imply that $\beta_D^{(m)}$ also converges to a finite value.

Step 4. By equilibrium condition (i), $\beta_D^{(m)}$ and $\beta_M^{(m)}$ are coefficients of the projection of v on the total demand $(\alpha^{(m)})^T(\theta+\xi^{(m)})+u$ and the market maker's signal θ_M:

$$v = \beta_D^{(m)}((\alpha^{(m)})^T(\theta+\xi^{(m)})+u) + (\beta_M^{(m)})^T\theta_M + \epsilon_{v,(\alpha^{(m)})^T(\theta+\xi^{(m)})+u,\theta_M}, \tag{OA.8}$$

where $\epsilon_{v,(\alpha^{(m)})^T(\theta+\xi^{(m)})+u,\theta_M}$ is independent of $(\alpha^{(m)})^T(\theta+\xi^{(m)})+u$ and θ_M.

Also, condition (ii), expressed via equation (17) in the proof of the special case of Theorem 2 in Appendix B of the main body of the paper,[4] can also be rewritten as a projection equation, of $v-\beta_D^{(m)}u-(\beta_M^{(m)})^T\theta_M$ (corresponding to the right-hand side of equation (17)) on $\theta+\xi^{(m)}+\widehat{\theta}^{(m)}$ (corresponding to the left-hand side of equation (17)):

$$v-\beta_D^{(m)}u-(\beta_M^{(m)})^T\theta_M = \beta_D^{(m)}(\alpha^{(m)})^T(\theta+\xi^{(m)}+\widehat{\theta}^{(m)}) + \epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}. \tag{OA.9}$$

We will now show that θ_M is independent of $\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}$. Because of joint normality, it is enough to show that $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},\theta_M)=0$. From equation (OA.9), and using the independence of $\widehat{\theta}^{(m)}$ and θ_M,

$$Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},\theta_M) =$$
$$Cov(v,\theta_M) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta,\theta_M) - Cov(\beta_D^{(m)}u,\theta_M) - Cov((\beta_M^{(m)})^T\theta_M,\theta_M).$$

From equation (OA.8), we get that

$$Cov(v,\theta_M) = Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta,\theta_M) + Cov(\beta_D^{(m)}u,\theta_M) + Cov((\beta_M^{(m)})^T\theta_M,\theta_M),$$

hence establishing the equality $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},\theta_M)=0$.

In the remainder of the proof, $p^{(m)}$ is the random variable that corresponds to the equilibrium price of the asset in market m, i.e.,

$$p^{(m)} = \beta_D^{(m)}(\alpha^{(m)})^T(\theta+\xi^{(m)}) + \beta_D^{(m)}u + (\beta_M^{(m)})^T\theta_M. \tag{OA.10}$$

Step 5. As $m\to\infty$, the behavior of the market depends on the sign of $Cov(u,v|\theta,\theta_M)$.

Step 5(a). Let us first consider the case in which $Cov(u,v|\theta,\theta_M)>0$.

[4] It is straightforward to check that the derivation of equation (17) does not depend on the additional assumptions imposed in the special case, and remains valid in the general case that we consider in the current proof.

In this case, $\beta_D^{(m)} \to Cov(u, v|\theta, \theta_M)/Var(u|\theta, \theta_M)$, which is immediately seen from the limits of the coefficients of the quadratic equation on $\beta_D^{(m)}$ in Step 3. We will show that $p^{(m)} \xrightarrow{L^2} E[v|\theta_M, \theta, u]$, i.e., $p^{(m)}$ converges to $E[v|\theta_M, \theta, u]$ in the L^2 sense.

We observe that

$$E[v|\theta_M, \theta, u] - p^{(m)} = E[v - p^{(m)}|\theta_M, \theta, u]$$

and, from equations (OA.9) and (OA.10),

$$v - p^{(m)} = \beta_D^{(m)}(\alpha^{(m)})^T \widehat{\theta}^{(m)} + \epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},$$

and since $\widehat{\theta}^{(m)}$ is independent of θ, θ_M, u, we get that

$$E[v|\theta_M, \theta, u] - p^{(m)} = E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta_M, \theta, u] = E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta, u],$$

where the second equality comes from the fact that $\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}$ and θ_M are independent.

By Lemma OA.1, to prove

$$E\left[E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta, u]^2\right] \to 0,$$

it is enough to show that, for almost every realization $(\widetilde{\theta}; \widetilde{u})$ of random vector $(\theta; u)$,

$$E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta = \widetilde{\theta}, u = \widetilde{u}] \to 0.$$

To show the latter convergence, it is in turn sufficient to show two convergences:

$$Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) \to 0$$

and $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, u) \to 0$.

We first show that $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) \to 0$. From equation (OA.9), and using the fact that $\widehat{\theta}^{(m)}$ and θ are independent, we have

$$\begin{aligned} &Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) \\ &\qquad = Cov(v, \theta) - Cov(\beta_D^{(m)} u, \theta) - Cov((\beta_M^{(m)})^T \theta_M, \theta) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T \theta, \theta). \end{aligned}$$

Recalling that $\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}$ and $\theta + \xi^{(m)} + \widehat{\theta}^{(m)}$ are independent and that $\widehat{\theta}^{(m)}$ and $\xi^{(m)}$ are independent of each other and of v, u, θ, and θ_M, we also have

$$\begin{aligned} Cov(v, \theta) &= Cov(v, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}) \\ &= Cov(\beta_D^{(m)}(\alpha^{(m)})^T \theta, \theta) + Cov(\beta_D^{(m)}(\alpha^{(m)})^T \widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \\ &\quad + Cov(\beta_D^{(m)}(\alpha^{(m)})^T \xi^{(m)}, \xi^{(m)}) + Cov(\beta_D^{(m)} u, \theta) + Cov((\beta_M^{(m)})^T \theta_M, \theta). \end{aligned}$$

Thus,

$$Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) = Cov(\beta_D^{(m)}(\alpha^{(m)})^T \widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) + Cov(\beta_D^{(m)}(\alpha^{(m)})^T \xi^{(m)}, \xi^{(m)}).$$

Now, since $\beta_D^{(m)}$ converges to a finite value, by Step 2 we have

$$Cov(\beta_D^{(m)}(\alpha^{(m)})^T \widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) = Cov((\widehat{A}_v^{(m)})^T \widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) - \beta_D^{(m)} Cov((\widehat{A}_u^{(m)})^T \widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \to 0.$$

Similarly, we have

$$Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, \xi^{(m)}) = Cov((\widehat{A}_v^{(m)})^T\xi^{(m)}, \xi^{(m)}) - \beta_D^{(m)} Cov((\widehat{A}_u^{(m)})^T\xi^{(m)}, \xi^{(m)}) \to 0.$$

And thus we get that

$$Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) \to 0.$$

Observe that to establish this convergence, we did not rely on the fact that $\beta_D^{(m)}$ converges to a positive value; we only used the fact that it converges to a finite value. We will use this observation in Step 5(b).

Next, we show that $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, u) \to 0$. From equation (OA.8), we get that

$$\begin{aligned}
&Cov(v, (\alpha^{(m)})^T\theta) + Cov(v, u) \\
&= Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, (\alpha^{(m)})^T\theta) + Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, (\alpha^{(m)})^T\xi^{(m)}) + Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, u) \\
&\quad + Cov(\beta_D^{(m)}u, (\alpha^{(m)})^T\theta) + Cov(\beta_D^{(m)}u, u) + Cov((\beta_M^{(m)})^T\theta_M, (\alpha^{(m)})^T\theta) + Cov((\beta_M^{(m)})^T\theta_M, u).
\end{aligned}$$

From equation (OA.9), it follows that

$$\begin{aligned}
&Cov(v, (\alpha^{(m)})^T\theta) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, (\alpha^{(m)})^T\theta) \\
&\qquad - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, (\alpha^{(m)})^T\xi^{(m)}) - Cov(\beta_D^{(m)}u, (\alpha^{(m)})^T\theta) \\
&\qquad\qquad - Cov((\beta_M^{(m)})^T\theta_M, (\alpha^{(m)})^T\theta) = Cov(\beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}, (\alpha^{(m)})^T\widehat{\theta}^{(m)}),
\end{aligned}$$

where the last term can be rewritten as

$$\begin{aligned}
&Cov(\beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}, (\alpha^{(m)})^T\widehat{\theta}^{(m)}) \\
&\quad = (\beta_D^{(m)})^{-1} Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) - 2Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) + \beta_D^{(m)} Var((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}).
\end{aligned}$$

As $\beta_D^{(m)}$ converges to a positive value, the limits established in Step 2 imply that all the terms on the right-hand side of the last equation converge to zero, and thus

$$Cov(\beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}, (\alpha^{(m)})^T\widehat{\theta}^{(m)}) \to 0,$$

and so

$$\begin{aligned}
&Cov(v, (\alpha^{(m)})^T\theta) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, (\alpha^{(m)})^T\theta) \\
&\qquad - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, (\alpha^{(m)})^T\xi^{(m)}) - Cov(\beta_D^{(m)}u, (\alpha^{(m)})^T\theta) \\
&\qquad\qquad - Cov((\beta_M^{(m)})^T\theta_M, (\alpha^{(m)})^T\theta) \to 0.
\end{aligned}$$

Thus

$$Cov(v, u) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, u) - Cov(\beta_D^{(m)}u, u) - Cov((\beta_M^{(m)})^T\theta_M, u) \to 0$$

and so $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, u) \to 0$.

Step 5(b). We now examine the remaining case: $Cov(u, v|\theta, \theta_M) \leq 0$. In this case, $\beta_D^{(m)} \to 0$, which again follows from the limits of the coefficients $a^{(m)}$, $b^{(m)}$, and $c^{(m)}$ in Step 3. We will show

that $p^{(m)} \xrightarrow{L^2} E[v|\theta, \theta_M]$.

Expressing v using equation (OA.9), and taking expectations on both sides conditional on θ and θ_M, we have

$$\begin{aligned} E[v|\theta,\theta_M] &= \beta_D^{(m)} E[u|\theta,\theta_M] + \left[(\beta_M^{(m)})^T\theta_M + \beta_D^{(m)}(\alpha^{(m)})^T(\theta+\xi^{(m)})\right] + E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta,\theta_M] \\ &= \beta_D^{(m)} E[u|\theta,\theta_M] + \left[p^{(m)} - \beta_D^{(m)} u\right] + E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta,\theta_M], \end{aligned}$$

and so we have

$$p^{(m)} - E[v|\theta,\theta_M] = \beta_D^{(m)}(u - E[u|\theta,\theta_M]) - E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta,\theta_M].$$

As $\beta_D^{(m)} \to 0$, the first term on the right-hand side, $\beta_D^{(m)}(u - E[u|\theta,\theta_M])$, converges to 0 (in L^2 sense). Also, in Step 4 we showed that $\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}$ is independent of θ_M, and therefore $E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta,\theta_M] = E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta]$. In Step 5(a), we showed that $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},\theta) \to 0$, implying that $E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta = \widetilde{\theta}, \theta_M = \widetilde{\theta}_M]$ converges to 0 for almost every realization $(\widetilde{\theta};\widetilde{\theta}_M)$ of random vector $(\theta;\theta_M)$. Therefore, by Lemma OA.1, $E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta,\theta_M] \xrightarrow{L^2} 0$. Thus, $p^{(m)} \xrightarrow{L^2} E[v|\theta,\theta_M]$.

6 Proof of Theorem 3 (General Case)

The proof of Theorem 3 in the main body of the paper applies to the special case in which the covariance matrix of random vector $(\theta_S;\theta_L;\theta_M;u)$ is full rank. In this section of the Online Appendix, we prove Theorem 3 for the general case, imposing only Assumptions 1H and 2H: $Cov(v,\theta_S|\theta_L,\theta_M) \neq 0$ and $Var(u|\theta_L,\theta_M) > 0$.

In addition to the markets indexed $m = 1, 2, \ldots,$ we consider the alternative market which includes s groups of traders $i = 1, \ldots, s$. The size of group i is ℓ_i. Each trader j of group i receives the same signal θ_i. In this alternative market, the market maker receives signal $(\theta_L;\theta_M)$. We use superscript (m) when we refer to the variables of the market m, and we use superscript (alt) when we refer to the variables in the alternative market. By Theorem 1 a unique linear equilibrium exists for each market m and for the alternative market.

For $i = s+1, \ldots, n$, we define $\xi_i^{(m)} = (\sum_j \xi_{i,j})/\ell_i^{(m)}$ and $\xi_L^{(m)} = (\xi_{s+1}^{(m)};\ldots;\xi_n^{(m)})$. For $i \leq s$, we let $\Sigma_i^{\xi} = 0$, $\xi_i^{(m)} = 0$, and $\xi_S^{(m)} = (\xi_1^{(m)};\ldots;\xi_s^{(m)})$. Let $\xi^{(m)} = (\xi_1^{(m)};\ldots;\xi_n^{(m)})$.

We also define

$$\widehat{\Sigma}_{diag}^{(alt)} = \begin{pmatrix} \frac{1}{\ell_1}\Sigma_{11} & 0 & \cdots & 0 \\ 0 & \frac{1}{\ell_2}\Sigma_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\ell_s}\Sigma_{ss} \end{pmatrix}$$

and

$$\widehat{\Sigma}_{diag}^{(m)} = \begin{pmatrix} \frac{1}{\ell_1^{(m)}}(\Sigma_{11}+\Sigma_1^{\xi}) & 0 & \cdots & 0 \\ 0 & \frac{1}{\ell_2^{(m)}}(\Sigma_{22}+\Sigma_2^{\xi}) & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\ell_n^{(m)}}(\Sigma_{nn}+\Sigma_n^{\xi}) \end{pmatrix}$$

where $\ell_i^{(m)} = \ell_i$ if $i \leq s$. Note that

$$\widehat{\Sigma}_{diag}^{(m)} \to \widehat{\Sigma}_{diag}^{(\infty)} := \begin{pmatrix} \widehat{\Sigma}_{diag}^{(alt)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly, we define

$$\Sigma^{\xi,(m)} = \begin{pmatrix} \frac{1}{\ell_1^{(m)}}\Sigma_1^{\xi} & 0 & \dots & \dots \\ 0 & \frac{1}{\ell_2^{(m)}}\Sigma_2^{\xi} & 0 & \dots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{\ell_n^{(m)}}\Sigma_n^{\xi} \end{pmatrix}.$$

Note that $\Sigma^{\xi,(m)}$ is the variance-covariance matrix of the random vector $\xi^{(m)}$, and that

$$\lim_{m\to\infty} \Sigma^{\xi,(m)} = 0.$$

We follow the methodology of the proof of the general case of Theorem 2 and the special case of Theorem 3. We introduce auxiliary random variables, and interpret various matrices in the proof as covariance matrices of combinations of these auxiliary random variables and the original random variables in the model.

Specifically, for each market m, we introduce a random vector $\widehat{\theta}^{(m)}$, which is independent of the other random variables in the model, and is distributed normally with mean 0 and covariance matrix $\widehat{\Sigma}_{diag}^{(m)}$. We also introduce a random vector $\widehat{\theta}_S$, which is independent of the other random variables in the model, and is distributed normally with mean 0 and covariance matrix $\widehat{\Sigma}_{diag}^{(alt)}$. Finally, we introduce a random vector $\widehat{\theta}^{(\infty)} = (\widehat{\theta}_S; 0)$, which is distributed normally with mean 0 and covariance matrix $\widehat{\Sigma}_{diag}^{(\infty)}$.

The remainder of the proof assumes that $Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S) \neq 0$. This assumption is without loss of generality: if the signals of the large groups are such that $Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S) = 0$, we can append a randomly and independently distributed signal to the common signals component of one large group, so that the conditional variance becomes nonzero. By Proposition OA.3 in Section 3 of this Online Appendix, traders never trade based on this additional, uninformative signal, and the equilibrium outcome is not impacted by its presence.

Finally, we define the following matrices and vectors, similarly as in the proof of Theorem 1.

For market m, we let

$$\widehat{\Lambda}^{(m)} = Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M),$$
$$\widehat{A}_u^{(m)} = (\widehat{\Lambda}^{(m)})^{-1} Cov(\theta, u|\theta_M),$$
$$\widehat{A}_v^{(m)} = (\widehat{\Lambda}^{(m)})^{-1} Cov(\theta, v|\theta_M).$$

For the alternative market, we let

$$\widehat{\Lambda}^{(alt)} = Var(\theta_S + \widehat{\theta}_S|\theta_M, \theta_L),$$
$$\widehat{A}_u^{(alt)} = (\widehat{\Lambda}^{(alt)})^{-1} Cov(\theta_S, u|\theta_M, \theta_L),$$
$$\widehat{A}_v^{(alt)} = (\widehat{\Lambda}^{(alt)})^{-1} Cov(\theta_S, v|\theta_M, \theta_L).$$

We remark that, for a vector $\widetilde{\theta}$ of the same dimensionality as θ,

$$(\widehat{A}_v^{(m)})^T \widetilde{\theta} = E[v \mid \theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = \widetilde{\theta}_L]$$

where we decompose $\widetilde{\theta}$ as $(\widetilde{\theta}_S; \widetilde{\theta}_L)$. We get an analogous expression for $\widehat{A}_u^{(m)}$. Similarly, we have

$$(\widehat{A}_v^{(alt)})^T \widetilde{\theta}_S = E[v \mid \theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S]$$

and we get an analogous expression for $\widehat{A}_u^{(alt)}$.

6.1 Change of Basis

As in the proof of the general case of Theorem 2 in Section 5 of this Online Appendix, to handle the problem of variance-covariance matrices that are not positive definite we perform a change of basis.

Let Φ be an orthogonal matrix such that

$$\Phi^T Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S)\Phi = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},$$

where M is a symmetric positive definite matrix whose size is the rank of $Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S)$. Since, by assumption, $Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S) \neq 0$, the size of matrix M is greater than zero.

Let $(\theta'_L; \theta''_L)$ be the random vector defined as $\Phi^T \theta_L$, where the dimensionality of θ'_L is equal to the rank of M. Note that $Var(\theta'_L|\theta_M, \theta_S + \widehat{\theta}_S) = M$ and that $\theta''_L = 0$. In a similar fashion, we let $((\widehat{\theta}_L^{(m)})'; (\widehat{\theta}_L^{(m)})'') = \Phi^T \widehat{\theta}_L^{(m)}$ and $((\xi_L^{(m)})'; (\xi_L^{(m)})'') = \Phi^T \xi_L^{(m)}$. For a vector $\widetilde{\theta}_L$ of the same dimensionality as θ_L, define $(\widetilde{\theta}'_L; \widetilde{\theta}''_L) = \Phi^T \widetilde{\theta_L}$.

We let $\chi_L^{(m)}$ be the residual of the projection of $(\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})'$ on $(\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})''$, i.e., $\chi_L^{(m)} = (\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})' - E[(\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})'|(\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})'']$. We have already shown in the general proof of Theorem 2 in Section 5 of this Online Appendix that $Var(\chi_L^{(m)}) \to 0$.

For a vector $\widetilde{\theta}_L$ of the same dimensionality as θ_L, we define

$$f(\widetilde{\theta}_L) = (\widetilde{\theta}_L)' - E[(\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})'|(\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})'' = (\widetilde{\theta}_L)''].$$

In addition, let

$$\widehat{B}_v^{(m)} = [Var((\theta_S + \widehat{\theta}_S; \theta'_L + \chi_L^{(m)})|\theta_M)]^{-1} Cov((\theta_S; \theta'_L), v \mid \theta_M)$$

and note that since $(\theta_S+\widehat{\theta}_S;\theta'_L;\theta_M)$ is not degenerate, and that $Var(\chi_L^{(m)}) \to 0$ and $Var(\widehat{\theta}_L^{(m)}) \to 0$, we have the following limit:

$$\lim_{m\to\infty} \widehat{B}_v^{(m)} = \widehat{B}_v^{(\infty)} := [Var((\theta_S+\widehat{\theta}_S;\theta'_L)|\theta_M]^{-1} Cov((\theta_S;\theta'_L), v|\theta_M).$$

Similarly, define

$$\widehat{B}_u^{(m)} = [Var((\theta_S+\widehat{\theta}_S;\theta'_L+\chi_L^{(m)})|\theta_M)]^{-1} Cov((\theta_S;\theta'_L), u \mid \theta_M)$$

and note that we have the following limit:

$$\lim_{m\to\infty} \widehat{B}_u^{(m)} = \widehat{B}_u^{(\infty)} := [Var((\theta_S+\widehat{\theta}_S;\theta'_L)|\theta_M)]^{-1} Cov((\theta_S;\theta'_L), u \mid \theta_M).$$

Finally, let

$$\widehat{C}_v = [Var((\theta_S+\widehat{\theta}_S;\theta_L)|\theta_M)]^{-1} Cov((\theta_S;\theta_L), v \mid \theta_M)$$

and

$$\widehat{C}_u = [Var((\theta_S+\widehat{\theta}_S;\theta_L)|\theta_M)]^{-1} Cov((\theta_S;\theta_L), u \mid \theta_M),$$

where the matrix inverse in the last two equations denote a Moore-Penrose pseudoinverse, because the random vector $(\theta_L;\theta_M)$ may be degenerate. Note that, for $\widetilde{\theta}_L$ a realization of θ_L, and any vector $\widetilde{\theta}_S$ of the same dimension as θ_S, we still have the equality

$$(\widehat{C}_v)^T(\widetilde{\theta}_S;\widetilde{\theta}_L) = E[v|\theta_M=0, \theta_S+\widehat{\theta}_S=\widetilde{\theta}_S, \theta_L=\widetilde{\theta}_L],$$

and similarly

$$(\widehat{C}_u)^T(\widetilde{\theta}_S;\widetilde{\theta}_L) = E[u|\theta_M=0, \theta_S+\widehat{\theta}_S=\widetilde{\theta}_S, \theta_L=\widetilde{\theta}_L].$$

6.2 Auxiliary Results

Before proceeding to the main body of the proof, we first state and prove several auxiliary results.

Auxiliary Result (a). First, we prove that

$$\lim_{m\to\infty} Cov(u,v|\theta_M, \theta_S+\widehat{\theta}_S, \theta_L+\widehat{\theta}_L^{(m)}+\xi_L^{(m)}) = Cov(u,v|\theta_M, \theta_S+\widehat{\theta}_S, \theta_L).$$

We have, noting that $\theta''_L = 0$,

$$\begin{aligned}
&Cov(u,v|\theta_M, \theta_S+\widehat{\theta}_S, \theta_L+\widehat{\theta}_L^{(m)}+\xi_L^{(m)}) \\
&\quad = Cov(u,v|\theta_M, \theta_S+\widehat{\theta}_S, \theta'_L+(\widehat{\theta}_L^{(m)})'+(\xi_L^{(m)})', (\widehat{\theta}_L^{(m)})''+(\xi_L^{(m)})'') \\
&\quad = Cov(u,v|\theta_M, \theta_S+\widehat{\theta}_S, \theta'_L+\chi_L^{(m)}).
\end{aligned}$$

The variance-covariance matrix of $(\theta_M, \theta_S+\widehat{\theta}_S, \theta'_L)$ is positive definite. As the variance of $\chi_L^{(m)}$ converges to zero, we get

$$\begin{aligned}
\lim_{m\to\infty} Cov(u,v|\theta_M, \theta_S+\widehat{\theta}_S, \theta'_L+\chi_L^{(m)}) &= Cov(u,v|\theta_M, \theta_S+\widehat{\theta}_S, \theta'_L) \\
&= Cov(u,v|\theta_M, \theta_S+\widehat{\theta}_S, \theta_L)
\end{aligned}$$

which yields the desired result.

Auxiliary Result (b). Next we prove that, for every vector $\widetilde{\theta}_S$ of the same dimensionality as θ_S, we have the limit

$$\lim_{m\to\infty} E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)} = 0] = E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0].$$

Indeed,

$$\begin{aligned}
&E\left[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)} = 0\right] \\
&\qquad = E\left[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L' + (\widehat{\theta}_L^{(m)})' + (\xi_L^{(m)})' = 0, (\widehat{\theta}_L^{(m)})'' + (\xi_L^{(m)})'' = 0\right] \\
&\qquad = E\left[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L' + \chi_L^{(m)} = 0, (\widehat{\theta}_L^{(m)})'' + (\xi_L^{(m)})'' = 0\right] \\
&\qquad = E\left[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L' + \chi_L^{(m)} = 0\right].
\end{aligned}$$

Observing that the variance-covariance matrix of $(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L')$ is positive definite and the variance of $\chi_L^{(m)}$ converges to zero yields the desired result.

Auxiliary Result (c). We get, in the same way, that for every vector $\widetilde{\theta}_S$ of the same dimensionality as θ_S, we have the limit

$$\lim_{m\to\infty} E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] = E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0]$$

and that for every vector $\widetilde{\theta}_M$ of the same dimensionality as θ_M, we have the limit

$$\lim_{m\to\infty} E[v|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] = E[v|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = 0],$$

$$\lim_{m\to\infty} E[u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] = E[u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = 0].$$

Auxiliary Result (d). Next, we prove that

$$\lim_{m\to\infty} Var((\widehat{A}_v^m)^T(\widehat{\theta}_S; \widehat{\theta}_L^{(m)})) = Var((\widehat{A}_v^{alt})^T\widehat{\theta}_S).$$

We have, using the independence of the two random vectors $\widehat{\theta}_S$ and $\widehat{\theta}_L^{(m)}$, the equality

$$Var((\widehat{A}_v^m)^T(\widehat{\theta}_S; \widehat{\theta}_L^{(m)})) = Var((\widehat{A}_v^m)^T(\widehat{\theta}_S; 0)) + Var((\widehat{A}_v^m)^T(0; \widehat{\theta}_L^{(m)})).$$

By Auxiliary Result (b),

$$\lim_{m\to\infty} Var((\widehat{A}_v^m)^T(\widehat{\theta}_S; 0)) = Var((\widehat{A}_v^{alt})^T\widehat{\theta}_S),$$

so it remains to prove that

$$\lim_{m\to\infty} Var((\widehat{A}_v^m)^T(0; \widehat{\theta}_L^{(m)})) = 0. \tag{OA.11}$$

We have

$$\begin{aligned}
Var((\widehat{A}_v^m)^T(0; \widehat{\theta}_L^{(m)})) &\le Var((\widehat{A}_v^m)^T(0; \widehat{\theta}_L^{(m)})) + Var((\widehat{A}_v^m)^T(0; \xi_L^{(m)})) \\
&= Var((\widehat{A}_v^m)^T(0; \widehat{\theta}_L^{(m)} + \xi_L^{(m)})).
\end{aligned}$$

We have, for any vector $\widetilde{\theta}_L$ of the same dimension as θ_L,

$$\begin{aligned}
&E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = \widetilde{\theta}_L] \\
&\quad = E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L' + (\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})' = (\widetilde{\theta}_L)', (\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})'' = (\widetilde{\theta}_L)''] \\
&\quad = E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L' + \chi_L^{(m)} = f(\widetilde{\theta}_L), (\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})'' = (\widetilde{\theta}_L)''] \\
&\quad = E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L' + \chi_L^{(m)} = f(\widetilde{\theta}_L)].
\end{aligned}$$

Thus,

$$\begin{aligned}
Var((\widehat{A}_v^{(m)})^T(0; \widehat{\theta}_L^{(m)} + \xi_L^{(m)})) &= Var((\widehat{B}_v^{(m)})^T(0; f(\widehat{\theta}_L^{(m)} + \xi_L^{(m)}))) \\
&= Var((\widehat{B}_v^{(m)})^T(0; f(\chi_L^{(m)}))) \\
&\to 0
\end{aligned}$$

since $\widehat{B}_v^{(m)} \to \widehat{B}_v^{(\infty)}$ and $Var(\chi_L^{(m)}) \to 0$. Limit (OA.11) follows.

Auxiliary Result (e). Next, we show that

$$(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} \to (\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)}. \tag{OA.12}$$

We note that

$$\begin{aligned}
(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} &= Cov((\widehat{A}_v^{(m)})^T(\widehat{\theta}_S; \widehat{\theta}_L^{(m)}), (\widehat{A}_u^{(m)})^T(\widehat{\theta}_S; \widehat{\theta}_L^{(m)})) \\
&= Cov((\widehat{B}_v^{(m)})^T(\widehat{\theta}_S; f(\widehat{\theta}_L^{(m)})), (\widehat{B}_u^{(m)})^T(\widehat{\theta}_S; f(\widehat{\theta}_L^{(m)}))), \\
&\to Cov((\widehat{B}_v^{(\infty)})^T(\widehat{\theta}_S; 0), (\widehat{B}_u^{(\infty)})^T(\widehat{\theta}_S; 0)),
\end{aligned}$$

using that, by linearity of f and independence of $\widehat{\theta}_L^{(m)}$ and $\xi_L^{(m)}$, $Var(f(\widehat{\theta}_L^{(m)})) \leq Var(f(\widehat{\theta}_L^{(m)} + \xi_L^{(m)})) = Var(\chi_L^m) \to 0$, and also, $\widehat{B}_u^{(m)} \to \widehat{B}_u^{(\infty)}$ and $\widehat{B}_v^{(m)} \to \widehat{B}_v^{(\infty)}$.

In addition, for any vector $\widetilde{\theta}_S$ of the same dimensionality as θ_S, we can write

$$(\widehat{B}_v^{(\infty)})^T(\widetilde{\theta}_S; 0) = E[v|\theta_M = 0, \theta_L = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S] = (\widehat{A}_v^{(alt)})^T \widetilde{\theta}_S$$

and

$$(\widehat{B}_u^{(\infty)})^T(\widetilde{\theta}_S; 0) = E[u|\theta_M = 0, \theta_L = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S] = (\widehat{A}_u^{(alt)})^T \widetilde{\theta}_S.$$

Thus,

$$Cov((\widehat{B}_v^{(\infty)})^T(\widehat{\theta}_S; 0), (\widehat{B}_u^{(\infty)})^T(\widehat{\theta}_S; 0)) = Cov((\widehat{A}_v^{(alt)})^T \widehat{\theta}_S, (\widehat{A}_u^{(\infty)})^T \widehat{\theta}_S)) = (\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)},$$

which allows to get Limit (OA.12).

Auxiliary Result (f). Next, we show that

$$\lim_{m \to \infty} Var((\widehat{A}_u^{(m)})^T(\theta_S; \theta_L + \xi_L^{(m)}) - u|\theta_M) = Var((\widehat{C}_u)^T(\theta_S; \theta_L) - u|\theta_M). \tag{OA.13}$$

Since $\xi_L^{(m)}$ is independent of $(\theta_S; \theta_L)$, u, and θ_M, we have

$$Var((\widehat{A}_u^{(m)})^T(\theta_S; \theta_L + \xi_L^{(m)}) - u|\theta_M) = Var((\widehat{A}_u^{(m)})^T(\theta_S; \theta_L) - u|\theta_M) + Var((\widehat{A}_u^{(m)})^T(0; \xi_L^{(m)})).$$

In Auxiliary Result (d) we have shown in Limit (OA.11) that $Var((\widehat{A}_v^{(m)})^T(0;\xi_L^{(m)})) \to 0$, and similarly we have $Var((\widehat{A}_u^{(m)})^T(0;\xi_L^{(m)})) \to 0$. It remains to show that

$$\lim_{m\to\infty} Var((\widehat{A}_u^{(m)})^T(\theta_S;\theta_L) - u|\theta_M) = Var((\widehat{C}_u)^T(\theta_S;\theta_L) - u|\theta_M). \tag{OA.14}$$

By the same argument as in Auxiliary Result (d), for any vector $\widetilde{\theta}_S$ of the same dimension as θ_S, and any vector $\widetilde{\theta}_L$ of the same dimension as θ_L,

$$\begin{aligned}(\widehat{B}_u^{(m)})^T(\widetilde{\theta}_S;\widetilde{\theta}_L) &= E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = \widetilde{\theta}_L] \\ &= E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta'_L + \chi_L^{(m)} = f(\widetilde{\theta}_L)],\end{aligned}$$

and since the random vector $(\theta_M, \theta_S + \widehat{\theta}_S, \theta'_L)$ is not degenerate, and $Var(\chi_L^{(m)}) \to 0$, we have the limit

$$\begin{aligned}&\lim_{m\to\infty} E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta'_L + \chi_L^{(m)} = f(\widetilde{\theta}_L)] \\ &\qquad = E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta'_L = f(\widetilde{\theta}_L)].\end{aligned}$$

Note that, since $\theta''_L = 0$, $f(\theta_L) = \theta'_L$. Thus, for any realization $\widetilde{\theta}_L$ of θ_L,

$$\begin{aligned}&E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta'_L = f(\widetilde{\theta}_L)] \\ &\qquad = E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = \widetilde{\theta}_L] = (\widehat{C}_u)^T(\widetilde{\theta}_S;\widetilde{\theta}_L).\end{aligned}$$

Therefore Limit (OA.14) obtains.

Auxiliary Result (g). This result is an equality. We show that

$$Var((\widehat{C}_u)^T(\theta_S;\theta_L) - u|\theta_M) = Var((\widehat{C}_u)^T(\theta_S;0) - u|\theta_M, \theta_L).$$

For any vectors $\widetilde{\theta}_M$, $\widetilde{\theta}_S$, $\widetilde{\theta}_L$, let

$$\eta(\widetilde{\theta}_M, \widetilde{\theta}_S, \widetilde{\theta}_L) = u - E[u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \widetilde{\theta}_L = \widetilde{\theta}_L].$$

Note that η is linear in its arguments. Also, since $\eta(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L)$ is the residual of the projection of u on $(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L)$, it is independent of the latter random vector. Therefore,

$$\begin{aligned}Var(u - (\widehat{C}_u)^T(\theta_S;\theta_L)|\theta_M) &= Var(\eta(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L) + (\widehat{C}_u)^T(\widehat{\theta}_S;0)|\theta_M) \\ &= Var(\eta(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L) + (\widehat{C}_u)^T(\widehat{\theta}_S;0)|\theta_M, \theta_L) \\ &= Var(\eta(0, \theta_S + \widehat{\theta}_S, 0) + (\widehat{C}_u)^T(\widehat{\theta}_S;0)|\theta_M, \theta_L) \\ &= Var(u - (\widehat{C}_u)^T(\theta_S + \widehat{\theta}_S;0) + (\widehat{C}_u)^T(\widehat{\theta}_S;0)|\theta_M, \theta_L) \\ &= Var(u - (\widehat{C}_u)^T(\theta_S;0)|\theta_M, \theta_L).\end{aligned}$$

6.3 Main Body of the Proof

We are now ready to prove Theorem 3. The proof proceeds in three steps.

Step 1. First let us focus on the linear equilibrium in market (m). Following Theorem 1, finding the linear equilibrium is equivalent to solving the quadratic equation

$$c^{(m)}(\beta_D^{(m)})^2 + b^{(m)}\beta_D^{(m)} + a^{(m)} = 0,$$

where

$$\begin{aligned}
a^{(m)} &= -(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_v^{(m)}, \\
b^{(m)} &= (\widehat{A}_v^{(m)})^T \left(2\widehat{\Sigma}_{diag}^{(m)} + \widehat{\Lambda}^{(m)}\right) \widehat{A}_u^{(m)} - Cov(u, v|\theta_M), \\
c^{(m)} &= Var((\widehat{A}^{(m)})_u^T(\theta + \xi^{(m)}) - u|\theta_M).
\end{aligned}$$

Similarly, there exists a unique linear equilibrium of the alternative market, and inding the linear equilibrium is equivalent to solving the quadratic equation

$$c^{(alt)}(\beta_D^{(alt)})^2 + b^{(alt)}\beta_D^{(alt)} + a^{(alt)} = 0,$$

where

$$\begin{aligned}
a^{(alt)} &= -(\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_v^{(alt)}, \\
b^{(alt)} &= (\widehat{A}_v^{(alt)})^T \left(2\widehat{\Sigma}_{diag}^{(alt)} + \widehat{\Lambda}^{(alt)}\right) \widehat{A}_u^{(alt)} - Cov(u, v|\theta_M, \theta_L), \\
c^{(alt)} &= Var((\widehat{A}^{(alt)})_u^T \theta_S - u|\theta_M, \theta_L).
\end{aligned}$$

The equilibrium price in market (m) is

$$\begin{aligned}
p^{(m)} &= (\beta_M^{(m)})^T\theta_M + \beta_D^{(m)}\left((\alpha^{(m)})^T\theta^{(m)} + u\right) \\
&= (\beta_M^{(m)})^T\theta_M + \beta_D^{(m)}\left((\alpha_S^{(m)})^T\theta_S + (\alpha_L^{(m)})^T(\theta_L^{(m)} + \xi_L^{(m)}) + u\right),
\end{aligned}$$

where we "decompose" the vector of coefficients $\alpha^{(m)}$ as $\alpha^{(m)} = \left(\alpha_S^{(m)}; \alpha_L^{(m)}\right)$.

The equilibrium price in the alternative market is

$$\begin{aligned}
p^{(alt)} &= (\beta_M^{(alt)})^T\theta_M^{(alt)} + \beta_D^{(alt)}\left((\alpha^{(alt)})^T\theta_S + u\right) \\
&= (\beta_{M,M}^{(alt)})^T\theta_M + (\beta_{M,L}^{(alt)})^T\theta_L + \beta_D^{(alt)}\left((\alpha^{(alt)})^T\theta_S + u\right),
\end{aligned}$$

where $\theta_M^{(alt)} = (\theta_M; \theta_L)$ and $\beta_M^{(alt)}$ is "decomposed" as $\beta_M^{(alt)} = (\beta_{M,M}^{(alt)}; \beta_{M,L}^{(alt)})$.

We will show in Step 2 that $\beta_D^{(m)} \to \beta_D^{(alt)}$, and then in Step 3 we will show that $\beta_M^{(m)} \to \beta_{M,M}^{(alt)}$, $\beta_D^{(m)}\alpha_L^{(m)} \to \beta_{M,L}^{(alt)}$, and $\beta_D^{(m)}\alpha_S^{(m)} \to \beta_D^{(alt)}\alpha^{(alt)}$. By the same argument as in Step 3 of the proof of the special case of Theorem 3 in Appendix C of the main body of the paper, showing these four convergence results is sufficient to prove the statement of the general case of Theorem 3.

Step 2. First, we show that the coefficients of the quadratic equation that $\beta_D^{(m)}$ satisfies converge to those of the quadratic equation that $\beta_D^{(alt)}$ satisfies. As the coefficient on $(\beta_D^{(alt)})^2$ in the latter equation is positive (as shown in Step 5 on the proof of Theorem 1 in Appendix A of the main body of the paper), this convergence implies that $\beta_D^{(m)}$ converges to $\beta_D^{(alt)}$.

Step 2(a). We first show that $a^{(m)} \to a^{(alt)}$, which is a direct consequence of Auxiliary Result (d) when observing that

$$a^{(m)} = -(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_v^{(m)} = -Var((\widehat{A}_v^{(m)})^T (\widehat{\theta}_S; \widehat{\theta}_L^{(m)}))$$

and

$$a^{(alt)} = -(\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_v^{(alt)} = -Var((\widehat{A}_v^{(alt)})^T \widehat{\theta}_S).$$

Step 2(b). Next, we show that $b^{(m)} \to b^{(alt)}$. We have

$$\begin{aligned}
b^{(m)} &= (\widehat{A}_v^{(m)})^T \left(2\widehat{\Sigma}_{diag}^{(m)} + \widehat{\Lambda}^{(m)}\right) \widehat{A}_u^{(m)} - Cov(u, v|\theta_M) \\
&= 2(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} + (\widehat{A}_v^{(m)})^T \widehat{\Lambda}^{(m)} \widehat{A}_u^{(m)} - Cov(u, v|\theta_M) \\
&= 2(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} \\
&\quad + Cov((\widehat{A}_v^{(m)})^T (\theta + \xi^{(m)} + \widehat{\theta}^{(m)}), (\widehat{A}_u^{(m)})^T (\theta + \xi^{(m)} + \widehat{\theta}^{(m)})) - Cov(u, v|\theta_M) \\
&= 2(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} - Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)}).
\end{aligned}$$

Similarly, we have

$$b^{(alt)} = 2(\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)} - Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L).$$

That $b^{(m)} \to b^{(alt)}$, is implied by

$$(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} \to (\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)},$$

and

$$\lim_{m \to \infty} Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)}) = Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L),$$

owing to Auxiliary Result (e) and (a), respectively.

Step 2(c). Finally, that $c^{(m)} \to c^{(alt)}$ comes from a successive application of Auxiliary Result (f) then (g).

Step 3. Step 2 of the current proof showed that $\beta_D^{(m)} \to \beta_D^{(alt)}$.

We now show that $\beta_M^{(m)} \to \beta_{M,M}^{(alt)}$, $\beta_D^{(m)} \alpha_L^{(m)} \to \beta_{M,L}^{(alt)}$, and $\beta_D^{(m)} \alpha_S^{(m)} \to \beta_D^{(alt)} \alpha^{(alt)}$.

First, for any vector $\widetilde{\theta}_S$ of the same dimension as θ_S,

$$\begin{aligned}
\beta_D^{(m)} (\alpha^{(m)})^T (\widetilde{\theta}_S; 0) &= \beta_D^{(m)} \left((\widehat{A}_v^{(m)})^T / \beta_D^{(m)} - (\widehat{A}_u^{(m)})^T \right) (\widetilde{\theta}_S; 0) \\
&= E[v - \beta_D^{(m)} u | \theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0],
\end{aligned}$$

and

$$\begin{aligned}
\beta_D^{(alt)} (\alpha^{(alt)})^T \widetilde{\theta}_S &= \beta_D^{(alt)} \left((\widehat{A}_v^{(alt)})^T / \beta_D^{(alt)} - (\widehat{A}_u^{(alt)})^T \right) \widetilde{\theta}_S \\
&= E[v - \beta_D^{(alt)} u | \theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0]
\end{aligned}$$

We have $\beta_D^{(m)} \to \beta_D^{(alt)}$, and by Auxiliary Result (b) and (c) respectively,

$$\lim_{m\to\infty} E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] = E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0]$$

and

$$\lim_{m\to\infty} E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] = E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0].$$

It follows that $\beta_D^{(m)}\alpha^{(m)} \to \beta_D^{(alt)}\alpha^{(alt)}$.

Next, we have the equality

$$\beta_M^{(m)} = \Sigma_{MM}^{-1}\left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v^{(m)}\right) - \beta_D^{(m)}\Sigma_{MM}^{-1}\left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u^{(m)}\right).$$

hence, for any $\widetilde{\theta}_M$,

$$(\beta_M^{(m)})^T\widetilde{\theta}_M = E[v - \beta_D^{(m)}u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0]. \tag{OA.15}$$

Similarly, as $\beta_D^{(m)} \to \beta_D^{(alt)}$, and by Auxiliary Result (b) and (c),

$$\lim_{m\to\infty} E[v|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] = E[v|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = 0]$$

and

$$\lim_{m\to\infty} E[u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] = E[u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = 0],$$

we get

$$(\beta_M^{(m)})^T\widetilde{\theta}_M \to E[v - \beta_D^{(alt)}u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = 0]. \tag{OA.16}$$

Finally, for any $\widetilde{\theta}_L$ realization of θ_L, we have

$$\beta_D^{(m)}(\alpha_L^{(m)})^T\widetilde{\theta}_L \to E[v - \beta_D^{(m)}u|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L = \widetilde{\theta}_L].$$

Thus,

$$(\beta_M^{(m)}; \beta_D^{(m)}\alpha_L^{(m)})^T(\widetilde{\theta}_M; \widetilde{\theta}_L) \to E[v - \beta_D^{(alt)}u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = \widetilde{\theta}_L],$$

and we observe that analogously to the expression for $(\beta_M^{(\infty)})^T\widetilde{\theta}_M$, we also have

$$(\beta_M^{(alt)})^T(\widetilde{\theta}_M; \widetilde{\theta}_L) = E[v - \beta_D^{(alt)}u|\theta_S + \widehat{\theta}_S = 0, (\theta_M; \theta_L) = (\widetilde{\theta}_M; \widetilde{\theta}_L)].$$

Thus $\beta_M^{(m)} \to \beta_{M,M}^{(alt)}$ and $\beta_D^{(m)}\alpha_L^{(m)} \to \beta_{M,L}^{(alt)}$. Combining those convergence results, and applying the same argument as in Step 3 of the proof of the special case of Theorem 3 in Appendix C of the main body of the paper, we conclude the proof.

7 Beauty Contest Games

In this section, we show that our information aggregation results do not extend to Beauty Contest games, even though such games share many of the features of the main model of our paper and of the model of Cournot competition (normally distributed signals, linear best responses, and the uniqueness of linear equilibrium). We first derive a closed-form linear equilibrium for a family of Beauty Contest games, and then use that characterization to illustrate information non-aggregation

in a specific example.

Our setup is in the general Beauty Contest framework of Morris and Shin (2002a), but with some key differences. First, we consider a game with a finite number of players (instead of a continuum), and then take a limit of such (still finite) games as the number of players becomes large. Second, we allow for an arbitrary, potentially asymmetric variance-covariance matrix of players' multidimensional signals (instead of assuming that players' signals are single-dimensional and symmetrically distributed).[5] These generalizations allow us to consider versions of Beauty Contest games with the same types of complex information structures as in the main model of our paper. In particular, in Example OA.5, we assume that some players are better informed than others.

Our formal model is as follows. There are $n \geq 2$ players, $i = 1, \ldots, n$. A "state of the world" $v \in \mathbb{R}$ is drawn at random. Each player i privately observes a signal $\theta_i \in \mathbb{R}^{k_i}$. Vector $(\theta_1; \ldots; \theta_n; v)$ is drawn from the normal distribution with mean $(0; \ldots; 0; \overline{v})$ and variance-covariance matrix Ω. As before, without loss of generality we assume that for each i, $Var(\theta_i)$ is full rank. We do not impose any other restrictions on matrix Ω.

After observing the signal, each player i independently chooses action $a_i \in \mathbb{R}$. Subsequently, the state of the world v is revealed, and each player i receives the payoff

$$\pi_i = -w(a_i - v)^2 - (1 - w)\left(a_i - \overline{a}_{-i}\right)^2,$$

where

$$\overline{a}_{-i} = \frac{\sum_{j \neq i} a_j}{n - 1}$$

and $w \in (0, 1)$.

As before, we focus on linear equilibria of the form $a_i(\theta_i) = \alpha_i^T \theta_i + \delta_i$.

Proposition OA.5 *In the Beauty Contest game, there exists a unique linear equilibrium.*

Proof. Fix some player i, and suppose every player $j \neq i$ plays according to the strategy $a_j(\theta_j) = \alpha_j^T \theta_j + \delta_j$, with $\alpha_j \in \mathbb{R}^{k_j}$ and $\delta_j \in \mathbb{R}$.

The expected payoff of player i is concave in a_i, and the best response is pinned down by the first-order condition:

$$-2w(a_i - E[v|\theta_i]) - 2(1 - w)(a_i - E[\overline{a}_{-i}|\theta_i]) = 0.$$

Thus,

$$a_i(\theta_i) = wE[v|\theta_i] + (1 - w)E[\overline{a}_{-i}|\theta_i],$$

with

$$E[\overline{a}_{-i}|\theta_i] = \frac{\sum_{j \neq i} \delta_j}{n - 1} + \frac{\sum_{j \neq i} \alpha_j^T E[\theta_j|\theta_i]}{n - 1}.$$

[5] See also Section 3 of Morris and Shin (2002b) for a two-player version of the Beauty Contest setting in which the signals have a general correlation strucutre, as well as the discussions in Bergemann and Morris (2013) on finite-player analogues of continuum-player quadratic economies (of which the Beauty Contest setting is a particular case).

Next,

$$E[\theta_j|\theta_i] = \Sigma_{ji}\Sigma_{ii}^{-1}\theta_i,$$
$$E[v|\theta_i] = \overline{v} + \Sigma_{vi}\Sigma_{ii}^{-1}\theta_i,$$

and so

$$E[\overline{a}_{-i}|\theta_i] = \frac{\sum_{j\neq i}\delta_j}{n-1} + \frac{\sum_{j\neq i}\alpha_j^T\Sigma_{ji}\Sigma_{ii}^{-1}}{n-1}\theta_i.$$

Therefore, we have

$$a_i(\theta_i) = w\overline{v} + w\Sigma_{vi}\Sigma_{ii}^{-1}\theta_i + (1-w)\sum_{j\neq i}\frac{\delta_j}{n-1} + (1-w)\left(\sum_{j\neq i}\frac{1}{n-1}\alpha_j^T\Sigma_{ji}\Sigma_{ii}^{-1}\right)\theta_i,$$

and so the best response of player i is linear: $a_i(\theta) = \alpha_i^T\theta_i + \delta_i$, with

$$\delta_i = w\overline{v} + (1-w)\sum_{j\neq i}\frac{\delta_j}{n-1}, \tag{OA.17}$$

$$\alpha_i^T = w\Sigma_{vi}\Sigma_{ii}^{-1} + (1-w)\sum_{j\neq i}\frac{1}{n-1}\alpha_j^T\Sigma_{ji}\Sigma_{ii}^{-1}. \tag{OA.18}$$

Therefore, all linear equilibria are characterized by vectors $\alpha_1, \ldots, \alpha_n$ and numbers $\delta_1, \ldots, \delta_n$, such that for all i, equations (OA.17) and (OA.18) are satisfied.

We immediately get $\delta_i = \overline{v}$ for all i. For vector $\alpha = (\alpha_1; \ldots; \alpha_n)$, multiplying equation (OA.18) by Σ_{ii} on the right, and combining the resulting equations for all i, we get

$$\alpha^T\Sigma_{diag} = w\Sigma_{v\theta} + \frac{1-w}{n-1}\alpha^T\left(\Sigma_{\theta\theta} - \Sigma_{diag}\right),$$

where Σ_{diag} is defined as before. Solving for α, we get[6]

$$\alpha = (n-1)w\Big((n-w)\Sigma_{diag} - (1-w)\Sigma_{\theta\theta}\Big)^{-1}\Sigma_{\theta v}.$$

■

We can now present a simple example of information non-aggregation in a large Beauty Contest game. In fact, the example has an even more striking feature: half of the agents directly observe the state of the world, without any noise. Yet in equilibrium, there is no information aggregation, and even the perfectly informed agents distort their actions relative to their signals.

[6]To see that matrix $(n-w)\Sigma_{diag} - (1-w)\Sigma_{\theta\theta}$ is positive definite (and thus invertible), note first that it is equal to $((n-w) - n(1-w))\Sigma_{diag} + (1-w)(n\Sigma_{diag} - \Sigma_{\theta\theta})$. The first part of the sum, $((n-w) - n(1-w))\Sigma_{diag}$, is positive definite, because Σ_{diag} is positive definite and $(n-w) - n(1-w) = (n-1)w > 0$. The second part, $(1-w)(n\Sigma_{diag} - \Sigma_{\theta\theta})$, is positive semi-definite. To see that, note first that $(1-w) > 0$. Next, to see that matrix $(n\Sigma_{diag} - \Sigma_{\theta\theta})$ is positive semi-definite, take any vector x of the same dimensionality as θ. We need to show that $x^T(n\Sigma_{diag} - \Sigma_{\theta\theta})x \geq 0$, or equivalently, that $nx^T\Sigma_{diag}x \geq x^T\Sigma_{\theta\theta}x$. Let ψ_1 be the scalar random variable equal to the dot product of θ_1 and the vector consisting of the first k_1 components of vector x, ψ_2 be the scalar random variable equal to the dot product of θ_2 and the vector consisting of the next k_2 components of vector x, and so on. Note that $\sum_{i=1}^n \psi_i = x^T\theta$ and that the inequality $nx^T\Sigma_{diag}x \geq x^T\Sigma_{\theta\theta}x$ can be equivalently rewritten as $n\sum_{i=1}^n Var(\psi_i) \geq Var(\sum_{i=1}^n \psi_i)$. The last inequality follows directly from the fact that for every $i, j \leq n$, we have $Cov(\psi_i, \psi_j) \leq \sqrt{Var(\psi_i)Var(\psi_j)} \leq (Var(\psi_i) + Var(\psi_j))/2$.

Example OA.5 *There are two groups of players, A and B. There are m players in each group. The state of the world v is distributed normally with mean zero and variance one. Every player in group A observes the true state of the world v. Every player j in group B observes signal $\theta_j^B = v+\epsilon_j$, where for each j, ϵ_j is distributed normally with mean zero and variance one, independently of v and of the signals of other players.*

By symmetry, in the unique linear equilibrium in Example OA.5, every player in group A chooses the same action $a^{(m)} = \alpha_A^{(m)} v$, and every player j in group B chooses action $b_j = \alpha_B^{(m)}(v+\epsilon_j)$ (since $\overline{v} = 0$, we also have $\delta = 0$.) From equation (OA.18) in the proof of Proposition OA.5, we have two equations that determine $\alpha_A^{(m)}$ and $\alpha_B^{(m)}$:

$$\alpha_A^{(m)} = w + \frac{1-w}{2m-1}\left((m-1)\alpha_A^{(m)} + m\alpha_B^{(m)}\right), \tag{OA.19}$$

$$\alpha_B^{(m)} = \frac{w}{2} + \frac{1-w}{2m-1}\left(m\frac{\alpha_A^{(m)}}{2} + (m-1)\frac{\alpha_B^{(m)}}{2}\right), \tag{OA.20}$$

solving which gives us

$$\alpha_A^{(m)} = \frac{w(4m-w-1)}{3mw+m-w(w+1)},$$
$$\alpha_B^{(m)} = \frac{w(2m-w)}{3mw+m-w(w+1)}.$$

In the limit, as $m \to \infty$, we get

$$\alpha_A^{(m)} \to \frac{4w}{3w+1},$$
$$\alpha_B^{(m)} \to \frac{2w}{3w+1}.$$

Thus, in the limit, even the players fully informed about the state of the world v (those in group A) end up choosing an action different from v, and the less informed players (those in group B) end up with actions that are, on average, even further away from v. Thus, in equilibrium, information dispersed among the agents does not get aggregated in any meaningful way.

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