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Appendix B from Hatfield et al., "Stability and Competitive Equilibrium in Trading Networks" (JPE, vol. 121, no. 5, p. 966)

In this appendix, we provide a proof of the maximal domain result (theorem 7) of Section III.C of the text. We also provide the counterexamples referenced in Section IV.B.

A. Proof of Theorem 7

THEOREM 7. Suppose that there are at least four agents and that the set of trades is exhaustive. Then if the preferences of some agent are not fully substitutable, there exist simple preferences for all other agents such that no stable outcome exists.

Proof. Suppose that the preferences of agent i are not fully substitutable and, in particular, that they fail the first part of the definition of the DFS condition with unique demands (Hatfield et al. 2013). Then there exist price vectors p and p' and trades ω and φ, with $b(\omega) = i$ and $p'_{-\omega} = p_{-\omega}$, $p'_{\omega} > p_{\omega}$, and $\{\Psi\} = D^i(p)$ and $\{\Psi'\} = D^i(p')$ and either

Case 1: $b(\varphi) = i$ and $\varphi \in \Psi$ but $\varphi \notin \Psi'$ or

Case 2: $s(\varphi) = i$ and $\varphi \notin \Psi$ but $\varphi \in \Psi'$.

In both cases, for each agent $g \neq i$, we let $K_g = |(\Omega_g \cap \Omega_i) \backslash \{\varphi, \omega\}|$ and partition $(\Omega_g \cap \Omega_i) \setminus \{\varphi, \omega\}$ into singleton sets $\Phi_g^1, \ldots, \Phi_g^{K_g}$. For any $k \in \{1, \ldots, K_g\}$ and any $\Xi \subseteq \Omega_g$, we let

$$u_g^k(\Xi \cap \Omega_g^k) = \begin{cases} -p_\xi & \text{if } (\Xi \cap \Phi_g^k) = \{\xi\} \text{ and } \xi \in \Omega_{g\to} \\ p_\xi & \text{if } (\Xi \cap \Phi_g^k) = \{\xi\} \text{ and } \xi \in \Omega_{\to g} \\ 0 & \text{if } (\Xi \cap \Phi_g^k) = \varnothing, \end{cases}$$

and let $u_g(\Xi) = \left(\sum_{k=1}^{Kg} u_g^k(\Xi \cap \Phi_g^k)\right) + \tilde{u}_g(\Xi \cap ([\Omega_g \backslash \Omega_i] \cup \{\varphi, \omega\}))$, where $\tilde{u}_g$ is as defined below.

Without further specification of agents' preferences, we can infer that whenever a stable outcome A exists, the outcome

$$\bar{A} = (A \backslash \{(\xi, q_\xi) : \xi \in [\tau(A_i) \backslash \{\varphi, \omega\}]; (\xi, q_\xi) \in A\}) \cup \{(\xi, p_\xi) : \xi \in [\tau(A_i) \backslash \{\varphi, \omega\}]\}$$

is also stable. To see this, note that if $(\xi, q_\xi) \in A$ for some $\xi \neq \varphi, \omega$ such that $b(\xi) = i$, then $q_\xi \geq p_\xi$. If $q_\xi > p_\xi$, then $\tilde{A} \equiv [A \backslash \{(\xi, q_\xi)\}] \cup \{(\xi, p_\xi)\}$ is also a stable match, as it is clearly individually rational as A is individually rational; and if Z was a blocking set for $\tilde{A}$, it would also be a blocking set for A. Similarly, if $(\xi, q_\xi) \in A$, $s(\xi) = i$, and $\xi \neq \varphi, \omega$, then $q_\xi \leq p_\xi$, and so $[A \backslash \{(\xi, q_\xi)\}] \cup \{(\xi, p_\xi)\}$ is also a stable match. The above claim now follows by induction.

It is helpful to define the marginal utility agent i obtains from having available trades in some set $\Phi \subseteq \{\varphi, \omega\}$ in addition to having trades in $\Omega_i \backslash \{\varphi, \omega\}$ at their prices according to the price vector p by

$$v^i(\Phi) \equiv \max_{\Xi \subseteq \Omega_i \backslash \{\varphi, \omega\}, \Phi' \subseteq \Phi} \left\{ u^i(\Xi \cup \Phi') + \sum_{\xi \in \Xi_{i\to}} p_\xi - \sum_{\xi \in \Xi_{\to i}} p_\xi \right\}.$$

We now proceed to discuss the two possible cases.

Case 1: $b(\varphi) = i$ and $\varphi \in \Psi$ but $\varphi \notin \Psi'$. Note that

$$v^i(\{\varphi, \omega\}) - v^i(\{\omega\}) > v^i(\{\varphi\}) - v^i(\varnothing) \geq 0, \tag{B1}$$

as otherwise we would have that $\varphi \in \Psi'$, as if

$$v^i(\{\varphi, \omega\}) - v^i(\{\omega\}) \leq v^i(\{\varphi\}) - v^i(\varnothing),$$

then i must demand φ at prices $(p_{-\omega}, p'_{\omega})$ as i demanded φ at prices p.

Now let $\hat{\varphi}$ and $\hat{\omega}$ be two trades such that $s(\varphi) = s(\hat{\varphi}) \equiv h$, $s(\omega) = s(\hat{\omega}) \equiv h'$, and $b(\hat{\varphi}) = b(\hat{\omega}) \equiv j \neq i$ (such a trade must exist as there are at least four agents and the set of trades is exhaustive).

We assume, first that $h \neq h'$. In this case, we set $\Phi_h^{K_h+1} = \{\varphi, \hat{\varphi}\}$ and, for any $\Xi \subseteq \Omega_h$, let

$$u_h^{K_h+1}(\Xi \cap \Phi_h^{K_h+1}) = \begin{cases} 0 & \text{if } |\Xi \cap \Phi_h^{K_h+1}| \leq 1 \\ -\infty & \text{if } |\Xi \cap \Phi_h^{K_h+1}| = 2. \end{cases}$$

Then, we let $\tilde{u}_h(\Xi \cap ([\Omega_h \setminus \Omega_i] \cup \{\varphi, \omega\})) \equiv u_h^{K_h+1}(\Xi \cap \Phi^{K_h+1})$. This construction ensures that the preferences of h are simple. The construction of simple preferences for h' is completely analogous to that for h.

If $h' = h$, then we first define $\Phi_h^{K_h+1}$ and $u_h^{K_h+1}$ as above. Then, we let $\Phi_h^{K_h+2} = \{\omega, \hat{\omega}\}$ and, for any $\Xi \subseteq \Omega_h$, let

$$u_h^{K_h+2}(\Xi \cap \Phi_h^{K_h+2}) = \begin{cases} 0 & \text{if } |\Xi \cap \Phi_h^{K_h+2}| \leq 1 \\ -\infty & \text{if } |\Xi \cap \Phi_h^{K_h+2}| = 2. \end{cases}$$

Then, we let

$$\tilde{u}_h(\Xi \cap ([\Omega_h \setminus \Omega_i] \cup \{\varphi, \omega\})) \equiv u_h^{K_h+1}(\Xi \cap \Phi_h^{K_h+1}) + u_h^{K_h+2}(\Xi \cap \Phi_h^{K_h+2}).$$

Finally, we finish the construction of the preferences of agent j. We define

$$\frac{2[v^i(\{\varphi, \omega\}) - v^i(\{\omega\})] + [v^i(\{\varphi\}) - v^i(\varnothing)]}{3} \equiv w(\varphi)$$

and

$$\frac{2[v^i(\{\varphi, \omega\}) - v^i(\{\varphi\})] + [v^i(\{\omega\}) - v^i(\varnothing)]}{3} \equiv w(\omega).$$

We let $\Phi_j^{K_j+1} = \{\hat{\varphi}, \hat{\omega}\}$ and, for any $\Xi \subseteq \Omega_j$, let

$$u_j^{K_j+1}(\Xi \cap \Phi_j^{K_j+1}) = \begin{cases} 0 & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \varnothing \\ w(\varphi) & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \{\hat{\varphi}\} \\ w(\omega) & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \{\hat{\omega}\} \\ -\infty & \text{if } |\Xi \cap \Phi_j^{K_j+1}| = 2, \end{cases}$$

and let $\tilde{u}_j(\Xi \cap ([\Omega_j \setminus \Omega_i] \cup \{\varphi, \omega\})) \equiv u_j^{K_j+1}(\Xi \cap \Phi_j^{K_j+1})$.

Then, by the above inequality, we must have

$$0 < w(\varphi) < v^i(\{\varphi, \omega\}) - v^i(\{\omega\}),$$
$$0 < w(\omega) < v^i(\{\varphi, \omega\}) - v^i(\{\varphi\}).$$

There are four subcases to consider to show that $\bar{A}$ cannot be stable.

Subcase 1: $\tau(\bar{A}) \cap \{\varphi, \omega\} = \varnothing$. If both $\hat{\varphi} \in \tau(\bar{A})$ and $\hat{\omega} \in \tau(\bar{A})$, then $\bar{A}$ is not individually rational for j. If $\hat{\varphi}, \hat{\omega} \notin \tau(\bar{A})$, then $\{(\hat{\varphi}, \epsilon)\}$ is a block for some sufficiently small ϵ. Hence, exactly one of $\hat{\varphi}$ and $\hat{\omega}$ is in $\tau(\bar{A})$. Suppose $(\hat{\varphi}, q_{\hat{\varphi}}) \in \bar{A}$ for some $q_{\hat{\varphi}} \in \mathbb{R}_{\geq 0}$. Individual rationality for j requires that

$$q_{\hat{\varphi}} \leq w(\varphi) < v^i(\{\varphi, \omega\}) - v^i(\{\omega\}).$$

But then

$$Z \equiv \{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in \Psi \setminus \tau(\bar{A}_i)\} \cup \{(\varphi, q_{\hat{\varphi}} + \epsilon), (\omega, \epsilon)\}$$

is a blocking set for some small $\epsilon > 0$. Note that (ω, ϵ) strictly increases by ϵ the utility of $s(\omega)$, no matter what other contracts $s(\omega)$ chooses. Similarly, for all $\xi \in \Psi \backslash \tau(\bar{A})$, $(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i})$ strictly increases by ϵ the utility of the agent other than i associated with this contract, no matter what other contracts that agent chooses. Agent $s(\hat{\varphi})$ chooses contract $(\varphi, q_{\hat{\varphi}} + \epsilon)$ and not $(\hat{\varphi}, q_{\hat{\varphi}})$, regardless of other contracts he chooses. Finally, the choice of agent i from $\bar{A} \cup Z$ is single-valued and includes Z, as the above inequality implies that if i chooses (ω, ϵ), he must also choose $(\varphi, q_{\hat{\varphi}} + \epsilon)$. We also have that $v^i(\{\omega\}) \geq v^i(\varnothing)$, implying that for ϵ small enough, i chooses both $(\varphi, q_{\hat{\varphi}} + \epsilon)$ and (ω, ϵ) from $\bar{A} \cup Z$, and hence i chooses all the contracts associated with trades in Ψ as a set containing those contracts is optimal at prices p.

If $(\hat{\omega}, q_{\hat{\omega}}) \in \bar{A}$ for some $q_{\hat{\omega}} \in \mathbb{R}_{\geq 0}$, we obtain a similar contradiction since individual rationality for j requires that $q_{\hat{\omega}} \leq w(\omega) < v^i(\{\varphi, \omega\}) - v^i(\{\varphi\})$.

Subcase 2: $(\varphi, q_\varphi) \in \bar{A}$ for some $q_\varphi \in \mathbb{R}_{\geq 0}$ and $\omega \notin \tau(\bar{A})$. In this case we must have $(\hat{\omega}, q_{\hat{\omega}}) \in \bar{A}$ for some $q_{\hat{\omega}} \in \mathbb{R}_{\geq 0}$, as otherwise $\{(\hat{\omega}, \epsilon)\}$ for some small $\epsilon > 0$ would be a blocking set since the incremental utility of j of signing $\hat{\omega}$ is $w(\omega) > 0$. Individual rationality for j requires

$$q_{\hat{\omega}} \leq w(\omega) < v^i(\{\varphi, \omega\}) - v^i(\{\omega\}).$$

Furthermore, we must have

$$q_\varphi \leq v^i(\{\varphi\}) - v^i(\varnothing)$$

as otherwise either $\bar{A}$ is not individually rational for i or

$$\{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in \Psi' \backslash \tau(\bar{A})\}$$

is a blocking set for $\epsilon > 0$ sufficiently small. However, these inequalities imply that

$$\{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in \Psi \backslash \tau(\bar{A})\} \cup \{(\omega, p_{\hat{\omega}} + \epsilon)\}$$

is a blocking set for some small $\epsilon > 0$.

Subcase 3: $(\omega, q_\omega) \in \bar{A}$ for some $q_\omega \in \mathbb{R}_{\geq 0}$ and $\varphi \notin \tau(\bar{A})$. The reasoning is analogous to that of the previous subcase.

Subcase 4: $\{(\varphi, q_\varphi), (\omega, q_\omega)\} \subseteq \bar{A}$ for some $q_\varphi, q_\omega \in \mathbb{R}_{\geq 0}$. It must be the case that

$$q_\varphi + q_\omega \leq v^i(\{\varphi, \omega\}) - v^i(\varnothing),$$

as otherwise $\bar{A}$ is not individually rational for i or

$$\{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in \Psi' \backslash \tau(\bar{A})\}$$

is a blocking set for some small $\epsilon > 0$. In order to prevent a block by $s(\varphi)$ and j (using $(\hat{\varphi}, q_\varphi + \epsilon)$ for some small $\epsilon > 0$), we must have $q_\varphi \geq w(\varphi)$. Similarly, to prevent a block by $s(\omega)$ and j, we must have $q_\omega \geq w(\omega)$. Simple algebra shows that $w(\varphi) + w(\omega) > v^i(\{\varphi, \omega\}) - v^i(\varnothing)$ is equivalent to the inequality (B1). Hence, we must have $q_\varphi + q_\omega > v^i(\{\varphi, \omega\}) - v^i(\varnothing)$, contradicting our earlier statement.

Case 2: $s(\varphi) = i$ and $\varphi \notin \Psi$ but $\varphi \in \Psi'$. Note that

$$v^i(\{\omega\}) - v^i(\varnothing) > v^i(\{\varphi, \omega\}) - v^i(\{\varphi\}), \tag{B2}$$

as otherwise we would have $\varphi \in \Psi'$, as if

$$v^i(\{\omega\}) - v^i(\{\varphi, \omega\}) \leq v^i(\varnothing) - v^i(\{\varphi\}),$$

then i must demand to sell φ at prices p if i demanded to sell φ at prices $(p_{-\omega}, p'_\omega)$.

As in case 1, we use the following conventions to simplify notation:

$$\frac{2[v^i(\{\omega\}) - v^i(\{\varphi, \omega\})] + [v^i(\varnothing) - v^i(\{\varphi\})]}{3} \equiv w(\varphi),$$
$$\frac{2[v^i(\{\omega\}) - v^i(\varnothing)] + [v^i(\{\varphi, \omega\}) - v^i(\{\varphi\})]}{3} \equiv w(\omega).$$

By (B2), we must have

$$0 < w(\varphi) < v^i(\{\omega\}) - v^i(\{\varphi, \omega\}),$$
$$0 < w(\omega) < v^i(\{\omega\}) - v^i(\varnothing).$$

We have to consider two subcases, depending on whether $s(\omega)$ is equal to $b(\varphi)$.

Subcase 1: $s(\omega) \neq b(\varphi)$.

Consider a trade $\hat{\omega}$ (which must exist by exhaustivity) for which $s(\hat{\omega}) = s(\omega) \equiv h$ and $b(\hat{\omega}) = b(\varphi) \equiv j$. Set $\Phi_h^{K_h+1} = \{\omega, \hat{\omega}\}$ and, for any $\Xi \subseteq \Omega_h$, let

$$u_h^{K_h+1}(\Xi \cap \Phi_h^{K_h+1}) = \begin{cases} 0 & \text{if } |\Xi \cap \Phi_h^{K_h+1}| \leq 1 \\ -\infty & \text{if } |\Xi \cap \Phi_h^{K_h+1}| = 2, \end{cases}$$

and let $\tilde{u}_h(\Xi \cap ([\Omega_h \setminus \Omega_i] \cup \{\varphi, \omega\})) \equiv u_h^{K_h+1}(\Xi \cap \Phi_h^{K_h+1})$. This construction ensures that the preferences of h are simple.

To extend the preferences of agent j, we set $\Phi_j^{K_j+1} = \{\omega, \hat{\omega}\}$ and, for any $\Xi \subseteq \Omega_j$, let

$$u_j^{K_j+1}(\Xi \cap \Phi_j^{K_j+1}) = \begin{cases} 0 & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \varnothing \\ w(\omega) & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \{\hat{\omega}\} \\ w(\varphi) & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \{\varphi\} \\ -\infty & \text{if } |\Xi \cap \Phi_j^{K_j+1}| = 2, \end{cases}$$

and let $\tilde{u}_j(\Xi \cap ([\Omega_j \setminus \Omega_i] \cup \{\varphi, \omega\})) \equiv u_j^{K_j+1}(\Xi \cap \Phi_j^{K_j+1})$.

Now, we show that no stable match can exist if preferences satisfy the above properties by distinguishing five possibilities.

a. $\{\varphi, \omega, \hat{\omega}\} \cap \tau(\bar{A}) = \varnothing$: In this case

$$\{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in (\Psi \backslash \tau(\bar{A})) \backslash \{\omega\}\} \cup \{(\omega, w(\omega))\}$$

is a blocking set for sufficiently small $\epsilon > 0$, as it increases the utility of each agent except i and $s(\omega)$ by at least ϵ, increases the utility of $s(\omega)$ by at least $w(\omega) > 0$, and increases the utility of i, since $w(\omega) < v^i(\{\omega\}) - v^i(\varnothing)$.

b. $(\hat{\omega}, q_{\hat{\omega}}) \in \bar{A}$ for some $q_{\hat{\omega}} \in \mathbb{R}_{\geq 0}$: Given our assumptions about preferences, individual rationality (for $s(\omega)$ and j) requires that $\varphi, \omega \notin \tau(\bar{A})$ and $q_{\hat{\omega}} \leq w(\omega)$. Since $w(\omega) < v^i(\{\omega\}) - v^i(\varnothing)$, this implies that

$$\{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in (\Psi \backslash \tau(\bar{A})) \backslash \{\omega\}\} \cup \{(\omega, q_{\hat{\omega}} + \epsilon)\}$$

is a blocking set for sufficiently small $\epsilon > 0$; this shows that we cannot have $\hat{\omega} \in \tau(\bar{A})$.

c. $(\omega, q_\omega) \in \bar{A}$ for some $q_\omega \in \mathbb{R}_{\geq 0}$ and $\varphi \notin \tau(\bar{A})$: In this case j obtains a utility of 0 under $\bar{A}$, and in order to prevent a block by $s(\omega)$ and j, we must have $q_\omega \geq w(\omega)$. But then

$$Z \equiv \{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in (\Psi' \backslash \tau(\bar{A})) \backslash \{\varphi\}\} \cup \{(\varphi, w(\varphi) - \epsilon)\}$$

is a blocking set for sufficiently small $\epsilon > 0$. To see this, note first that j chooses all of his contracts in the blocking set since each of these contracts increases his utility by $\epsilon > 0$. Note that the utility of i after the block is $v^i(\{\varphi\}) + w(\varphi) - |Z|\epsilon$, while his utility before the block is at most $v^i(\{\omega\}) - w(\omega)$. Subtracting the former expression from the latter, we obtain

$$\frac{[v^i(\{\omega\}) - v^i(\varnothing)] - [v^i(\{\omega,\varphi\}) - v^i(\{\varphi\})]}{3} - |Z|\epsilon,$$

which is positive for $\epsilon > 0$ sufficiently small.

d. $\{(\varphi, q_\varphi), (\omega, q_\omega)\} \subseteq \bar{A}$ for some $q_\varphi, q_\omega \in \mathbb{R}_{\geq 0}$: We must have that

$$q_\varphi \geq v^i(\{\omega\}) - v^i(\{\varphi,\omega\}),$$

since otherwise either $\bar{A}$ would not be individually rational for i or

$$\{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in \Psi \backslash \tau(\bar{A})\}$$

would be a blocking set. Similarly, we must have

$$q_\omega \leq v^i(\{\omega,\varphi\}) - v^i(\{\varphi\}).$$

We claim that $\{(\hat{\omega}, q_\omega + \epsilon)\}$ is a blocking set for $\epsilon > 0$ sufficiently small; it is clearly chosen by $s(\omega)$, and $b(\varphi)$ obtains a utility increase of at least

$$[w(\omega) - (q_\omega + \epsilon)] - [w(\varphi) - q_\varphi].$$

Substituting and using the price inequalities we just derived, we find that this expression is greater than or equal to

$$[v^i(\{\omega\}) - v^i(\varnothing)] - [v^i(\{\varphi,\omega\}) - v^i(\{\varphi\})] - \epsilon,$$

which is positive for ϵ sufficiently small.

e. $(\varphi, q_\varphi) \in \bar{A}$ for some $q_\varphi \in \mathbb{R}$ and $\omega \notin \tau(\bar{A})$: If $\{(\hat{\omega}, \epsilon)\}$ is not a blocking set for $\bar{A}$, then

$$\begin{aligned} q_\varphi &\leq w(\varphi) - w(\omega) \\ &\leq v^i(\varnothing) - v^i(\{\varphi,\omega\}). \end{aligned}$$

But then

$$Z \equiv \{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in (\Psi \backslash \tau(\bar{A})) \backslash \{\omega\}\} \cup \{(\omega, \epsilon)\}$$

is a blocking set for $\epsilon > 0$ sufficiently small, as $s(\omega)$ chooses a set containing $Z_{s(\omega)}$, and the utility of i before is at most

$$v^i(\{\varphi\}) + v^i(\varnothing) - v^i(\{\varphi,\omega\})$$

and choosing from $\bar{A} \cup Z$, i obtains

$$v^i(\{\omega\}) - |Z|\epsilon.$$

Subtracting the former expression from the latter, we obtain

$$[v^i(\{\omega\}) - v^i(\varnothing)] - [v^i(\{\varphi,\omega\}) - v^i(\{\varphi\})] - |Z|\epsilon,$$

which is positive for $\epsilon > 0$ sufficiently small.

Subcase 2: $s(\omega) = b(\varphi) \equiv j$.

To extend the preferences of agent j, set $\Phi_j^{K_j+1} = \{\omega, \varphi\}$ and, for any $\Xi \subseteq \Omega_j$, we let

$$u_j^{K_j+1}(\Xi \cap \Phi_j^{K_j+1}) = \begin{cases} 0 & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \varnothing \\ -w(\omega) & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \{\omega\} \\ w(\varphi) - w(\omega) & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \{\varphi, \omega\} \\ -\infty & \text{if } (\Xi \cap \Phi_j^{K_j+1}) = \{\varphi\}, \end{cases}$$

and let $\tilde{u}_j(\Xi \cap ([\Omega_j \setminus \Omega_i] \cup \{\varphi, \omega\})) \equiv u_j^{K_j+1}(\Xi \cap \Phi_j^{K_j+1})$.

There are four possibilities to consider to show that $\bar{A}$ cannot be stable.

a. $\tau(\bar{A}) \cap \{\varphi, \omega\} = \varnothing$: The argument from case 2.1(a) can be used to show that i and $j = s(\omega)$ have an incentive to block.

b. $(\omega, q_\omega) \in \bar{A}$ for some $q_\omega \in \mathbb{R}_{\geq 0}$: Suppose that $\varphi \notin \tau(\bar{A})$. Individual rationality for j requires that $q_\omega \geq w(\omega)$. The argument from case 2.1(c) can then be used to establish that

$$\{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in (\Psi' \backslash \tau(\bar{A})) \backslash \{\varphi\}\} \cup \{(\varphi, w(\varphi) - \epsilon)\}$$

is a blocking set for sufficiently small $\epsilon > 0$.

c. $\{(\varphi, q_\varphi), (\omega, q_\omega)\} \subseteq \bar{A}$ for some $q_\varphi, q_\omega \in \mathbb{R}_{\geq 0}$: We must have

$$q_\varphi \geq v^i(\{\omega\}) - v^i(\{\varphi, \omega\}),$$

as otherwise either $\bar{A}$ would not be individually rational for i or

$$\{(\xi, p_\xi + \epsilon \mathbf{1}_{b(\xi)=i} - \epsilon \mathbf{1}_{s(\xi)=i}) : \xi \in \Psi \backslash \tau(\bar{A})\}$$

would be a blocking set. Similarly, we must have

$$q_\omega \leq v^i(\{\omega, \varphi\}) - v^i(\{\varphi\}).$$

The first inequality implies that $\bar{A}$ cannot be individually rational for j since the incremental utility of signing φ on top of ω is

$$w(\varphi) - q_\varphi \leq w(\varphi) - [v^i(\{\omega\}) - v^i(\{\varphi, \omega\})] < 0.$$

d. $\varphi \in \tau(\bar{A})$ and $\omega \notin \tau(\bar{A})$: This clearly cannot be individually rational for j given that he obtains $-\infty$ utility if he signs φ but not ω.

The argument in the case that the preferences of i do not satisfy the second part of the DFS definition (Hatfield et al. 2013) is analogous to that presented above for the first part. QED

B. Examples Omitted from the Main Text

1. *An Economy with Stable Outcomes and Core Outcomes but No Stable Core Outcome*

There are two agents, i and j, and two trades, φ and χ, where $s(\varphi) = s(\chi) = i$ and $b(\varphi) = b(\chi) = j$. Agents' valuations are as follows.

Ψ	$\varnothing$	$\{\varphi\}$	$\{\chi\}$	$\{\varphi, \chi\}$
$u_i(\Psi)$	0	−2	−2	−6
$u_j(\Psi)$	0	0	0	7

The set of core outcomes is given by $\{\{(\varphi, p_\varphi), (\chi, p_\chi)\} : 6 \leq p_\varphi + p_\chi \leq 7\}$. However, the unique stable outcome is $\varnothing$: any outcome of the form $\{(\varphi, p_\varphi)\}$ or $\{(\chi, p_\chi)\}$ is not individually rational, and any outcome $\{(\varphi, p_\varphi), (\chi, p_\chi)\}$ can be individually rational only if $p_\varphi \geq 4$, $p_\chi \geq 4$, and $p_\varphi + p_\chi \leq 7$.

2. *An Economy with an Outcome That Is Stable and in the Core but Not Strongly Group Stable*

Let $I = \{i, j\}$, $\Omega = \{\varphi, \chi, \psi\}$, and $s(\varphi) = s(\chi) = s(\psi) = i$ and $b(\varphi) = b(\chi) = b(\psi) = j$. Furthermore, let agents' valuations be given as follows.

Ψ	$\varnothing$	$\{\varphi\}$	$\{\chi\}$	$\{\psi\}$	$\{\varphi, \chi\}$	$\{\varphi, \psi\}$	$\{\chi, \psi\}$	$\{\varphi, \chi, \psi\}$
$u_i(\Psi)$	0	0	−2	−2	−2	−2	−9	−20
$u_j(\Psi)$	0	2	1	1	3	3	2	15

In this case, any outcome of the form $\{(\varphi, p_\varphi)\}$ such that $0 \leq p_\varphi \leq 2$ is both stable and in the core. At the same time, any such outcome is not strongly group stable, as $\{(\chi, 6), (\psi, 6)\}$ constitutes a block.

3. *An Economy with Fully Substitutable Preferences in Which a Core Outcome Is Not Stable*

Consider again the setting of Section B.1 but let preferences be as follows.

Ψ	$\varnothing$	$\{\varphi\}$	$\{\chi\}$	$\{\varphi, \chi\}$
$u_i(\Psi)$	0	0	0	−3
$u_j(\Psi)$	0	5	5	9

In this case, $\{(\varphi, 2), (\chi, 2)\}$ is a core outcome but is not individually rational for agent i; he will choose to drop one of the two contracts. We therefore see that the set of imputed utilities of a core outcome may not correspond to the set of imputed utilities for any stable outcome: in this example, the payoff of agent i in any stable outcome is at least 3, while it is only 1 in the core outcome above.