Stability and Competitive Equilibrium in Trading Networks

John William Hatfield

University of Texas at Austin

Scott Duke Kominers

Harvard University and University of Chicago

Alexandru Nichifor

University of St Andrews

Michael Ostrovsky

Stanford University

Alexander Westkamp

Maastricht University

We introduce a model in which agents in a network can trade via bilateral contracts. We find that when continuous transfers are allowed and utilities are quasi-linear, the full substitutability of preferences is sufficient to guarantee the existence of stable outcomes for any underlying network structure. Furthermore, the set of stable outcomes is essentially equivalent to the set of competitive equilibria, and all stable outcomes are in the core and are efficient. By contrast, for any domain of preferences strictly larger than that of full substitutability, the existence of stable outcomes and competitive equilibria cannot be guaranteed.

We thank Susan Athey, Peter Cramton, Vincent P. Crawford, Drew Fudenberg, Albert Gallegos, Çağatay Kayı, Paul R. Milgrom, Benny Moldovanu, Alvin E. Roth, Michael Schwarz, William Thomson, Utku Ünver, E. Glen Weyl, many seminar and conference participants,

[*Journal of Political Economy*, 2013, vol. 121, no. 5]
© 2013 by The University of Chicago. All rights reserved. 0022-3808/2013/12105-0002$10.00

I. Introduction

The analysis of markets with heterogeneous agents and personalized prices has a long tradition in economics, which began with the canonical one-to-one assignment model of Koopmans and Beckmann (1957), Gale (1960), and Shapley and Shubik (1971). In this model, agents on one side of the market are matched to objects (or agents) on the other side, and each "match" generates a pair-specific surplus. Agents' utilities are quasi-linear in money, and arbitrary monetary transfers between the two sides are allowed. In this case, the efficient assignment—the one that maximizes the sum of all involved parties' payoffs—can be supported by the price mechanism as a competitive equilibrium outcome. Moreover, several solution concepts (competitive equilibrium, core, and pairwise stability) essentially coincide.

Crawford and Knoer (1981) extend the assignment model to a richer setting, in which heterogeneous firms form matches with heterogeneous workers. In the Crawford and Knoer setting, one firm can be matched to multiple workers, but each worker can be matched to at most one firm. Crawford and Knoer assume that preferences are separable across pairs; that is, the payoff from a particular firm-worker pair is independent of the other matches the firm forms. Crawford and Knoer do not rely on the linear programming duality theory used in previous work; instead, they use a modification of the deferred-acceptance algorithm of Gale and Shapley (1962) to prove their results, thus demonstrating a close link between the concepts of pairwise stability and competitive equilibrium. Kelso and Crawford (1982) then extend the previous results, showing that the restrictive assumption of the separability of preferences across pairs is inessential: it is enough that firms view workers as *substitutes* for each other.

In this paper, we show that the results from the two-sided models described above continue to hold in a much richer environment in which a network of heterogeneous agents can trade indivisible goods or services via bilateral contracts. Some agents can be involved in production, buying inputs from other agents, turning them into outputs at some cost, and

the anonymous referees, and the editor, Philip Reny, for helpful comments and suggestions. Most of this work was conducted while Hatfield was at Stanford University and Westkamp was at the University of Bonn. Hatfield and Nichifor appreciate the hospitality of Harvard Business School; Kominers appreciates the hospitality of the Harvard Program for Evolutionary Dynamics; and Westkamp appreciates the hospitality of the Stanford Graduate School of Business. Kominers thanks the National Science Foundation (grant CCF-1216095 and a graduate research fellowship), the Yahoo! Key Scientific Challenges Program, the John M. Olin Center (a Terence M. Considine Fellowship), the American Mathematical Society, and the Simons Foundation for support. Nichifor thanks the Netherlands Organisation for Scientific Research (grant VIDI-452-06-013) and the Scottish Institute for Research in Economics for support. Ostrovsky thanks the Alfred P. Sloan Foundation for support. Westkamp thanks the German Science Foundation for support.

then selling the outputs. We find that if all agents' preferences satisfy a suitably generalized substitutability condition, then stable outcomes and competitive equilibria are guaranteed to exist and are efficient. Moreover, in that case, the sets of competitive equilibria and stable outcomes are in a sense equivalent. These results apply to arbitrary trading networks and do not require any assumptions on the network structure such as two-sidedness or acyclicity.

In particular, our framework does not require a "vertical" network structure. Consider, for example, the market for used cars—a $300 billion market in the United States alone.[1] The participants in this market are the sellers, who no longer need their old cars; the buyers, who want to purchase used cars; and the car dealers, who buy, refurbish, and resell used cars. Sellers and buyers can trade directly with each other or they can trade with dealers. If all trade flowed in one direction (i.e., sellers sold cars only to dealers and buyers, and dealers sold cars only to buyers), this market would fit naturally into the vertical network model of Ostrovsky (2008). However, an important feature of the used car market is trade *among* dealers. For instance, of the 15.6 million used cars sold by franchised dealers in the United States in 2011, almost half (6.9 million) were sold "wholesale," that is, to dealers rather than to individual customers (NADA 2012).[2] Among independent dealers, more than two-thirds reported selling cars to other dealers (NIADA 2011).[3] Such trades are explicitly ruled out in the vertical network setting.[4] By contrast, the generality of our model—specifically, the accommodation of fully general trading network structures—makes it possible to study stable outcomes and competitive equilibria in settings like the used car market, where trade can flow not only "vertically" but also "horizontally."[5]

The presence of continuously transferable utility is essential for our results. Hatfield and Kominers (2012) show that without continuous trans-

[1] See http://www.census.gov/compendia/statab/2012/tables/12s1058.pdf, table 1058.

[2] Some of these interdealer trades may comprise cycles. Consider, e.g., a BMW dealer who receives a used Lexus as a trade-in. For this dealer, it may be more profitable to resell the traded-in car to a Lexus dealer instead of an individual customer because the Lexus dealer can have Lexus-trained mechanics inspect and refurbish the car, assign it a "Certified Pre-Owned" status, provide a Lexus-backed warranty, and offer other valuable services and add-ons that the BMW dealer cannot provide. Likewise, a Lexus dealer may prefer to sell a traded-in BMW to a BMW dealer instead of an individual customer.

[3] "Franchised" dealers are typically associated with a specific car manufacturer or a small number of manufacturers and sell both new and used cars. "Independent" dealers sell only used cars. Trade among dealers includes transactions that take place at wholesale auctions, where only dealers are allowed to purchase cars (Tadelis and Zettelmeyer 2011; Larsen 2013), and direct dealer-to-dealer transactions (NIADA 2011).

[4] See Sec. IV.C for a formal discussion of the restrictions imposed in the prior literature.

[5] Other examples of markets in which horizontal trade and subcontracting are common include reinsurance and securities underwriting, construction, and materials fabrication (Kamien, Li, and Samet 1989; Spiegel 1993; Baake, Oechssler, and Schenk 1999; Gale, Hausch, and Stegeman 2000; Patrik 2001; Powers and Shubik 2001; Marion 2013).

fers, in markets that lack a vertical structure, stable outcomes may not exist. Even in vertical trading networks, without continuously transferable utility, stable outcomes are not guaranteed to be Pareto efficient (Blair 1988; Westkamp 2010). Another key assumption, which is also essential for the existence of stable outcomes in the previous matching literature, is the substitutability of preferences: we prove a "maximal domain" theorem showing that if any agent's preferences are not substitutable, then substitutable preferences can be found for other agents such that neither competitive equilibria nor stable outcomes exist. We discuss the economic content of the substitutability assumption in Section II.B after formally defining it.[6]

In our model, contracts specify a buyer, a seller, provision of a good or service, and a monetary transfer. An agent may be involved in some contracts as a seller and in other contracts as a buyer. Agents' preferences are defined by cardinal utility functions over sets of contracts and are quasilinear with respect to the numeraire. To incorporate technological feasibility constraints (e.g., a baker cannot produce bread without buying any flour), we allow agents' utilities for certain production plans to be unboundedly negative. We say that preferences are *fully substitutable* if contracts are substitutes for each other in a generalized sense; that is, whenever an agent gains a new purchase opportunity, he becomes both less willing to make other purchases and more willing to make sales, and whenever he gains a new sales opportunity, he becomes both less willing to make other sales and more willing to make purchases. This intuitive substitutability condition has appeared in the literature on matching in vertical networks (Ostrovsky 2008; Westkamp 2010; Hatfield and Kominers 2012) and generalizes the classical notions of substitutability in two-sided settings (Kelso and Crawford 1982; Roth 1984; Hatfield and Milgrom 2005). Full substitutability is equivalent to the *gross substitutes and complements* condition of the literature on competitive equilibria in exchange economies with indivisible objects (Gul and Stacchetti 1999, 2000; Sun and Yang 2006, 2009). Full substitutability is also equivalent to the submodularity of the indirect utility function (Gul and Stacchetti 1999; Ausubel and Milgrom 2002).[7]

Our main results are as follows. We first show that when preferences are fully substitutable, competitive equilibria are guaranteed to exist. Our proof is constructive. Its key idea is to consider an associated two-sided many-to-one matching market in which "firms" are the agents and "workers" are the possible trades in the original economy. Fully substitutable utilities of the agents in the original economy give rise to substitutable (in the Kelso-Crawford sense) preferences of the firms in the associated two-

[6] In that section, we also argue that full substitutability is a natural assumption on the preferences of sellers, buyers, and dealers in the used car setting.

[7] The stated equivalences are shown in our companion paper (Hatfield et al. 2013).

sided market, and the equilibrium outcome in the associated market can be mapped back to a competitive equilibrium of the original economy.[8] While the construction of the associated market is conceptually natural, it involves several additional steps that deal with the potentially unbounded utilities in the original economy and ensure that the equilibrium of the associated economy is "full employment" (as this is required for mapping it back into an equilibrium of the original economy). Having established the existence of competitive equilibria, we then use standard techniques to demonstrate analogues of the first and second welfare theorems, as well as the lattice structure of the set of competitive equilibrium prices. While these properties are of independent interest, we also use them in proving some of our subsequent results.

We then turn to our key results establishing the connections between competitive equilibria and stable outcomes. First, we show that (even when preferences are not fully substitutable) any competitive equilibrium induces a stable outcome. The proof of this result is similar in spirit to the standard arguments showing that competitive equilibrium outcomes are in the core, but it is more subtle. Unlike the core, stability also rules out the possibility that agents may profitably recontract while maintaining some of their prior contractual relationships with other agents. Second, we prove a converse: under fully substitutable preferences, any stable outcome corresponds to a competitive equilibrium. These two results establish an essential equivalence between the two solution concepts under full substitutability. While this equivalence is analogous to a similar finding of Kelso and Crawford (1982) for two-sided many-to-one matching markets, it is more complex. In the setting of Kelso and Crawford, one can construct "missing" prices for unrealized trades simply by considering those trades one by one, because in that setting each worker can be employed by at most one firm. In our setting, that simple procedure would not work because each agent can be involved in multiple trades. To get around this difficulty, for a given stable outcome, we consider a new economy consisting of trades that are *not* part of the stable outcome and modified utilities that assume that the agents have access to the trades that *are* part of the stable outcome. We then show that preferences in the modified economy are fully substitutable and use our earlier results to establish the existence of a competitive equilibrium in the modified economy. Finally, we use the prices for the trades in the competitive equilibrium of the modified economy to construct a competitive equilibrium in the original economy.

[8] This technique is a generalization of the construction of Sun and Yang (2006), which maps an exchange economy with two classes of goods (with preferences satisfying the gross substitutes and complements condition over these two classes) to an exchange economy in which preferences satisfy the Kelso-Crawford substitutability condition. In Secs. IV.B and IV.C, we discuss in more detail the connection of our results with those of Sun and Yang.

Thus, fully substitutable preferences are sufficient for the existence of stable outcomes and competitive equilibria and for the essential equivalence of these two concepts. Our final main result establishes that full substitutability is also necessary, in the maximal domain sense: if any agent's preferences are not fully substitutable, then fully substitutable preferences can be found for other agents such that no stable outcome exists.[9]

After presenting our main results, we analyze the relationship between stability as defined in this paper and several other solution concepts. Generalizing the results of Shapley and Shubik (1971) and Sotomayor (2007), we show that all stable outcomes are in the core (although, in contrast to the basic one-to-one assignment model, the converse is not true in our setting). We then consider the strong group stability solution concept and show that, in contrast to the results of Echenique and Oviedo (2006) and Klaus and Walzl (2009) for matching markets without transfers, in our setting the set of stable outcomes coincides with the set of strongly group stable outcomes.[10]

Finally, we show that our model embeds the more common setting in which agents are indifferent over their trading partners. We introduce a condition on utilities formalizing this idea and show that, under this condition, a competitive equilibrium with "anonymous"—rather than personalized—prices always exists. Our framework also allows for a hybrid case, in which prices are personalized for some goods and anonymous for others.

The remainder of this paper is organized as follows. In Section II, we formalize our model. In Section III, we present our main results. In Sec-

[9] In the setting of two-sided many-to-one matching with transfers, Kelso and Crawford (1982) show that substitutability is sufficient for the existence of stable outcomes and competitive equilibria; Gul and Stacchetti (1999) and Hatfield and Kojima (2008) prove corresponding necessity results. In a setting in which two types of indivisible objects need to be allocated to consumers, Sun and Yang (2006) show that competitive equilibria are guaranteed to exist if consumers view objects of the same type as substitutes and view objects of different types as complements (see also Sec. IV.B). Sufficiency and necessity of fully substitutable preferences also obtain in settings of many-to-many matching with and without contracts (Roth [1984], Echenique and Oviedo [2006], Klaus and Walzl [2009], and Hatfield and Kominers [2013a] prove sufficiency results; Hatfield and Kojima [2008] and Hatfield and Kominers [2013a] prove necessity results) and matching in vertical networks (Ostrovsky [2008] and Hatfield and Kominers [2012] prove sufficiency; Hatfield and Kominers [2012] prove necessity). Substitutable preferences are sufficient for the existence of a stable outcome in the setting of many-to-one matching with contracts (Hatfield and Milgrom 2005) but are not necessary (Hatfield and Kojima 2008, 2010; Hatfield and Kominers 2013b). In work subsequent to our paper, Baldwin and Klemperer (2013) use the techniques of tropical geometry to obtain alternative proofs of the sufficiency and necessity of full substitutability for the existence of competitive equilibria in indivisible goods economies. They also use these techniques to explore more general classes of preferences.

[10] In the companion paper (Hatfield et al. 2013), we also consider chain stability, extending the definition of Ostrovsky (2008). While chain stability is logically weaker than stability, we show that the two concepts are equivalent when agents' preferences are fully substitutable. Hatfield and Kominers (2012) prove an analogous result for the setting of Ostrovsky.

tion IV, we study the relationships among competitive equilibria, stable outcomes, and other solution concepts. We present conclusions in Section V. Except where mentioned otherwise, the proofs of all results are presented in Appendix A.

II. Model

There is a finite set I of agents in the economy. These agents can participate in bilateral trades. Each trade ω is associated with a *buyer* $b(\omega) \in I$ and a *seller* $s(\omega) \in I$, with $b(\omega) \neq s(\omega)$. The set of possible trades, denoted Ω, is finite and exogenously given. The set Ω may contain multiple trades that have the same buyer and the same seller. For instance, a worker (seller) may be hired by a firm (buyer) in a variety of capacities with different job conditions and characteristics, and each possible type of job may be represented by a different trade. One firm may sell multiple units of a good (or several different goods) to another firm, with each unit represented by a separate trade. Furthermore, a firm may be the seller in one trade and the buyer in another trade with the same partner; formally, the set Ω can contain trades ω and ψ such that $s(\omega) = b(\psi)$ and $s(\psi) = b(\omega)$.[11]

It is convenient to think of a trade as representing the nonpecuniary aspects of a transaction between a seller and a buyer (although in principle it could include some "financial" terms and conditions as well). The purely financial aspect of a transaction associated with a trade ω is represented by a price p_ω; the complete vector of prices for all trades in the economy is denoted by $p \in \mathbb{R}^{|\Omega|}$. Formally, a *contract* x is a pair (ω, p_ω), with $\omega \in \Omega$ denoting the trade and $p_\omega \in \mathbb{R}$ denoting the price at which the trade occurs. The set of available contracts is $X \equiv \Omega \times \mathbb{R}$. For any set of contracts $Y \subseteq X$, we denote by $\tau(Y)$ the set of trades involved in contracts in Y:

$$\tau(Y) \equiv \{\omega \in \Omega : (\omega, p_\omega) \in Y \text{ for some } p_\omega \in \mathbb{R}\}.$$

For a contract $x = (\omega, p_\omega)$, we denote by $b(x) \equiv b(\omega)$ and $s(x) \equiv s(\omega)$ the buyer and seller associated with the trade ω of contract x. Consider any set of contracts $Y \subseteq X$. We denote by $Y_{\to i}$ the set of "upstream" contracts for i in Y, that is, the set of contracts in Y in which agent i is the buyer: $Y_{\to i} \equiv \{y \in Y : i = b(y)\}$. Similarly, we denote by $Y_{i\to}$ the set of "downstream" contracts for i in Y, that is, the set of contracts in Y in which agent i is the seller: $Y_{i\to} \equiv \{y \in Y : i = s(y)\}$. We denote by Y_i the set of contracts in Y in which agent i is involved as the buyer or the seller:

[11] Such a pair of trades constitutes a cycle of length 2; since the model places no restrictions on the structure of the set of trades, longer cycles may also be present in the economy. The incorporation of cycles into the model is what allows us to accommodate markets with horizontal trading relationships such as the used car market discussed in the introduction.

$Y_i \equiv Y_{\to i} \cup Y_{i\to}$. We use analogous notation to denote the subsets of trades associated with some agent i. We let $a(Y) \equiv \cup_{y \in Y}\{b(y), s(y)\}$ denote the set of agents involved in contracts in Y as buyers or sellers.

We say that the set of contracts Y is *feasible* if there is no trade ω and prices p_ω and $\hat{p}_\omega$ with $p_\omega \neq \hat{p}_\omega$ such that both contracts (ω, p_ω) and $(\omega, \hat{p}_\omega)$ are in Y; that is, a set of contracts is feasible if each trade is associated with at most one contract in that set. An *outcome* $A \subseteq X$ is a feasible set of contracts.[12] Thus, an outcome specifies which trades are executed and what the associated prices are but does not specify prices for trades that do not take place. An *arrangement* is a pair $[\Psi; p]$, where $\Psi \subseteq \Omega$ is a set of trades and $p \in \mathbb{R}^{|\Omega|}$ is a vector of prices for all trades in the economy. We denote by $\kappa([\Psi; p]) \equiv \cup_{\psi \in \Psi}\{(\psi, p_\psi)\}$ the set of contracts induced by the arrangement $[\Psi; p]$. Note that $\kappa([\Psi; p])$ is an outcome and that $\tau(\kappa([\Psi; p])) = \Psi$.

A. *Preferences*

Each agent i has a valuation function u_i over sets of trades $\Psi \subseteq \Omega_i$; we extend u_i to Ω by taking $u_i(\Psi) \equiv u_i(\Psi_i)$ for any $\Psi \subseteq \Omega$. The valuation u_i gives rise to a quasi-linear utility function U_i over sets of trades and the associated transfers. We formalize this in two different ways. First, for any outcome Y, we say that

$$U_i(Y) \equiv u_i(\tau(Y)) + \sum_{(\omega, p_\omega) \in Y_{i\to}} p_\omega - \sum_{(\omega, p_\omega) \in Y_{\to i}} p_\omega.$$

Second, for any arrangement $[\Psi; p]$, we say that

$$U_i([\Psi; p]) \equiv u_i(\Psi) + \sum_{\psi \in \Psi_{i\to}} p_\psi - \sum_{\psi \in \Psi_{\to i}} p_\psi.$$

Note that, by construction, $U_i([\Psi; p]) = U_i(\kappa([\Psi; p]))$.

We allow $u_i(\Psi)$ to take the value $-\infty$ for some sets of trades Ψ in order to incorporate various technological constraints.[13] However, we also as-

[12] In the literature on matching with contracts, the term "allocation" has been used to refer to a set of contracts. Unfortunately, the term "allocation" is also used in the competitive equilibrium literature to denote an assignment of goods, without specifying transfers. For this reason, to avoid confusion, we use the term "outcome" to refer to a feasible set of contracts.

[13] For instance, if agent i requires an input to produce the output associated with trade ω and cannot produce that output without that input, then $u_i(\{\omega\}) = -\infty$. Incorporating such constraints is essential for modeling economies with intermediate production. Note that such constraints are ruled out in the literature on exchange economies with indivisible goods (see, e.g., Bikhchandani and Mamer 1997; Gul and Stacchetti 1999), which assumes that every bundle of goods is acceptable to every agent.

sume that for all i, the outside option is finite: $u_i(\varnothing) \in \mathbb{R}$. That is, no agent is "forced" to sign any contracts at extremely unfavorable prices; he always has an outside option of completely withdrawing from the market at some potentially high but finite price.

The utility function U_i gives rise to both demand and choice correspondences. The *choice correspondence* of agent i given a set of contracts $Y \subseteq X$ is defined as the collection of sets of contracts maximizing the utility of agent i:

$$C_i(Y) \equiv \underset{Z \subseteq Y_i;\, Z \text{ is feasible}}{\arg\max}\ U_i(Z).$$

The *demand correspondence* of agent i given a price vector $p \in \mathbb{R}^{|\Omega|}$ is defined as the collection of sets of trades maximizing the utility of agent i under prices p:

$$D_i(p) \equiv \underset{\Psi \subseteq \Omega_i}{\arg\max}\ U_i([\Psi; p]).$$

Note that while the demand correspondence always contains at least one (possibly empty) set of trades, the choice correspondence may be empty-valued (e.g., if Y consists of all contracts with prices strictly between 0 and 1). If the set Y is finite, then the choice correspondence is also guaranteed to contain at least one set of contracts.

We can now introduce the *full substitutability* concept for our setting: When presented with additional contractual opportunities to purchase, an agent both rejects any previously rejected purchase opportunities and continues to choose any previously chosen sale opportunities. Analogously, when presented with additional contractual opportunities to sell, an agent rejects any previously rejected sale opportunities and continues to choose any previously chosen purchase opportunities. Formally, we define full substitutability in the language of sets and choices, adapting and merging the same-side substitutability and cross-side complementarity conditions of Ostrovsky (2008).

Definition 1. The preferences of agent i are *fully substitutable* if

1. for all sets of contracts Y, $Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{i\rightarrow} = Z_{i\rightarrow}$, and $Y_{\rightarrow i} \subseteq Z_{\rightarrow i}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $Y_{\rightarrow i} \setminus Y^*_{\rightarrow i} \subseteq Z_{\rightarrow i} \setminus Z^*_{\rightarrow i}$ and $Y^*_{i\rightarrow} \subseteq Z^*_{i\rightarrow}$;
2. for all sets of contracts Y, $Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{\rightarrow i} = Z_{\rightarrow i}$, and $Y_{i\rightarrow} \subseteq Z_{i\rightarrow}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $Y_{i\rightarrow} \setminus Y^*_{i\rightarrow} \subseteq Z_{i\rightarrow} \setminus Z^*_{i\rightarrow}$ and $Y^*_{\rightarrow i} \subseteq Z^*_{\rightarrow i}$.

In other words, the choice correspondence C_i is fully substitutable if (once attention is restricted to sets for which C_i is single-valued), when

the set of opportunities available to i on one side expands, i both rejects a (weakly) larger set of contracts on that side and selects a (weakly) larger set of contracts on the other side.

B. *Discussion of the Full Substitutability Condition*

While definition 1 is natural and intuitive, it does rule out some economically important cases. First, it rules out the possibility of large fixed costs, which, for example, may make an agent willing to sell several units of its product at a particular price p but unwilling to sell just one such unit at the same price. More generally, it rules out economies of scale and complementarities in production or consumption. (Of course, these cases are also ruled out by the usual Kelso-Crawford substitutability condition in two-sided markets.) In addition, the full substitutability condition rules out the possibility that an intermediary has aggregate capacity constraints while able to produce multiple types of output, each requiring a different type of input. For instance, suppose that agent i (a bakery) can make white or brown bread from white or brown flour, respectively. Suppose that i is profitably producing and selling white bread and gains an opportunity to sell brown bread profitably. If i is capacity constrained, he may shift some of his capacity from producing white bread to producing brown bread, thus buying less white flour (or perhaps not buying it at all). In this case, the preferences of agent i are not fully substitutable as the expansion of the set of options available to i on one side leads i to drop some of his contracts on the other side.[14] Note that our domain maximality result (theorem 7) implies that in all the cases in which preferences are not fully substitutable, the existence of stable outcomes and competitive equilibria cannot be guaranteed.

At the same time, the full substitutability condition holds for a variety of important classes of production and utility functions. The most straightforward case in which full substitutability holds is the case of homogeneous goods, with diminishing marginal utilities of consumption and increasing marginal costs of production. For example, suppose that some agents in the market participate only as consumers (they do not sell anything in the market), and their payoffs depend only on the number of units of the good that they purchase, with each additional unit being less valuable than the previous one. Some agents participate only as sellers (they do not buy anything in the market), and their production costs depend only on the number of units that they sell, with each additional unit being more expensive to produce than the previous one. Finally, some agents are intermediaries who both buy units of an input good and pro-

[14] We thank a referee for this example.

duce units of an output good. They require one unit of input to produce one unit of output and incur a manufacturing cost, which depends only on the number of units "transformed," with each additional unit being more expensive to "transform" than the previous one. In this economy, all preferences are fully substitutable. Full substitutability also holds in various generalizations of this model, incorporating, for example, heterogeneous transportation costs or the possibility that some intermediaries may derive utility from consuming some of the inputs or have the capability to produce some outputs without buying the corresponding inputs.

For a richer class of fully substitutable preferences that involves "substantively" heterogeneous goods, we return to the used car setting discussed in the introduction. Buyers and sellers of used cars typically want to trade at most one car; thus, their preferences trivially satisfy the full substitutability condition.[15] The preferences of dealers are more complex. Consider a dealer d. The dealer's goal is to maximize the difference between the prices at which he sells used cars and the amounts he pays to acquire and refurbish them. Formally, let Y be a set of contracts, representing the options available to dealer d. The set $Y_{\rightarrow d} \subseteq Y$ is the set of car offers available to dealer d, in which each element (φ, p_φ) specifies the characteristics of the offered car and its price. The set $Y_{d\rightarrow} \subseteq Y$ is the set of requests for cars available to dealer d, in which each element (ψ, p_ψ) specifies the characteristics of the requested car and its price. Note that these offers and requests can come from other dealers or from individual sellers or buyers.

Dealer d knows whether any given car offer φ and request ψ are compatible, that is, whether the characteristics of car offer φ match the characteristics of request ψ (ignoring prices).[16] The dealer also knows the cost $c_{\varphi,\psi}$ of preparing a given car φ for resale to satisfy a compatible re-

[15] Important exceptions are financial leasing companies selling off-lease vehicles and rental car agencies selling fleet vehicles (Tadelis and Zettelmeyer 2011; Larsen 2013). In both of these cases, sellers' payoffs are essentially additive across cars; hence, their preferences satisfy the full substitutability condition.

[16] For instance, a blue Toyota Camry of a particular year and mileage would be compatible with a request for a Toyota Camry with a matching year and mileage range but would not be compatible with a request for a blue Honda Accord or for a blue Camry with the "wrong" year or mileage range. Note that we do not require a buyer of a used car to have demand only for a specific make-model-year-mileage-option combination; a buyer's preferences can specify, e.g., that the value of a Toyota Camry to him is \$2,000 higher than the value of a Honda Accord with the same characteristics, or that each additional 1,000 miles on the car's odometer decreases that car's value by \$150. In other words, each request ψ is detailed enough that the buyer has the same value for any car that matches the request ψ, and the buyer's preferences are represented by a set of requests that he is indifferent over ("I am willing to pay \$15,000 for a Toyota Camry with such-and-such characteristics *or* \$14,500 for a Toyota Camry with so-and-so characteristics *or* \$13,000 for a Honda Accord with such-and-such characteristics *or* . . .").

quest ψ.[17] The dealer's objective is to match some of the car offers in $Y_{\to d}$ with some of the requests in $Y_{d\to}$ in a way that maximizes his profit, $\sum_{(\varphi,\psi)\in\mu}(p_\psi - p_\varphi - c_{\varphi,\psi})$, where μ denotes the set of compatible car offer-request pairs that the dealer selects.

Formally, define a matching, μ, as a set of pairs of trades (φ, ψ) such that φ is an element of $\Omega_{\to d}$ (i.e., a car available to dealer d), ψ is an element of $\Omega_{d\to}$ (i.e., a car request received by dealer d), φ and ψ are compatible, and each trade in Ω_d belongs to at most one pair in μ. Slightly abusing notation, let the cost of matching μ, $c(\mu)$, be equal to the sum of the costs of pairs involved in μ (i.e., $c(\mu) = \sum_{(\varphi,\psi)\in\mu} c_{\varphi,\psi}$).

For a set of trades $\Xi \subseteq \Omega_d$, let $M(\Xi)$ denote the set of matchings μ of elements of Ξ such that every element of Ξ belongs to exactly one pair in μ.[18] Then the valuation of dealer d over sets of trades $\Xi \subseteq \Omega_d$ is given by

$$u_d(\Xi) = \begin{cases} -\min_{\mu\in M(\Xi)} c(\mu) & \text{if } M(\Xi) \neq \varnothing \\ -\infty & \text{if } M(\Xi) = \varnothing; \end{cases}$$

that is, it is equal to the cost of the cheapest way of matching all car requests and offers in Ξ if such a matching is possible and is equal to $-\infty$ otherwise.[19] (Note that $u_d(\varnothing) = 0$.) The utility function of d over feasible sets of contracts is induced by valuation u_d in the standard way, formalized in the beginning of Section II.A.

PROPOSITION 1. The preferences of dealer d are fully substitutable.

For intuition, suppose that a new request (ψ, p_ψ) is added to the set of options Y available to dealer d (resulting in a new set of options $Z = Y \cup \{(\psi, p_\psi)\}$), and the dealer reoptimizes; denote the corresponding optimal choices by Y^* and Z^*. If the new request (ψ, p_ψ) remains unfilled after reoptimization ($(\psi, p_\psi) \notin Z^*$) or it is satisfied by a car offer (φ, p_φ) that was not previously a part of the optimal choice ($(\varphi, p_\varphi) \notin Y^*$), then all other car offers and requests in the optimal solution remain unaffected and the conditions of definition 1 are immediately satisfied. If, on the other hand, this new request (ψ, p_ψ) is matched to a car offer (φ, p_φ) that was previously a part of the optimal choice of dealer d ($(\varphi, p_\varphi) \in Y^*$), then

[17] This cost may involve inspecting the car, repairing it, detailing it, and so on. Note that the cost may be specific to request ψ; e.g., a car sold to an individual buyer may need to be repaired and detailed, while the same car sold to another dealer may not require these extra costs.

[18] Of course, $M(\Xi)$ can be empty; e.g., it is empty if the number of car offers in Ξ is not equal to the number of car requests or if there are some requests in Ξ that are not compatible with any car offers in Ξ.

[19] This assumption ensures that any set chosen by dealer d contains an equal number of car offers and car requests. In principle, we could consider a more general (yet still fully substitutable) valuation function in which a dealer has utility for a car that he does not resell. In that case, the dealer may end up choosing more car offers than car requests.

the remaining contracts in the optimal solution are affected in exactly the same way as they would be affected if contract (φ, p_φ) were simply removed from the set of options Y and the dealer were asked to reoptimize. Thus, if the preferences of dealer d satisfy the requirements of full substitutability for option sets of size k, they also satisfy these requirements for option sets of size $k + 1$. This observation is the key inductive step in the proof of proposition 1.[20]

Concluding the discussion of full substitutability, we note that definition 1 restricts attention to sets of contracts for which choices are single-valued. In the companion paper (Hatfield et al. 2013), we show that this definition is equivalent to more general versions that explicitly deal with indifferences and multivalued correspondences. In addition, this definition is equivalent to several conditions, including a generalization of the "gross substitutes and complements" condition on demand functions (Sun and Yang 2006) and the submodularity of the indirect utility function $V_i(p) \equiv \max_{\Psi \subseteq \Omega_i} U_i([\Psi; p])$. Our proofs rely on several equivalent definitions of full substitutability developed in the companion paper; we indicate in Appendix A wherever this is the case.

C. *Stability and Competitive Equilibrium*

The main solution concepts that we study are *stability* and *competitive equilibrium*. Both concepts specify which trades are executed and what the associated prices are. Competitive equilibria also specify prices for trades that are not formed.

Definition 2. An outcome A is *stable* if it is

1. *individually rational:* $A_i \in C_i(A)$ for all i;
2. *unblocked:* there is no feasible nonempty *blocking set* $Z \subseteq X$ such that
   a. $Z \cap A = \varnothing$, and
   b. for all $i \in a(Z)$, for all $Y \in C_i(Z \cup A)$, we have $Z_i \subseteq Y$.

Individual rationality requires that no agent can become strictly better off by dropping some of the contracts that he is involved in. This is a standard requirement in the matching literature. The second condition states that when presented with a stable outcome A, one cannot propose a new set of contracts Z such that, for every agent i involved in these new

[20] Note that the definition of the valuation function u_d of dealer d implicitly rules out the complications listed in the beginning of Sec. II.B: fixed costs, economies of scale, and capacity constraints. In the presence of such complications, the preferences of dealer d may not be fully substitutable. For another example of an intermediary with fully substitutable preferences over "substantively" heterogeneous goods, see the iron ore/scrap/steel plant example at the end of Sec. I.A of Ostrovsky (2008). For a related class of rich substitutable preferences of agents who form contracts only on one side (i.e., only buy or only sell), see the class of "endowed assignment valuations" discussed by Hatfield and Milgrom (2005).

contracts, Z_i is a subset of any optimal choice from $Z_i \cup A_i$. This requirement is a natural adaptation of the stability condition of Hatfield and Kominers (2012) to the current setting. We discuss the relationship between our concept of stability and several other stability concepts considered in the matching literature, such as the core and strong stability, in Section IV.A.

Our second solution concept is competitive equilibrium.

DEFINITION 3. An arrangement $[\Psi; p]$ is a *competitive equilibrium* if, for all $i \in I$, $\Psi_i \in D_i(p)$.

This is the standard concept of competitive equilibrium, adapted to the current setting: market clearing is "built in," because each trade in Ψ carries with it the corresponding buyer and seller, and in competitive equilibrium each agent is (weakly) optimizing given market prices. Note that here we implicitly allow for "personalized" prices: identical goods may be sold by a seller to two different buyers at two different prices. In many settings, sellers may not care whom they sell their goods to, and buyers may not care whom they buy from; hence, it is natural to talk about "anonymous," good-specific prices rather than personalized prices. Indeed, this is how the classical models of competitive equilibrium are usually set up and interpreted. In Section IV.B we show how to embed the anonymous-price setting in our framework.

III. Main Results

We now present our three main contributions. First, we show that when preferences are fully substitutable, competitive equilibria are guaranteed to exist and have a number of interesting properties, analogous to those of competitive equilibria in two-sided settings. We then show that under full substitutability, the set of competitive equilibria essentially coincides with the set of stable outcomes. Finally, we show that if preferences are not fully substitutable, then stable outcomes and competitive equilibria need not exist.

A. *Existence and Properties of Competitive Equilibria*

THEOREM 1. Suppose that agents' preferences are fully substitutable. Then there exists a competitive equilibrium.

A key idea of the proof of theorem 1 is to associate to the original market a two-sided many-to-one matching market with transfers, in which each agent corresponds to a "firm" and each trade corresponds to a "worker." The valuation of firm i for hiring a set of workers $\Psi \subseteq \Omega_i$ in the associated two-sided market is given by

$$v_i(\Psi) \equiv u_i(\Psi_{\to i} \cup (\Omega \backslash \Psi)_{i \to}). \tag{1}$$

Intuitively, we think of the firm as employing all the workers associated with trades that the firm buys and with trades that the firm *does not* sell. We show that v_i satisfies the gross substitutes condition of Kelso and Crawford (1982) as long as u_i is fully substitutable.[21] Workers strongly prefer to work rather than being unemployed, and their utilities are monotonically increasing in wages. Also, every worker ω has a strong preference for being employed by $b(\omega)$ or $s(\omega)$ rather than some other firm $i \in I \setminus \{b(\omega), s(\omega)\}$. With these definitions, we have a two-sided market of the type studied by Kelso and Crawford (1982). In this market, a competitive equilibrium is guaranteed to exist, and in every equilibrium, every worker ω is matched to $b(\omega)$ or $s(\omega)$.

We then transform this competitive equilibrium back into a set of trades and prices for the original economy as follows: Trade ω is included in the set of executed trades in the original economy if the worker ω is hired by $b(\omega)$ in the associated market, and trade ω is not included in the set of executed trades if worker ω is hired by $s(\omega)$. We use the wages in the associated market as prices in the original market. We thus obtain a competitive equilibrium of the original economy: Given the prices generated, a trade ω is demanded by its buyer if and only if it is also demanded by its seller (i.e., *not* demanded by the seller in the associated market).

This construction also provides an algorithm for finding a competitive equilibrium. For instance, once we have transformed the original economy into an associated market, we can use an ascending auction for workers to find the minimal-price competitive equilibrium of the associated market; we may then map that competitive equilibrium back to a competitive equilibrium of the original economy.

An important issue that we need to address in the above construction is that the modified valuation function in equation (1) may in principle be unbounded and take the value $-\infty$ for some sets of trades, violating the assumptions of Kelso and Crawford (1982). To deal with this issue, we further modify the valuation function by bounding it in a way that preserves full substitutability and at the same time ensures that the equilibrium derived from the "bounded" economy remains an equilibrium of the original economy. We also need to ensure that the equilibrium in the associated two-sided market exhibits full employment in order to be able to map an equilibrium of the associated economy to an equilibrium of the original one.

We now turn to the properties of competitive equilibria in this economy. While these properties, as well as their proofs, are similar to those

[21] This construction is analogous to the one Sun and Yang (2006) use to transform an exchange economy with two types of goods, which are substitutable within each type and complementary across types, into an economy in which preferences satisfy the gross substitutes condition of Kelso and Crawford (1982).

of competitive equilibria in two-sided settings (Gul and Stacchetti 1999; Sun and Yang 2006), it is important to verify that they continue to hold in this richer environment. We also rely on some of these properties in the proofs of our subsequent results.

We first note an analogue of the first welfare theorem in our economy.

THEOREM 2. Suppose that $[\Psi; p]$ is a competitive equilibrium. Then Ψ is an efficient set of trades; that is, $\sum_{i \in I} u_i(\Psi) \geq \sum_{i \in I} u_i(\Psi')$ for any $\Psi' \subseteq \Omega$.

Our next result can be viewed as a strong version of the second welfare theorem for our setting, providing a converse to theorem 2: For any efficient set of trades Ψ and any competitive equilibrium price vector p, the arrangement $[\Psi; p]$ is a competitive equilibrium. Generically, the efficient set of trades is unique, in which case this statement follows immediately from theorem 2. We show that it also holds when there are multiple efficient sets of trades.

THEOREM 3. Suppose that agents' preferences are fully substitutable. Then for any competitive equilibrium $[\Xi; p]$ and efficient set of trades Ψ, $[\Psi; p]$ is also a competitive equilibrium.

The result of theorem 3 implies that the notion of a *competitive equilibrium price vector* is well defined. Our next result shows that the set of such vectors is a lattice.

THEOREM 4. Suppose that agents' preferences are fully substitutable. Then the set of competitive equilibrium price vectors is a lattice.

The lattice structure of the set of competitive equilibrium prices is analogous to the lattice structure of the set of stable outcomes for economies without transferable utility. In those models, there is a buyer-optimal and a seller-optimal stable outcome. In our model, the lattice of equilibrium prices may in principle be unbounded. If the lattice is bounded (e.g., if all valuations u_i are bounded), then there exist lowest-price and highest-price competitive equilibria.

B. *The Relationship between Competitive Equilibria and Stable Outcomes*

We now show how the sets of stable outcomes and competitive equilibria are related. First, we show that for every competitive equilibrium $[\Psi; p]$, the associated outcome $\kappa([\Psi; p])$ is stable.

THEOREM 5. Suppose that $[\Psi; p]$ is a competitive equilibrium. Then $\kappa([\Psi; p])$ is stable.

If for some competitive equilibrium $[\Psi; p]$ the outcome $\kappa([\Psi; p])$ is not stable, then either it is not individually rational or it is blocked. If it is not individually rational for some agent i, then $\kappa([\Psi; p])_i \notin C_i(\kappa([\Psi; p]))$. Hence, $\Psi_i \notin D_i(p)$, and so $[\Psi; p]$ is not a competitive equilibrium. If $\kappa([\Psi; p])$ admits a blocking set Z, then all the agents with contracts in Z are strictly better off after the deviation. It follows that at the original price vector p, there exists an agent $i \in a(Z)$ who is strictly better off

combining trades from $\tau(Z)$ with (some or all of) his holdings in $\tau(\kappa([\Psi; p])) = \Psi$. Hence, $\Psi_i \notin D_i(p)$, so $[\Psi; p]$ is not a competitive equilibrium. Note that this result does not rely on full substitutability.

However, it is not generally true that all stable outcomes correspond to competitive equilibria. To see this, consider the following example.

Example 1. There are two agents, i and j, and two trades, χ and φ, where $s(\chi) = s(\varphi) = i$ and $b(\chi) = b(\varphi) = j$. Agents' valuations are

$$\begin{aligned} u_i(\varnothing) &= u_j(\varnothing) = 0, \\ u_i(\{\chi\}) &= u_i(\{\varphi\}) = u_i(\{\chi, \varphi\}) = -4, \\ u_j(\{\chi\}) &= u_j(\{\varphi\}) = u_j(\{\chi, \varphi\}) = 3. \end{aligned}$$

In this case, $\varnothing$ is stable. Since $\varnothing$ is the only efficient set of trades, by theorem 3 any competitive equilibrium must be of the form $[\varnothing; p]$. However, we must then have $p_\chi + p_\varphi \leq 4$, as otherwise i will choose to sell at least one of φ or χ. Moreover, we must have $p_\chi, p_\varphi \geq 3$, as otherwise j will buy at least one of φ or χ. Clearly, all three inequalities cannot jointly hold. Hence, while $\varnothing$ is stable, there is no corresponding competitive equilibrium.

The key issue is that an outcome A specifies prices *only* for the trades in $\tau(A)$, while a competitive equilibrium must specify prices for *all* trades (including those trades that do not transact). Hence, in the presence of complementarities, it is possible that, while an outcome A is stable, one cannot assign prices to trades outside of $\tau(A)$ in such a way that $\tau(A)_i$ is an optimal set of trades for every agent i given those prices. Note that in example 1, the preferences of agent j are fully substitutable, but those of agent i are not.

If, however, the preferences of all agents are fully substitutable, then for any stable outcome A, we can in fact find a supporting set of prices p such that $[\tau(A); p]$ is a competitive equilibrium and the prices of trades that transact are the same as in A.

Theorem 6. Suppose that agents' preferences are fully substitutable and that A is a stable outcome. Then there exists a price vector $p \in \mathbb{R}^{|\Omega|}$ such that $[\tau(A); p]$ is a competitive equilibrium, and if $(\omega, \bar{p}_\omega) \in A$, then $p_\omega = \bar{p}_\omega$.

To construct a competitive equilibrium from a stable outcome A, we need to find appropriate prices for the trades that are not part of the stable outcome, that is, trades $\omega \in \Omega \backslash \tau(A)$. In the case of two-sided markets, this can be done on a trade-by-trade basis, because it is sufficient to verify that the price assigned to a trade will not make this trade desirable for either its buyer or its seller given the prices of the trades that those agents execute. In our setting, this approach does not work because the willingness of a buyer to make a new purchase may also depend on the

prices assigned to the trades in which he is a potential seller. Thus, equilibrium prices for trades in $\Omega \backslash \tau(A)$ are interdependent and need to be assigned simultaneously in a consistent manner.

To prove theorem 6, we start with the original market and the stable outcome A and then construct a modified market. In this modified market, the set of available trades is $\Omega \backslash \tau(A)$, and the valuation of each player i for a set of trades $\Psi \subseteq \Omega \backslash \tau(A)$ is equal to the highest value that he can attain by combining the trades in Ψ_i with various subsets of A_i. We first show that the corresponding preferences of each player i are fully substitutable; the modified market thus has a competitive equilibrium by theorem 1. We then show that at least one such equilibrium has to be of the form $[\varnothing; \hat{p}]$ for some vector $\hat{p} \in \mathbb{R}^{|\Omega \backslash \tau(A)|}$; otherwise, as we show, there must exist a nonempty set that blocks A in the original economy (the proof of this statement relies on theorems 2 and 3, our "first" and "second" welfare theorems). Assigning the prices specified by $\hat{p}$ to the trades that are not part of A, we obtain a competitive equilibrium of the original economy.

C. *Full Substitutability as a Maximal Domain*

We now show a maximal domain result: if the preferences of any one agent are not fully substitutable, then stable outcomes need not exist. In fact, in that case we can construct *simple* preferences for other agents such that no stable outcome exists.

DEFINITION 4. The preferences of agent i are *simple* if it is possible to (a) partition the set Ω_i into sets $\Phi^1, \ldots, \Phi^K$ such that for each $k = 1, \ldots, K$, $|\Phi^k| \leq 2$, and (b) define functions u_i^k on subsets of Φ^k such that for each $\Xi \subseteq \Omega_i$,

$$u_i(\Xi) = \sum_{k=1}^{K} u_i^k(\Xi \cap \Phi^k),$$

and each u_i^k satisfies the following conditions:

- if $\Phi^k \subseteq \Omega_{\rightarrow i}$ or $\Phi^k \subseteq \Omega_{i\rightarrow}$ (i.e., all trades in Φ^k are on the same side), then $u_i^k(\Xi \cap \Phi^k) \neq -\infty$ if $|\Xi \cap \Phi^k| \leq 1$, and $u_i^k(\Xi \cap \Phi^k) = -\infty$ if $|\Xi \cap \Phi^k| = 2$;
- if $\Phi^k = \{\omega, \psi\}$ such that $\omega \in \Omega_{\rightarrow i}$ and $\psi \in \Omega_{i\rightarrow}$, then $u_i^k(\varnothing) \neq -\infty$, $u_i^k(\{\omega, \psi\}) \neq -\infty$, and at least one of $u_i^k(\{\omega\})$ and $u_i^k(\{\psi\})$ is equal to $-\infty$.

Simple preferences play a role similar to that of *unit-demand preferences*, used in the Gul and Stacchetti (1999) result characterizing the maximal domain for the existence of competitive equilibria in exchange economies. However, in our setting we must allow an individual agent to act as a set of unit-demand consumers, unit-supply producers, and intermedi-

aries who can transform exactly one unit of input into exactly one unit of output. This is necessary as each contract specifies both a buyer and a seller, and the violation of substitutability may occur for an agent i only when he holds multiple contracts with another agent. Note that simple preferences are fully substitutable.

Our maximal domain result also requires sufficient "richness" of the set of trades. Specifically, we require that the set of trades Ω is *exhaustive*, that is, that for each distinct $i, j \in I$ there exist $\psi, \omega \in \Omega$ such that $b(\psi) = s(\omega) = i$ and $b(\omega) = s(\psi) = j$.

THEOREM 7. Suppose that there are at least four agents and that the set of trades is exhaustive. Then if the preferences of some agent are not fully substitutable, there exist simple preferences for all other agents such that no stable outcome exists.[22]

To understand the intuition behind theorem 7, consider the following example.

EXAMPLE 2. Agent i is just a buyer and has perfectly complementary preferences over the trades χ and φ ($s(\chi) \neq s(\varphi)$) and is not interested in other trades, that is, $u_i(\{\chi, \varphi\}) = 1$ and $u_i(\{\chi\}) = u_i(\{\varphi\}) = u_i(\varnothing) = 0$.

Suppose that $s(\chi)$ and $s(\varphi)$ also have trades $\hat{\chi}$ and $\hat{\varphi}$ (where $s(\hat{\chi}) = s(\chi)$ and $s(\hat{\varphi}) = s(\varphi)$) with another agent $j \neq i$. Let the valuations of these agents be given by

$$u_{s(\chi)}(\{\hat{\chi}\}) = u_{s(\chi)}(\{\chi\}) = u_{s(\chi)}(\varnothing) = 0, \quad u_{s(\chi)}(\{\chi, \hat{\chi}\}) = -\infty,$$

$$u_{s(\varphi)}(\{\hat{\varphi}\}) = u_{s(\varphi)}(\{\varphi\}) = u_{s(\varphi)}(\varnothing) = 0, \quad u_{s(\varphi)}(\{\varphi, \hat{\varphi}\}) = -\infty,$$

$$u_j(\{\hat{\chi}, \hat{\varphi}\}) = u_j(\{\hat{\chi}\}) = u_j(\{\hat{\varphi}\}) = \frac{3}{4}, \quad u_j(\varnothing) = 0.$$

Then in any stable outcome, $s(\chi)$ will sell at most one of χ and $\hat{\chi}$, and $s(\varphi)$ will sell at most one of φ and $\hat{\varphi}$. It cannot be that $\{\chi, \varphi\}$ is part of a stable outcome, as the total price of χ and φ is at most 1; this means that at least one of these trades has a price less than or equal to 1/2. Suppose without loss of generality that $p_\varphi \leq 1/2$; we then have that $\{(\hat{\varphi}, 5/8)\}$ is a blocking set. It also cannot be the case that $\{(\hat{\chi}, p_{\hat{\chi}})\}$ or $\{(\hat{\varphi}, p_{\hat{\varphi}})\}$ is stable: in the former case, $p_{\hat{\chi}}$ must be less than 3/4, in which case $\{(\chi, 7/8), (\varphi, 1/16)\}$ is a blocking set. An analogous construction addresses the latter case.

The proof of theorem 7 essentially generalizes example 2 and can be found in online Appendix B. As (for any preferences, by theorem 5) all

[22] The proof of this result also shows that, for two-sided markets with transferable utility, if any agent's preferences are not fully substitutable, then if there exists at least one other agent on the same side of the market, simple preferences can be constructed such that no stable outcome exists.

competitive equilibria generate stable outcomes and (by theorem 7) stable outcomes may not exist when preferences are not fully substitutable, we have the following corollary.

COROLLARY 1. Suppose that there are at least four agents and that the set of trades is exhaustive. Then, if the preferences of some agent are not fully substitutable, there exist simple preferences for all other agents such that no competitive equilibrium exists.

IV. Other Solution Concepts and Frameworks

In this section, we describe the relationships between competitive equilibrium, stability, and other solution concepts that have played important roles in the literature and discuss the connections between our setting and several earlier frameworks.

A. *The Core and Strong Group Stability*

We start by introducing a classical solution concept: the *core*.

DEFINITION 5. An outcome A is in the *core* if it is *core unblocked*: there does not exist a set of contracts Z such that, for all $i \in a(Z)$, $U_i(Z) > U_i(A)$.

The definition of the core differs from that of stability in two ways. First, a core block requires all the agents with contracts in the blocking set to drop their contracts with other agents; this is a more stringent restriction than that of stability, which allows agents with contracts in the blocking set to retain previous relationships. Second, a core block does not require that $Z_i \in C_i(Z \cup A)$ for all $i \in a(Z)$; rather, it requires only the less stringent condition that $U_i(Z) > U_i(A)$.

DEFINITION 6. An outcome A is *strongly group stable* if it is

1. *individually rational*;
2. *strongly unblocked*: there does not exist a nonempty feasible $Z \subseteq X$ such that
   a. $Z \cap A = \varnothing$ and
   b. for all $i \in a(Z)$, there exists a $Y^i \subseteq Z \cup A$ such that $Z \subseteq Y^i$ and $U_i(Y^i) > U_i(A)$.

Strong group stability is more stringent than both stability and core as strong unblockedness (1) allows for the possibility that when considering a block Z, agents may retain previously held contracts (as in the definition of stability, but not in the definition of the core), and (2) requires only that the new set of contracts for each agent be an improvement (as in the definition of the core, but not in the definition of stability, the improvement does not have to be optimal).

Strong group stability is more stringent than the *strong stability* concept of Hatfield and Kominers (2013a) as strong stability imposes the additional requirement that each Y^i must be individually rational. Strong group stability is also more stringent than the *group stability* concept introduced by Roth and Sotomayor (1990) and extended to the setting of many-to-many matching by Konishi and Ünver (2006), as group stability imposes the additional requirement that if $y \in Y^{b(y)}$, then $y \in Y^{s(y)}$, that is, that the deviating agents agree on the contracts from the original allocation kept after the deviation. Strong stability and group stability themselves strengthen the concept of setwise stability introduced by Echenique and Oviedo (2006) and Klaus and Walzl (2009), which imposes both of the above requirements.[23]

Given these definitions, the following result is immediate.

THEOREM 8. Any strongly group stable outcome is stable and in the core. Furthermore, any core outcome is efficient.

Without additional assumptions on preferences, no additional structure need be present.[24]

For models without continuously transferable utility (see, e.g., Sotomayor 1999; Echenique and Oviedo 2006; Klaus and Walzl 2009; Westkamp 2010; Hatfield and Kominers 2013a), strong group stability is strictly more stringent than stability. However, in the presence of continuously transferable utility and fully substitutable preferences, these solution concepts coincide.

THEOREM 9. If preferences are fully substitutable and A is a stable outcome, then A is strongly group stable and in the core. Moreover, for any core outcome A, there exists a stable outcome $\hat{A}$ such that $\tau(A) = \tau(\hat{A})$.[25]

B. *Competitive Equilibria without Personalized Prices*

The competitive equilibrium concept studied in this paper treats trades as the basic unit of analysis; a price vector specifies one price for each trade. For example, if agent i has one object to sell, a competitive equilibrium price vector generally specifies a different price for each possible buyer, allowing for personalized pricing. Personalized prices arise natu-

[23] The setwise stability concept used in these works is slightly stronger than the definition of setwise stability introduced by Sotomayor (1999); Klaus and Walzl (2009) discuss the subtle differences between these two definitions.

[24] In online App. B, we present examples showing (i) that it may be the case that both stable and core outcomes exist for a given set of preferences, while no outcome is both stable and in the core, and (ii) an outcome that is both stable and in the core need not be strongly group stable.

[25] This result is in a sense sharp: In online App. B, we present an example that shows that even for fully substitutable preferences, the core may be strictly larger than the set of stable outcomes.

rally in decentralized markets, reflecting the idea that agents have access to different trading opportunities.

By contrast, for markets in which all trading opportunities can be thought of as being universally available, it is natural to assume that the identity of the trading partner is irrelevant; in that case, the convention is to study notions of competitive equilibrium that assign a single, uniform price to each object (see, e.g., Gul and Stacchetti 1999; Sun and Yang 2006). Our next result shows that the standard uniform pricing model studied in the prior literature embeds into our model.

DEFINITION 7. Consider an arbitrary agent $i \in I$.

1. The trades in some set $\Psi \subseteq \Omega_i$ are *mutually incompatible for i* if for all $\Xi \subseteq \Omega_i$ such that $|\Xi \cap \Psi| \geq 2$, $u_i(\Xi) = -\infty$.
2. The trades in some set $\Psi \subseteq \Omega_i$ are *perfect substitutes for i* if for all $\Xi \subseteq \Omega_i \backslash \Psi$ and all $\omega, \omega' \in \Psi$, $u_i(\Xi \cup \{\omega\}) = u_i(\Xi \cup \{\omega'\})$.

THEOREM 10. Suppose that agents' preferences are fully substitutable. Suppose further that for agent i, trades in $\Psi \subseteq \Omega_i$ are mutually incompatible and perfect substitutes, and let $[\Xi; p]$ be an arbitrary competitive equilibrium.

a. If $\Psi \subseteq \Omega_{i\rightarrow}$, define q by $q_\varphi = \max_{\chi \in \Psi} p_\chi$ for all $\varphi \in \Psi$ and $q_\varphi = p_\varphi$ for all $\varphi \in \Omega \backslash \Psi$. Then $[\Xi; q]$ is a competitive equilibrium.
b. If $\Psi \subseteq \Omega_{\rightarrow i}$, define q by $q_\varphi = \min_{\chi \in \Psi} p_\chi$ for all $\varphi \in \Psi$ and $q_\varphi = p_\varphi$ for all $\varphi \in \Omega \backslash \Psi$. Then $[\Xi; q]$ is a competitive equilibrium.

As the preferences of agent i are fully substitutable, a trade $\omega \in \Omega_{i\rightarrow}$ cannot perfectly substitute for a trade $\omega' \in \Omega_{\rightarrow i}$. Hence, the two cases in the theorem are exhaustive.

This result allows us to embed the more standard competitive equilibrium frameworks of Gul and Stacchetti (1999) and Sun and Yang (2006) as special cases of our model. In an economy in the sense of Sun and Yang, a finite set S of indivisible objects needs to be allocated among a finite set J of agents with quasi-linear utilities. Objects are partitioned into two groups, S^1 and S^2. Agents' preferences satisfy the *gross substitutes and complements* (GSC) condition: Objects in the same group are substitutes and objects belonging to different groups are complements. The setting of Gul and Stacchetti can be interpreted as the special case in which $S^2 = \varnothing$. To embed a Sun and Yang economy into our model, one can view each object in S^1 as an agent who can "sell" trades to agents in J and each object in S^2 as an agent who can "buy" trades from agents in J.[26] Each agent in

[26] Thus, the set I of agents in the constructed economy is equal to $J \cup S^1 \cup S^2$. The set of possible trades Ω consists of $|S^1| \times |J| + |J| \times |S^2|$ trades: those in which agents in S^1 are

$S = S^1 \cup S^2$ has reservation utility of 0 from not trading, is allowed to form at most one contract, and cares only about the price of that contract.[27] Agents in J can form multiple contracts, and the valuation u_j of agent $j \in J$ from a set of trades with agents in set $S \subseteq S^1 \cup S^2$ is equal to the valuation of agent j in the original economy from the set of objects S. Note that an agent j forming a contract with agent $o \in S^1$ at price p in the network economy corresponds to agent j buying object o at price p in the original economy, while agent j forming a contract with agent $o \in S^2$ at price p corresponds to agent j buying object o at price $-p$ in the original economy. With this embedding, GSC in the original economy is equivalent to full substitutability in the network economy, and thus all our results apply immediately. Since, for every agent in S, all trades are mutually incompatible and are perfect substitutes, theorem 1 and theorem 10 together imply the existence result of Sun and Yang for uniform-price competitive equilibria. Note also that this embedding makes it transparent why the construction of Sun and Yang works for markets with two groups of complementary goods but does not work for markets with three or more groups: the former case can be reinterpreted in our framework by making one group of objects "sellers" in the market and the other group of objects "buyers," while the latter case cannot.

C. *Relation to Previous Models*

In this subsection, we discuss how our model extends the frameworks considered in the earlier literature. To make the discussion concrete, we focus on the used car market example we discussed in the introduction and Section II.B.

First, recall that the set of possible trades among dealers can contain cycles. Because of this possibility, such a market cannot be modeled using the vertical supply chain matching framework of Ostrovsky (2008), which explicitly rules out cycles. More generally, if the set of contractual opportunities is finite, that is, if prices are not allowed to vary continuously, as in the frameworks of Ostrovsky (2008) and Hatfield and Kominers (2012), stable outcomes may fail to exist when cycles are present (see Hatfield and Kominers 2012, theorem 5). Thus, the earlier models of matching in networks are not suitable for studying markets such as the used car market, in which horizontal trading relationships are allowed.

---

sellers and agents in J are buyers and those in which agents in J are sellers and agents in S^2 are buyers. Each pair $(j, o) \in J \times (S^1 \cup S^2)$ is involved in exactly one possible trade in Ω, and so we can identify the set Ω with the set $J \times (S^1 \cup S^2)$.

[27] Thus, agents in S^1 will be willing to participate only in contracts with nonnegative prices, while agents in S^2 will be willing to participate only in contracts with nonpositive prices. In any equilibrium, all prices paid by agents in J to agents in S^1 will be nonnegative, while all prices "paid" by agents in S^2 to agents in J will be nonpositive.

Second, note that used cars can be traded directly from a seller to a buyer as well as indirectly through a dealer. Because of this possibility, such a market cannot be modeled using the framework of Sun and Yang (2006). To see this, consider the following example. The market consists of one seller, i, one dealer, j, and one buyer, k. The set Ω of possible trades consists of three trades: trade $\omega_{i\to j}$ from seller i to dealer j, trade $\omega_{i\to k}$ from seller i to buyer k, and trade $\omega_{j\to k}$ from dealer j to buyer k. The seller i cares only about the price he receives for the car. The dealer j cares only about the difference between the price he has to pay i to acquire the car and the price at which he can resell the car to k. The buyer k cares about the quality of the car and additional services provided by the dealer and the price he has to pay. All agents' preferences are fully substitutable and thus satisfy the conditions of our model. However, there is no partition of the three trades into two groups such that *all* agents view trades in the same group as substitutes and trades in different groups as complements, as required by Sun and Yang. To see this, suppose that there exists such a partition $\Omega = \Omega^1 \cup \Omega^2$. From the perspective of i, selling the car to j (trade $\omega_{i\to j}$) is a substitute for selling the car to k (trade $\omega_{i\to k}$). Therefore, these two trades have to be in the same element of the partition; without loss of generality, suppose that element is Ω^1. Similarly, since for k buying from j (trade $\omega_{j\to k}$) is a substitute for buying from i (trade $\omega_{i\to k}$), these two trades also have to be in the same element of the partition. Hence, $\Omega^1 = \Omega$ and $\Omega^2 = \varnothing$. But then the GSC condition of Sun and Yang requires dealer j to view trades $\omega_{i\to j}$ and $\omega_{j\to k}$ as substitutes, violating the assumptions of the example. Thus, in order to model the used car market (or other intermediated markets) in Sun and Yang's framework, one would have to either rule out intermediated trade through dealers or exclude direct trade between individual sellers and buyers.

Finally, note that features such as the presence of cycles and the possibility of intermediated trade make the frameworks of Kelso and Crawford (1982), Bikhchandani and Mamer (1997), and Gul and Stacchetti (1999) inapplicable to the analysis of the used car market and other markets with those features. Furthermore, Kelso and Crawford assume that every firm finds every set of workers acceptable, and, analogously, Bikhchandani and Mamer and Gul and Stacchetti assume that all possible bundles of trades are acceptable to every agent (at least at sufficiently low prices). By contrast, by allowing the valuations of bundles of trades to take the value $-\infty$, our framework makes it possible to incorporate production feasibility constraints. Of course, a key construction in our existence proof, just as in the proof of the existence result of Sun and Yang (2006), is the reduction from our richer setting to Kelso and Crawford's framework, with suitable modifications and adaptations. Hence, while the results and techniques of these earlier papers are not

directly applicable to our framework, they play an important role in our analysis of matching in trading networks.

V. Conclusion

We have introduced a general model in which a network of agents can trade via bilateral contracts. In this setting, when continuous transfers are allowed and agents' preferences are quasi-linear, full substitutability of preferences is sufficient and (in the maximal domain sense) necessary for the guaranteed existence of stable outcomes. Furthermore, full substitutability implies that the set of stable outcomes is equivalent to the set of competitive equilibria and that all stable outcomes are in the core and are efficient.

Viewing these results in light of the previous matching literature leads to two additional observations.

First, stability may be a natural extension of the notion of competitive equilibrium for some economically important settings in which competitive equilibria do not exist. If the underlying network structure of a market does not contain cycles, then stable outcomes exist even if there are restrictions on the contracts that agents are allowed to form, as long as agents' preferences are fully substitutable (Ostrovsky 2008). For instance, a price floor (or ceiling) may prevent markets from clearing and thus lead to the nonexistence of competitive equilibria. When studying a market for a single good, the classical supply-demand diagram may be sufficient for reasoning about the effects of the price floor. However, in more complicated cases, such as supply chain networks or two-sided markets with multiple goods, a simple diagram is no longer sufficient. The results of this paper suggest that stability may be an appropriate extension of competitive equilibrium for those cases: When contractual arrangements are not restricted, the notions of stability and competitive equilibrium are equivalent, while when contracting restrictions exist, stability continues to make predictions. Recent evidence suggests that these predictions are experimentally supported in multigood markets in which competitive equilibria do not exist when price floors are present (Hatfield, Plott, and Tanaka 2012a, 2012b).

Second, contrasting our results for general networks with previous findings presents a puzzle. Typically, in the matching literature, there are strong parallels between the existence and properties of stable outcomes in markets with fully transferable utility and those in which transfers either are not allowed or are restricted. (This similarity was first observed by Shapley and Shubik [1971] for the basic one-to-one matching model and continues to hold for increasingly complex environments, up to the case of vertical networks.) Our results show that this relationship breaks

down for networks with cycles (in which agents' preferences are fully substitutable): with continuous transfers, stable outcomes are guaranteed to exist, while without them, the set of stable outcomes may be empty (Hatfield and Kominers 2012). It is an open question why the presence of a continuous numeraire can replace the assumption of supply chain structure in ensuring the existence of stable outcomes in trading networks.

Appendix A

Proof of Proposition 1

Consider a dealer d, a set of trades Φ in which d can be involved as a buyer, and a set of trades Ψ in which d can be involved as a seller. For every trade $\varphi \in \Phi$ and trade $\psi \in \Psi$, dealer d knows whether φ and ψ are compatible. The payoff of dealer d from a feasible set of contracts (trades and associated prices) is as given in Section II.B.

We first introduce an auxiliary definition. We say that a set of contracts $Y \subseteq X_d$ is *generic* if (*a*) it is finite (i.e., it contains a finite number of elements) and (*b*) for every subset $Y' \subseteq Y$, $|C_d(Y')| = 1$ (i.e., the choice of d from any subset of Y is single-valued). For a generic set of contracts Y, we denote by Y^* the (unique) choice of d from Y.

Next, we prove the following lemma (by induction on m).

Lemma A.1. For every positive integer m,

1. for all generic sets of contracts Y, $Z \subseteq X_d$ such that $|Y| + 1 = |Z| \leq m$, $Y_{d\rightarrow} = Z_{d\rightarrow}$, and $Y_{\rightarrow d} \subsetneq Z_{\rightarrow d}$, we have $Y_{\rightarrow d} \setminus Y^*_{\rightarrow d} \subseteq Z_{\rightarrow d} \setminus Z^*_{\rightarrow d}$ and $Y^*_{d\rightarrow} \subseteq Z^*_{d\rightarrow}$;
2. for all generic sets of contracts Y, $Z \subseteq X_d$ such that $|Y| + 1 = |Z| \leq m$, $Y_{\rightarrow d} = Z_{\rightarrow d}$, and $Y_{d\rightarrow} \subsetneq Z_{d\rightarrow}$, we have $Y_{d\rightarrow} \setminus Y^*_{d\rightarrow} \subseteq Z_{d\rightarrow} \setminus Z^*_{d\rightarrow}$ and $Y^*_{\rightarrow d} \subseteq Z^*_{\rightarrow d}$.

In other words, the lemma says that the choice function of a dealer satisfies the requirements of the full substitutability condition when applied to generic sets of size at most m and just one new contract is added to the choice set.

Proof. For $m = 1$, statements 1 and 2 are both clearly true since both Y^* and Z^* are empty.

Suppose that statements 1 and 2 are true for all $m \leq k$. We prove them for $m = k + 1$. Specifically, we prove statement 2; the proof of statement 1 is completely analogous.

Consider sets Y and Z satisfying the conditions of statement 2. (In the language of the used car example, Z has one additional request for a used car relative to Y, and both sets contain the same offers of cars.) If $Y^* = Z^*$ (i.e., the optimal choice of dealer d is unaffected by the new request), then the conclusion of statement 2 is clearly true.

Otherwise (i.e., if $Y^* \neq Z^*$), let (ψ, p_ψ) be the new request in Z (i.e., the unique element in $Z \setminus Y$). It must be the case that $(\psi, p_\psi) \in Z^*$ (because otherwise this new request could not have affected the optimal choice of d).

We now consider two cases: (1) Y^* contains a contract that involves trade ψ at some price $p'_\psi \neq p_\psi$ and (2) Y^* does not contain such a contract.

Case 1: It must be the case that $p'_\psi < p_\psi$, as if $p'_\psi > p_\psi$, then request (ψ, p_ψ) is never chosen by dealer d when (ψ, p'_ψ) is also available. If, when choosing from Z, dealer d simply replaces (ψ, p'_ψ) in Y^* with (ψ, p_ψ), his payoff goes up by $p_\psi - p'_\psi$ (relative to that from Y^*). Note that there cannot be a subset of Z containing (ψ, p_ψ) that gives dealer d a strictly higher payoff than that because otherwise replacing (ψ, p_ψ) in that subset with (ψ, p'_ψ) would result in a subset of Y that gives dealer d a higher payoff than Y^*. Finally, since, by assumption, sets Y and Z are generic, all choice functions are single-valued, and thus we must have

$$Z^* = (Y^* \backslash \{(\psi, p'_\psi)\}) \cup \{(\psi, p_\psi)\}.$$

It is now immediate that the conclusion of statement 2 holds.

Case 2: Consider the input contract (φ, p_φ) to which request (ψ, p_ψ) is matched when d is choosing from Z. If $(\varphi, p_\varphi) \notin Y^*$ (i.e., the car to which request ψ is matched was not involved in the optimal choice from set Y), then it must be the case that the remaining matches are unaffected; hence,

$$Z^* = Y^* \cup \{(\varphi, p_\varphi)\} \cup \{(\psi, p_\psi)\},$$

and the conclusion of statement 2 holds.

Suppose instead that (φ, q_φ) was matched to some request when dealer d was choosing from Y. (This subcase is the heart of the proof of lemma A.1 and proposition 1; this is the part that relies on the use of the inductive hypothesis.) Let W^* be the choice of dealer d from set $W = Y \backslash \{(\varphi, p_\varphi)\}$. Crucially, it must be the case that $Z^* = W^* \cup \{(\varphi, p_\varphi)\} \cup \{(\psi, p_\psi)\}$: by assumption, in the optimal choice Z^*, contract (φ, p_φ) is matched to request (ψ, p_ψ), and thus the remaining chosen contracts are simply those that maximize the payoff of dealer d when choosing from the remaining set of options, $Y \backslash \{(\varphi, p_\varphi)\}$. Now, we can apply the inductive hypothesis to sets W and Y (which are, respectively, one element smaller than sets Y and Z). By statement 1 of the inductive hypothesis (which is now the relevant statement), $W_{\to d} \backslash W^*_{\to d} \subseteq Y_{\to d} \backslash Y^*_{\to d}$ and $W^*_{d\to} \subseteq Y^*_{d\to}$. Combining these two set inclusions with the relationships identified above, we now have

$$Y^*_{\to d} = Y_{\to d} \backslash (Y_{\to d} \backslash Y^*_{\to d}) \subseteq Y_{\to d} \backslash (W_{\to d} \backslash W^*_{\to d}) = W^*_{\to d} \cup \{(\varphi, p_\varphi)\} = Z^*_{\to d}$$

and

$$Y_{d\to} \backslash Y^*_{d\to} = W_{d\to} \backslash Y^*_{d\to} \subseteq W_{d\to} \backslash W^*_{d\to} = Z_{d\to} \backslash Z^*_{d\to},$$

concluding the proof of lemma A.1. QED

We now use lemma A.1 to prove that the preferences of dealer d are fully substitutable. Consider dealer d and sets $Y, Z \subseteq X_d$ satisfying the assumptions of part 2 of definition 1. (The proof for part 1 is completely analogous.) Let $\hat{Y} = Y \cap (Y^* \cup Z^*)$ and $\hat{Z} = Z \cap (Y^* \cup Z^*)$; that is, $\hat{Y}$ and $\hat{Z}$ are the subsets of Y and Z that contain all possible contracts relevant for choices of d from Y and Z. Clearly, $\hat{Y}_{\to d} = \hat{Z}_{\to d}$, $\hat{Y}_{d\to} \subseteq \hat{Z}_{d\to}$, and sets $\hat{Y}$ and $\hat{Z}$ are finite and inherit the property of Y and Z that the choices of d from those sets are single-valued (and those choices are Y^* and Z^*, respectively). Sets $\hat{Y}$ and $\hat{Z}$ are not necessarily generic;

however, we can slightly perturb prices in contracts in $\hat{Y}$ and $\hat{Z}$ in such a way that the resulting sets $\tilde{Y}$ and $\tilde{Z}$ are generic, the relationships $\tilde{Y}_{\to d} = \tilde{Z}_{\to d}$ and $\tilde{Y}_{d\to} \subseteq \tilde{Z}_{d\to}$ are preserved, and the optimal choices of dealer d from those sets, $\tilde{Y}^*$ and $\tilde{Z}^*$, are the perturbed original choices Y^* and Z^* (i.e., they involve the same trades, along with the perturbed prices).[28]

We now show that $\tilde{Y}_{d\to} \setminus \tilde{Y}^*_{d\to} \subseteq \tilde{Z}_{d\to} \setminus \tilde{Z}^*_{d\to}$ and $\tilde{Y}^*_{\to d} \subseteq \tilde{Z}^*_{\to d}$, which imply the same relationships for sets $\hat{Y}$, $\hat{Z}$, Y^*, and Z^*, which in turn imply the same relationships for sets Y, Z, Y^*, and Z^*, and those relationships are precisely the conclusions in part 2 of definition 1 that we need to prove.

If $\tilde{Y} = \tilde{Z}$, then it is immediate that $\tilde{Y}_{d\to} \setminus \tilde{Y}^*_{d\to} \subseteq \tilde{Z}_{d\to} \setminus \tilde{Z}^*_{d\to}$ and $\tilde{Y}^*_{\to d} \subseteq \tilde{Z}^*_{\to d}$. Otherwise, let $n = |\tilde{Z} \setminus \tilde{Y}|$ and consider an increasing sequence of sets

$$\tilde{Y} = Y^0 \subsetneq Y^1 \subsetneq \cdots \subsetneq Y^n = \tilde{Z},$$

in which each set contains exactly one extra contract relative to the previous set in the sequence. By lemma A.1, for every $l = 0, \ldots, n-1$, we have $Y^l_{d\to} \setminus Y^{l,*}_{d\to} \subseteq Y^{l+1}_{d\to} \setminus Y^{l+1,*}_{d\to}$ and $Y^{l,*}_{\to d} \subseteq Y^{l+1,*}_{\to d}$. This implies that $\tilde{Y}_{d\to} \setminus \tilde{Y}^*_{d\to} = Y^0_{d\to} \setminus Y^{0,*}_{d\to} \subseteq Y^n_{d\to} \setminus Y^{n,*}_{d\to} = \tilde{Z}_{d\to} \setminus \tilde{Z}^*_{d\to}$ and $\tilde{Y}^*_{\to d} = Y^{0,*}_{\to d} \subseteq Y^{n,*}_{\to d} = \tilde{Z}^*_{\to d}$.

Proof of Theorem 1

The proof consists of four steps: (1) transforming the original valuations into bounded ones, (2) constructing a two-sided many-to-one matching market with transfers based on the network market with bounded valuations, (3) picking a full-employment competitive equilibrium in the two-sided market, and (4) using that equilibrium to construct a competitive equilibrium in the original market. Throughout the proof, we will refer to valuation functions and utility functions that give rise to fully substitutable preferences as fully substitutable.

Step 1: We first transform a fully substitutable but potentially unbounded from below valuation function u_i into a fully substitutable and *bounded* valuation function $\hat{u}_i$. For this purpose, we now introduce a very high price Π. Specifically, for

[28] To formally construct such a perturbation, let Δ be the smallest positive difference between the utilities of dealer d from two different feasible subsets of $\hat{Z}$. Let $f = |\hat{Z}|$. Randomly order contracts in $\hat{Z}$, and add $\Delta/2$ to the price in the first contract, $\Delta/4$ to the price in the second contract, $\ldots$, $\Delta/2^f$ to the price in the last contract. Let $\tilde{Z}$ be the resulting set of contracts with perturbed prices. Since $\hat{Y}$ is a subset of $\hat{Z}$ (and $\tilde{Y}$ must be a subset of $\tilde{Z}$), prices in the perturbed set $\tilde{Y}$ are automatically pinned down. To see that set $\tilde{Z}$ is generic, consider any two distinct sets $\tilde{Z}^1, \tilde{Z}^2 \subseteq \tilde{Z}$ such that $U_d(\tilde{Z}^1) > -\infty$ and $U_d(\tilde{Z}^2) > -\infty$, and also consider the corresponding distinct sets $\hat{Z}^1, \hat{Z}^2 \subseteq \hat{Z}$ (the utilities from which are therefore also finite). If $U_d(\hat{Z}^1) \neq U_d(\hat{Z}^2)$, then by construction $|U_d(\hat{Z}^1) - U_d(\hat{Z}^2)| \geq \Delta$, and thus we also have $U_d(\tilde{Z}^1) \neq U_d(\tilde{Z}^2)$ because, by construction, the sum of perturbations of any set of prices is less than or equal to $\sum_{l=1}^{f} \Delta/2^l$, which is strictly less than Δ. If $U_d(\hat{Z}^1) = U_d(\hat{Z}^2)$, then, since the two sets are distinct, it must also be the case that $U_d(\tilde{Z}^1) \neq U_d(\tilde{Z}^2)$. To see that, consider the first contract (according to the random order constructed above, using which the perturbations were constructed) that belongs to one of these sets but not to the other, and consider the size of its perturbation. Again, by construction, any sum of the *remaining* perturbations is less than this first one, and thus $U_d(\tilde{Z}^1) \neq U_d(\tilde{Z}^2)$. Hence, no two subsets of $\tilde{Z}$ give the same finite utility to dealer d, which implies that $\tilde{Z}$ is generic, as required. Since $\hat{Y} \subseteq \hat{Z}$, it immediately follows that set $\tilde{Y}$ is also generic. Note that the above argument also implies that, as required, the optimal choices of dealer d from sets $\tilde{Y}$ and $\tilde{Z}$ are the perturbed optimal choices of dealer d from sets $\hat{Y}$ and $\hat{Z}$, respectively.

each agent i, let $\bar{u}_i$ be the highest possible absolute valuation of agent i from a combination of trades, that is, $\bar{u}_i = \max_{\{\Psi \subseteq \Omega_i : |u_i(\Psi)| < \infty\}} |u_i(\Psi)|$. Then set $\Pi = 2\sum_{i \in I} \bar{u}_i + 1$. Consider the following *modified economy*. Assume that for every trade, the buyer of that trade can always purchase a perfect substitute for that trade for Π and the seller of that trade can always produce this trade at the cost of Π with no inputs needed. Formally, for each agent i, for a set of trades $\Psi \subseteq \Omega_i$, let

$$\hat{u}_i(\Psi) = \max_{\Psi' \subseteq \Psi} [u_i(\Psi') - \Pi \cdot |\Psi \backslash \Psi'|].$$

For the economy with valuations $\hat{u}_i$, let $\hat{U}_i$ denote the utility function of agent i and let $\hat{D}_i$ denote the resulting demand correspondence. Note that by the choice of Π, for any $\Psi \subseteq \Omega_i$,

$$\bar{u}_i \geq \hat{u}_i(\Psi) \geq \max\{u_i(\varnothing) - \Pi \cdot |\Psi|, u_i(\Psi)\}$$

and that $\hat{u}_i(\Psi) = u_i(\Psi)$ whenever $u_i(\Psi) \neq -\infty$. We use these facts throughout the proof.

The rest of step 1 consists of proving the following lemma.

Lemma A.2. The utility function $\hat{U}_i$ is fully substitutable.

Proof. Take any fully substitutable valuation function u_i. Take any trade $\varphi \in \Omega_{i\rightarrow}$. Consider a modified valuation function u_i^φ:

$$u_i^\varphi(\Psi) = \max\{u_i(\Psi), u_i(\Psi \backslash \{\varphi\}) - \Pi\}.$$

That is, this valuation function allows (but does not require) agent i to pay Π instead of forming one particular trade, φ. With this definition, the valuation function u_i^φ is fully substitutable.

To see this, consider utility U_i^φ and demand D_i^φ corresponding to valuation u_i^φ. We show that D_i^φ satisfies the IFS condition of Hatfield et al. (2013), one of the equivalent definitions of full substitutability.[29] Fix two price vectors p and p' such that $p \leq p'$ and $|D_i^\varphi(p)| = |D_i^\varphi(p')| = 1$. Take $\Psi \in D_i^\varphi(p)$ and $\Psi' \in D_i^\varphi(p')$. We need to show that for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$, $e_\omega(\Psi) \leq e_\omega(\Psi')$.

Let price vector q coincide with p for all trades other than φ, and set $q_\varphi = \min\{p_\varphi, \Pi\}$. Note that if $p_\varphi < \Pi$, then $p = q$ and $D_i^\varphi(p) = D_i(p)$. If $p_\varphi > \Pi$, then under utility U_i^φ, agent i always wants to form trade φ at price p_φ, and the only decision is whether to "buy it out" or not at the cost Π; that is, the agent's effective demand is the same as under price vector q. Thus, $D_i^\varphi(p) = \{\Xi \cup \{\varphi\} : \Xi \in D_i(q)\}$. Finally, if $p_\varphi = \Pi$, then $p = q$ and $D_i^\varphi(p) = D_i(p) \cup \{\Xi \cup \{\varphi\} : \Xi \in D_i(p)\}$. Construct price vector q' corresponding to p' analogously.

Now, if $p_\varphi \leq p'_\varphi < \Pi$, then $D_i^\varphi(p) = D_i(p)$, $D_i^\varphi(p') = D_i(p')$, and thus $e_\omega(\Psi) \leq e_\omega(\Psi')$ follows directly from the IFS condition for demand D_i.

[29] The definition of the IFS condition is as follows. For agent i and any set of trades $\Psi \subseteq \Omega_i$, define the (*generalized*) *indicator function* $e(\Psi) \in \{-1, 0, 1\}^{|\Omega|}$ to be the vector with component $e_\omega(\Psi) = 1$ for each upstream trade $\omega \in \Psi_{\rightarrow i}$, $e_\omega(\Psi) = -1$ for each downstream trade $\omega \in \Psi_{i\rightarrow}$, and $e_\omega(\Psi) = 0$ for each trade $\omega \notin \Psi$. (The interpretation of $e(\Psi)$ is that an agent buys a strictly positive amount of a good if he is the buyer in a trade in Ψ and "buys" a strictly negative amount if he is the seller of such a trade.) Then we say that the preferences of agent i are *indicator-language fully substitutable* (IFS) if for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \leq p'$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $e_\omega(\Psi) \leq e_\omega(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

If $\Pi \leq p_\varphi \leq p'_\varphi$, then (since we assumed that D_i^φ was single-valued at p and p') it has to be the case that D_i is single-valued at the corresponding price vectors q and q'. Let $\Xi \in D_i(q)$ and $\Xi' \in D_i(q')$. Then $\Psi = \Xi \cup \{\varphi\}$, $\Psi' = \Xi' \cup \{\varphi\}$, and the statement follows from the IFS condition for demand D_i because $q \leq q'$.

Finally, if $p_\varphi < \Pi \leq p'_\varphi$, then $p = q$, Ψ is the unique element in $D_i(p)$, and $\Psi' = \Xi' \cup \{\varphi\}$, where Ξ' is the unique element in $D_i(q')$. Then for $\omega \neq \varphi$, the statement follows from IFS for demand D_i because $p \leq q'$. For $\omega = \varphi$, the statement does not need to be checked because $p_\varphi < p'_\varphi$.

Thus, when $\varphi \in \Omega_{i\rightarrow}$, the valuation function u_i^φ is fully substitutable. The proof for the case when $\varphi \in \Omega_{\rightarrow i}$ is completely analogous.

To complete the proof of the lemma, it is now enough to note that valuation function $\hat{u}_i(\Psi) = \max_{\Psi' \subseteq \Psi}[u_i(\Psi') - \Pi \cdot |\Psi \backslash \Psi'|]$ can be obtained from the original valuation u_i by allowing agent i to "buy out" all of his trades, one by one, and since the preceding argument shows that each such transformation preserves substitutability, $\hat{u}_i$ is substitutable as well. QED

Step 2: We now transform the modified economy with bounded and fully substitutable valuations $\hat{u}_i$ into an associated two-sided many-to-one matching market with transfers, which satisfies the assumptions of Kelso and Crawford (1982). The set of *firms* in this market is I, and the set of *workers* is Ω.

Worker ω can be matched to at most one firm. His utility is defined as follows: If he is matched to firm $i \in \{b(\omega),\ s(\omega)\}$, then his utility is equal to the monetary transfer that he receives from that firm, that is, his salary $p_{i,\omega}$, which can in principle be negative. If he is matched to any other firm i, his utility is equal to $-\Pi - 1 + p_{i,\omega}$, where Π is as defined in step 1 and $p_{i,\omega}$ is the salary firm i pays him. If worker ω remains unmatched, his utility is equal to $-2\Pi - 2$.

Firm i can be matched to any set of workers, but only its matches to workers $\omega \in \Omega_i$ have an impact on its valuation. Formally, the valuation of firm i from hiring a set of workers $\Psi \subseteq \Omega$ is given by

$$\tilde{u}_i(\Psi) = \hat{u}_i(\Psi_{\rightarrow i} \cup (\Omega \backslash \Psi)_{i\rightarrow}) - \hat{u}_i(\Omega_{i\rightarrow}),$$

where the second term in the difference is simply a constant, which ensures that $\tilde{u}_i(\varnothing) = 0$, so that valuation function $\tilde{u}_i$ satisfies the NFL assumption of Kelso and Crawford. Hiring a set of workers $\Psi \subseteq \Omega$ when the salary vector is $p \in \mathbb{R}^{|I| \times |\Omega|}$ yields i a utility of

$$\tilde{U}_i([\Psi; p]) \equiv \tilde{u}_i(\Psi_i) - \sum_{\omega \in \Psi} p_{i,\omega}.$$

The associated demand correspondence is denoted by

$$\tilde{D}_i \equiv \arg\max_{\Psi \subseteq \Omega_i} \tilde{U}_i([\Psi; p]).$$

The MP assumption of Kelso and Crawford requires that any firm's change in valuation from adding a worker, ω, to any set of other workers is at least as large as the lowest salary worker ω would be willing to accept from the firm when his only alternative is to remain unmatched. This assumption is also satisfied in our

market: A worker's utility from remaining unmatched is $-2\Pi - 2$, while his valuation, excluding salary, from matching with any firm is at least $-\Pi - 1$, and so he would strictly prefer to work for any firm for negative salary $-\Pi$ instead of remaining unmatched. At the same time, the change in valuation of any firm i from adding worker ω to a set of workers Ψ is equal to $\tilde{u}_i(\Psi \cup \{\omega\}) - \tilde{u}_i(\Psi) \geq -\bar{u}_i - \bar{u}_i > -\Pi$, and thus every firm i would also always strictly prefer to hire worker ω for the negative salary $-\Pi$.

Finally, we show that the preferences of i in this market satisfy the gross substitutes (GS) condition of Kelso and Crawford. Take two salary vectors $p, p' \in \mathbb{R}^{|I|\times|\Omega|}$ such that $p \leq p'$ and $|\tilde{D}_i(p)| = |\tilde{D}_i(p')| = 1$. Let $\Psi \in \tilde{D}_i(p)$ and $\Psi' \in \tilde{D}_i(p')$. Denote by $q = (p_{i,\omega})_{\omega\in\Omega}$ and $q' = (p'_{i,\omega})_{\omega\in\Omega}$ the vectors of salaries that i faces under p and p', respectively. Note that $\Psi \in \tilde{D}_i(p)$ if and only if $\Psi_{\to i} \cup (\Omega\backslash\Psi)_{i\to} \in \hat{D}_i(q)$ and $\Psi' \in \tilde{D}_i(p')$ if and only if $\Psi'_{\to i} \cup (\Omega\backslash\Psi')_{i\to} \in \hat{D}_i(q')$. In particular, $|\hat{D}_i(q)| = |\hat{D}_i(q')| = 1$. Since $q \leq q'$ and $\hat{D}_i$ is fully substitutable, the IFS condition implies that for any $\omega \in \Psi_{\to i}$ such that $q_\omega = q'_\omega$, we have $\omega \in \Psi'_{\to i}$, and for any $\omega \notin \Omega_{i\to}\backslash\Psi_{i\to}$ such that $q_\omega = q'_\omega$, we have $\omega \notin \Omega_{i\to}\backslash\Psi'_{i\to}$. In other words, for every $\omega \in \Psi$ such that $q_\omega = q'_\omega$, we have $\omega \in \Psi'$, and thus the GS condition is satisfied for all salary vectors for which demand $\tilde{D}_i$ is single-valued. As shown by Hatfield et al. (2013), this implies that the GS condition is satisfied for all salary vectors.

Step 3: By the results of Kelso and Crawford (theorem 2 and the discussion in Sec. II), there exists a full-employment competitive equilibrium of the associated two-sided market constructed in step 2. Take one such equilibrium, and for every ω and i, let $\mu(\omega)$ denote the firm matched to ω in this equilibrium and let $r_{i,\omega}$ denote the equilibrium salary of ω at i.

Note that in this equilibrium, it must be the case that every worker ω is matched to either $b(\omega)$ or $s(\omega)$. Indeed, suppose that ω is matched to some other firm $i \notin \{b(\omega), s(\omega)\}$. Since by definition, for any $\Psi \subseteq \Omega$, $\tilde{u}_i(\Psi \cup \{\omega\}) - \tilde{u}_i(\Psi) = 0$, it must be the case that $r_{i,\omega} \leq 0$. Then, for worker ω to weakly prefer to work for i rather than $b(\omega)$, it must be the case that $r_{b(\omega),\omega} \leq -\Pi - 1$. But at that salary, firm $b(\omega)$ strictly prefers to hire ω, contradicting the assumption that ω is not matched to $b(\omega)$ in this equilibrium.

Note also that if $\mu(\omega) = b(\omega)$, then $r_{b(\omega),\omega} \geq r_{s(\omega),\omega}$, and if $\mu(\omega) = s(\omega)$, then $r_{s(\omega),\omega} \geq r_{b(\omega),\omega}$ (otherwise, worker ω would strictly prefer to change his employer). Now, define prices $p_{i,\omega}$ as follows: if $i \neq b(\omega)$ and $i \neq s(\omega)$, then $p_{i,\omega} = r_{i,\omega}$. Otherwise, $p_{i,\omega} = \max\{r_{b(\omega),\omega}, r_{s(\omega),\omega}\}$. Note that matching μ and associated prices $p_{i,\omega}$ also constitute a competitive equilibrium of the two-sided market.

Step 4: We can now construct a competitive equilibrium for the original economy. Let $p^* \in \mathbb{R}^{|\Omega|}$ be defined as $p^*_\omega \equiv p_{\mu(\omega),\omega}$ for each $\omega \in \Omega$, that is, the salary that ω actually receives in the equilibrium of the two-sided market. Let $\Psi^* \equiv \{\omega \in \Omega : \mu(\omega) = b(\omega)\}$, that is, the set of trades/workers who in the equilibrium of the two-sided market are matched to their buyers (and thus not matched to their sellers!).

We now claim that $[\Psi^*; p^*]$ is a competitive equilibrium of the network economy with bounded valuations $\hat{u}_i$. Take any set of trades $\Psi \in \Omega_i$. We show that $\hat{U}_i([\Psi^*; p^*]) \geq \hat{U}_i([\Psi; p^*])$. By construction, for any $\omega \in \Omega_{\to i}$, we have $\omega \in \Psi^*$ if and only if $i = \mu(\omega)$, and for any $\omega \in \Omega_{i\to}$, we have $\omega \in \Psi^*$ if and only if $i \neq \mu(\omega)$. Thus, in the equilibrium of the two-sided market, firm i is matched to the set of work-

ers $\Psi^*_{\to i} \cup (\Omega_{i\to} \backslash \Psi^*_{i\to})$, which implies that

$$\begin{aligned} &\tilde{u}_i(\Psi^*_{\to i} \cup (\Omega_{i\to} \backslash \Psi^*_{i\to})) - \sum_{\omega \in \Psi^*_{\to i}} p_{i,\omega} - \sum_{\omega \in \Omega_{i\to} \backslash \Psi^*_{i\to}} p_{i,\omega} \\ &\geq \tilde{u}_i(\Psi_{\to i} \cup (\Omega_{i\to} \backslash \Psi_{i\to})) - \sum_{\omega \in \Psi_{\to i}} p_{i,\omega} - \sum_{\omega \in \Omega_{i\to} \backslash \Psi_{i\to}} p_{i,\omega}. \end{aligned} \tag{A1}$$

Now, for any set $\Phi \subseteq \Omega_{i\to}$, we have

$$\sum_{\omega \in \Omega_{i\to} \backslash \Phi} p_{i,\omega} = \left(\sum_{\omega \in \Omega_{i\to}} p_{i,\omega} \right) - \left(\sum_{\omega \in \Phi} p_{i,\omega} \right).$$

Using this fact and the definition of $\tilde{u}_i$, we can rewrite the inequality (A1) as

$$\begin{aligned} &\hat{u}_i(\Psi^*_{\to i} \cup \Psi^*_{i\to}) - \sum_{\omega \in \Psi^*_{\to i}} p_{i,\omega} + \sum_{\omega \in \Psi^*_{i\to}} p_{i,\omega} \\ &\geq \hat{u}_i(\Psi_{\to i} \cup \Psi_{i\to}) - \sum_{\omega \in \Psi_{\to i}} p_{i,\omega} + \sum_{\omega \in \Psi_{i\to}} p_{i,\omega}, \end{aligned}$$

which in turn can be rewritten as

$$\begin{aligned} \hat{U}_i([\Psi^*; p^*]) &= \hat{u}_i(\Psi^*_i) - \sum_{\omega \in \Psi^*_{\to i}} p^*_\omega + \sum_{\omega \in \Psi^*_{i\to}} p^*_\omega \\ &\geq \hat{u}_i(\Psi) - \sum_{\omega \in \Psi_{\to i}} p^*_\omega + \sum_{\omega \in \Psi_{i\to}} p^*_\omega \\ &= \hat{U}([\Psi; p^*]). \end{aligned}$$

We now show that $[\Psi^*; p^*]$ is an equilibrium of the original economy with valuations u_i. Suppose to the contrary that there exist an agent i and a set of trades $\Xi \subseteq \Omega_i$ such that $U_i([\Xi; p^*]) > U_i([\Psi^*; p^*])$. Since $\hat{U}_i([\Xi; p^*]) \leq \hat{U}_i([\Psi^*; p^*])$ and, by the construction of $\hat{u}_i$, $\hat{U}_i([\Xi; p^*]) \geq U_i([\Xi; p^*])$, it follows that $\hat{U}_i([\Psi^*; p^*]) > U_i([\Psi^*; p^*])$. This, in turn, implies that for some nonempty set $\Phi \subseteq \Psi^*_i$, we have

$$\hat{u}_i(\Psi^*_i) = u_i(\Psi^*_i \backslash \Phi) - \Pi \cdot |\Phi| \leq \bar{u}_i - \Pi.$$

This implies that

$$\begin{aligned} \sum_{j \in I} \hat{u}_j(\Psi^*) &= \hat{u}_i(\Psi^*) + \sum_{j \neq i} \hat{u}_j(\Psi^*) \\ &\leq \bar{u}_i - \Pi + \sum_{j \neq i} \bar{u}_j \\ &= \sum_{j \in I} \bar{u}_j - \Pi \\ &= -\sum_{j \in I} \bar{u}_j - 1 \\ &< \sum_{j \in I} u_j(\varnothing), \end{aligned}$$

contradicting theorem 2. (The proof of theorem 2 is entirely self-contained.)

Proof of Theorem 2

If $[\Psi; p]$ is a competitive equilibrium, then for any $\Xi \subseteq \Omega$, we have

$$\begin{aligned} u_i(\Psi) + \sum_{\omega \in \Psi_{i\mapsto}} p_\omega - \sum_{\omega \in \Psi_{\mapsto i}} p_\omega &= U_i([\Psi; p]) \\ &\geq U_i([\Xi; p]) \\ &= u_i(\Xi) + \sum_{\omega \in \Xi_{i\mapsto}} p_\omega - \sum_{\omega \in \Xi_{\mapsto i}} p_\omega \end{aligned}$$

for every $i \in I$. By summing these inequalities over all $i \in I$, we find that

$$\sum_{i \in I} u_i(\Psi) \geq \sum_{i \in I} u_i(\Xi).$$

Proof of Theorem 3

We use an approach analogous to the one Gul and Stacchetti (1999) use to prove their lemma 6. Suppose that $[\Xi; p]$ is a competitive equilibrium and that $\Psi \subseteq \Omega$ is an efficient set of trades. Since Ψ is efficient, we have

$$\begin{aligned} \sum_{i \in I} \left[u_i(\Psi) + \sum_{\omega \in \Psi_{i\mapsto}} p_\omega - \sum_{\omega \in \Psi_{\mapsto i}} p_\omega \right] &= \sum_{i \in I} U_i([\Psi; p]) \\ &\geq \sum_{i \in I} U_i([\Xi; p]) \\ &= \sum_{i \in I} \left[u_i(\Xi) + \sum_{\omega \in \Xi_{i\mapsto}} p_\omega - \sum_{\omega \in \Xi_{\mapsto i}} p_\omega \right]. \end{aligned} \tag{A2}$$

As $[\Xi; p]$ is a competitive equilibrium, we have for each $i \in I$ that

$$\begin{aligned} u_i(\Xi) + \sum_{\omega \in \Xi_{i\mapsto}} p_\omega - \sum_{\omega \in \Xi_{\mapsto i}} p_\omega &= U_i([\Xi; p]) \\ &\geq U_i([\Psi; p]) \\ &= u_i(\Psi) + \sum_{\omega \in \Psi_{i\mapsto}} p_\omega - \sum_{\omega \in \Psi_{\mapsto i}} p_\omega. \end{aligned}$$

We therefore see that (A2) can hold only if, for each $i \in I$, $U_i([\Xi; p]) = U_i([\Psi; p])$. Therefore, for all $i \in I$, we have that $\Psi_i \in D_i(p)$; thus, $[\Psi; p]$ is a competitive equilibrium.

Proof of Theorem 4

Our approach extends the proof of theorem 3 of Sun and Yang (2009) to the network setting. Given a price vector p, let $V(p) \equiv \sum_{i \in I} V_i(p)$, where (as defined in Sec. II.B) $V_i(p) \equiv \max_{\Psi \subseteq \Omega_i} U_i([\Psi; p])$. Let $\Psi^* \subseteq \Omega$ be any efficient set of trades and let $U^* = \sum_{i \in I} u_i(\Psi^*)$. Note that for any competitive equilibrium price vector p^*, $V(p^*) = U^*$.

We first prove an analogue of lemma 1 of Sun and Yang (2009).

LEMMA A.3. A price vector $p' \in \mathbb{R}^{|\Omega|}$ is a competitive equilibrium price vector if and only if $p' \in \arg\min_p V(p)$.

Proof. To prove the first implication of the lemma, we let p' be a competitive equilibrium price vector and let p be an arbitrary price vector. For each agent i, consider some arbitrary $\Psi^i \in D_i(p)$. By construction, we have

$$\begin{aligned} V(p) = \sum_{i \in I} V_i(p) &= \sum_{i \in I} \left[u_i(\Psi^i) + \sum_{\omega \in \Psi^i_{i \mapsto}} p_\omega - \sum_{\omega \in \Psi^i_{\to i}} p_\omega \right] \\ &\geq \sum_{i \in I} \left[u_i(\Psi^*) + \sum_{\omega \in \Psi^*_{i \mapsto}} p_\omega - \sum_{\omega \in \Psi^*_{\to i}} p_\omega \right] \\ &= \sum_{i \in I} u_i(\Psi^*) = U^* = V(p'), \end{aligned}$$

where the inequality follows from utility maximization. This proves that $p' \in \arg\min_p V(p)$.

Now, to prove the other implication of the lemma, let p' be any price vector that minimizes V (and thus satisfies $V(p') = U^*$). We claim that $[\Psi^*; p']$ is a competitive equilibrium. To see this, note that the definition of V_i implies that

$$V_i(p') \geq u_i(\Psi^*) + \sum_{\omega \in \Psi^*_{i \mapsto}} p'_\omega - \sum_{\omega \in \Psi^*_{\to i}} p'_\omega. \tag{A3}$$

Summing (A3) across $i \in I$ gives

$$\sum_{i \in I} V_i(p') \geq \sum_{i \in I} \left[u_i(\Psi^*) + \sum_{\omega \in \Psi^*_{i \mapsto}} p'_\omega - \sum_{\omega \in \Psi^*_{\to i}} p'_\omega \right] = \sum_{i \in I} u_i(\Psi^*) = U^*, \tag{A4}$$

with equality holding exactly when (A3) holds with equality for every i. If (A3) were strict for any i, we would obtain $V(p') > U^*$ from (A4), contradicting the assumption that p' minimizes V (and thus p' satisfies $V(p') = U^*$). Thus, for all $i \in I$, equality holds in (A3), and thus $[\Psi^*; p']$ is a competitive equilibrium. QED

Now, suppose that p and q are two competitive equilibrium price vectors, and let $p \wedge q$ and $p \vee q$ denote their meet and join, respectively. Note that

$$\begin{aligned} 2U^* &\leq V(p \wedge q) + V(p \vee q) \\ &\leq V(p) + V(q) = 2U^*, \end{aligned}$$

where the first inequality follows because (by lemma A.3) U^* is the minimal value of V, the second inequality follows from the submodularity of V (which holds because each agent's preferences are fully substitutable, implying that V_i is submodular for every $i \in I$), and the equality follows from lemma A.3 because p and q are competitive equilibrium price vectors. Since we also know that $V(p \wedge q) \geq U^*$ and $V(p \vee q) \geq U^*$, it has to be the case that $V(p \wedge q) = V(p \vee q) = U^*$, and so by lemma A.3, $p \wedge q$ and $p \vee q$ are competitive equilibrium price vectors.

Proof of Theorem 5

Let $A \equiv \kappa([\Psi; p])$. Suppose that A is not stable; then either it is not individually rational or there exists a blocking set.

If A is not individually rational, then $A_i \notin C_i(A)$ for some $i \in I$. Hence, $A_i \notin \arg\max_{Z \subseteq A_i} U_i(Z)$, and therefore, $\tau(A_i) = \Psi_i \notin D_i(p)$, contradicting the assumption that $[\Psi; p]$ is a competitive equilibrium.

Suppose now that there exists a set Z blocking A, and let $J = a(Z)$ be the set of agents involved in contracts in Z. For any trade ω involved in a contract in Z, let $\tilde{p}_\omega$ be the price for which $(\omega, \tilde{p}_\omega) \in Z$. For each $j \in J$, pick a set $Y^j \in C_j(Z \cup A)$. As Z blocks A, (by definition) we have $Z_j \subseteq Y^j$. Since $Z \cap A = \varnothing$ and $Z_j \subseteq Y$ for all $Y \in C_j(Z \cup A)$, we have that $A_j \notin C_j(Z \cup A)$. Hence, for all $j \in J$,

$$U_j(A) < U_j(Y^j) = u_j(\tau(Y^j)) + \sum_{\omega \in \tau(Z)_{j\to}} \tilde{p}_\omega - \sum_{\omega \in \tau(Z)_{\to j}} \tilde{p}_\omega + \sum_{\omega \in \tau(Y^j \setminus Z)_{j\to}} p_\omega - \sum_{\omega \in \tau(Y^j \setminus Z)_{\to j}} p_\omega.$$

Summing these inequalities over all $j \in J$, we have

$$\begin{aligned}
\sum_{j \in J} U_j(A) &< \sum_{j \in J} \left[u_j(\tau(Y^j)) + \sum_{\omega \in \tau(Y^j \setminus Z)_{j\to}} p_\omega - \sum_{\omega \in \tau(Y^j \setminus Z)_{\to j}} p_\omega \right] \\
&= \sum_{j \in J} \left[u_j(\tau(Y^j)) + \sum_{\omega \in \tau(Z)_{j\to}} p_\omega - \sum_{\omega \in \tau(Z)_{\to j}} p_\omega \right. \\
&\qquad \left. + \sum_{\omega \in \tau(Y^j \setminus Z)_{j\to}} p_\omega - \sum_{\omega \in \tau(Y^j \setminus Z)_{\to j}} p_\omega \right] \\
&= \sum_{j \in J} \left[u_j(\tau(Y^j)) + \sum_{\omega \in \tau(Y^j)_{j\to}} p_\omega - \sum_{\omega \in \tau(Y^j)_{\to j}} p_\omega \right] \\
&= \sum_{j \in J} U_j(Y^j),
\end{aligned}$$

where we repeatedly apply the fact that for every trade ω in $\tau(Z)$, the price (first $\tilde{p}_\omega$ and then p_ω) of this trade is added exactly once and subtracted exactly once in the summation over all agents.

Now, the preceding inequality says that the sum of the utilities of agents in J given prices p would be strictly higher if each $j \in J$ chose Y^j instead of A_j. It therefore must be the case that for some $j \in J$, we have $U_j([\tau(Y^j); p]) > U_j([A; p])$. It follows that $A_j \notin D_j(p)$; hence, $[\Psi; p]$ cannot be a competitive equilibrium.

Proof of Theorem 6

Consider a stable outcome $A \subseteq X$. For every agent $i \in I$, define a modified valuation function $\hat{u}_i$ on sets of trades $\Psi \subseteq \Omega \setminus \tau(A)$:

$$\hat{u}_i(\Psi) = \max_{Y \subseteq A_i} \left[u_i(\Psi \cup \tau(Y)) + \sum_{(\omega,\bar{p}_\omega) \in Y_{i\to}} \bar{p}_\omega - \sum_{(\omega,\bar{p}_\omega) \in Y_{\to i}} \bar{p}_\omega \right].$$

In other words, the modified valuation $\hat{u}_i(\Psi)$ of Ψ is equal to the maximal value attainable by agent i by combining the trades in Ψ_i with various subsets of A_i. We denote the utility function associated to $\hat{u}_i$ by $\hat{U}_i$. Since the original utilities are fully substitutable, and thus their demand correspondences D_i satisfy the DEFS condition of Hatfield et al. (2013), the demand correspondences $\hat{D}_i$ for utility functions $\hat{U}_i$ also satisfy the DEFS condition, and thus every $\hat{U}_i$ is also fully substitutable.

Now, consider a *modified economy* for the set of agents I, in which the set of trades is $\Omega \backslash \tau(A)$, and utilities are given by $\hat{U}$. If there is a competitive equilibrium of the modified economy of the form $[\varnothing; \hat{p}_{\Omega \backslash \tau(A)}]$, that is, involving no trades, then we are done: We can combine the prices in this competitive equilibrium with the prices in A to obtain the price vector p as

$$p_\omega = \begin{cases} \bar{p}_\omega & \text{if } (\omega, \bar{p}) \in A \\ \hat{p}_\omega & \text{otherwise.} \end{cases}$$

It is clear that in this case $[\tau(A); p]$ is a competitive equilibrium of the original economy: Since $\varnothing \in \hat{D}_i(\hat{p})$ for every i, no agent strictly prefers to add trades not in $\tau(A)$, and by the individual rationality of A, no agent strictly prefers to drop any trades in $\tau(A)$.

Now suppose that there is not a competitive equilibrium of the modified economy in which no trades occur. By theorem 1, the modified economy has at least one competitive equilibrium $[\hat{\Psi}; \hat{p}]$. By theorems 2 and 3, we know that $\hat{\Psi}$ is efficient and $\varnothing$ is not. It follows that

$$\frac{\sum_{i \in I} \hat{u}_i(\hat{\Psi}) - \sum_{i \in I} \hat{u}_i(\varnothing)}{2|\Omega| + 1} > 0;$$

we denote this value by δ.

Now, we consider a second modification of the valuation functions, obtained by taking

$$\tilde{u}_i(\Psi) = \hat{u}_i(\Psi) - \delta|\Psi_i|.$$

We show next that the utility functions $\tilde{U}_i$ corresponding to the valuations $\tilde{u}_i$ are fully substitutable: Take agent i and any two price vectors p' and p''. Construct a new price vector $\tilde{p}'$ as follows. For every trade $\omega \in \Omega \backslash \tau(A)$, $\tilde{p}'_\omega = p_\omega + \delta$ if $b(\omega) = i$, $\tilde{p}'_\omega = p_\omega - \delta$ if $s(\omega) = i$, and $\tilde{p}'_\omega = 0$ if $\omega \notin \Omega_i$. Construct price vector $\tilde{p}''$ analogously, starting with p''. Note that for any set of trades $\Psi \subseteq \Omega \backslash \tau(A)$, we have $\tilde{U}_i([\Psi; p']) = \hat{U}_i([\Psi; \tilde{p}'])$ and $\tilde{U}_i([\Psi; p'']) = \hat{U}_i([\Psi; \tilde{p}''])$, and therefore, for the corresponding indirect utility functions, we have $\tilde{V}_i(p') = \hat{V}_i(\tilde{p}')$ and $\tilde{V}_i(p'') = \hat{V}_i(\tilde{p}'')$.

Now, by the submodularity of $\hat{V}_i$ (which follows from the full substitutability of $\hat{U}_i$), we have

$$\hat{V}_i(\tilde{p}' \wedge \tilde{p}'') + \hat{V}_i(\tilde{p}' \vee \tilde{p}'') \leq \hat{V}_i(\tilde{p}') + \hat{V}_i(\tilde{p}''),$$

and therefore,

$$\tilde{V}_i(p' \wedge p'') + \tilde{V}_i(p' \vee p'') \leq \tilde{V}_i(p') + \tilde{V}_i(p'').$$

Hence, $\tilde{V}_i$ is submodular, and therefore, $\tilde{U}_i$ is fully substitutable.

Now, by our choice of δ, we have $\sum_{i \in I} \tilde{u}_i(\hat{\Psi}) > \sum_{i \in I} \tilde{u}_i(\varnothing)$. Thus, $\varnothing$ is not efficient under the valuations $\tilde{u}$ and therefore cannot be supported in a competitive equilibrium under those valuations. Take any competitive equilibrium $[\tilde{\Psi}, q]$ of the economy with agents I, trades $\Omega \backslash \tau(A)$, and utilities $\tilde{U}$. We know that $\tilde{\Psi} \neq \varnothing$. Moreover, since $\tilde{\Psi}_i \in \tilde{D}_i(q)$ for every i (where $\tilde{D}_i$ is the demand correspondence induced by $\tilde{U}_i$), we know that for any $\Phi \subsetneq \tilde{\Psi}_i$, $\tilde{U}_i([\tilde{\Psi}_i; q]) \geq \tilde{U}_i([\Phi; q])$, and thus $\hat{U}_i([\tilde{\Psi}_i; q]) > \hat{U}_i([\Phi; q])$. This then implies that for all i, in the original economy with trades Ω and utility functions U_i, the set of trades $\{(\psi, q_\psi) : \psi \in \tilde{\Psi}_i\}$ is a subset of every $Y \in C_i(A \cup \{(\psi, q_\psi) : \psi \in \tilde{\Psi}_i\})$. Thus, $\{(\psi, q_\psi) : \psi \in \tilde{\Psi}\}$ is a blocking set for A, contradicting the assumption that A is stable.

Proof of Theorem 9

Suppose that A is a stable outcome. By theorem 6, there is a vector of prices p such that $[\tau(A); p]$ is a competitive equilibrium. Now note that the second part of the proof of theorem 5 actually shows that any outcome associated with a competitive equilibrium, in particular A, is strongly group stable.

To see that for any core outcome A there is a stable outcome $\hat{A}$ such that $\tau(A) = \tau(\hat{A})$, note that, by theorem 8, every core outcome A has an efficient set of trades $\tau(A)$. By theorem 3, we can find a competitive equilibrium corresponding to any efficient set of trades, in particular, $\tau(A)$. Finally, by theorem 5, this competitive equilibrium induces a stable outcome.

Proof of Theorem 10

We prove part *a*; the proof of part *b* is completely analogous.

First, we show that $\Xi \in D_i(q)$. Note that $|\Xi_i \cap \Psi| \in \{0, 1\}$ by mutual incompatibility of the trades in Ψ. If $\Xi_i \cap \Psi = \varnothing$, then let $\xi \in \arg\max_{\psi \in \Psi} p_\psi$. Now consider $\Phi \in D_i(q)$. As $|\Phi_i \cap \Psi| \in \{0, 1\}$ by mutual incompatibility of the trades in Ψ, there are two subcases to consider: If $\Phi_i \cap \Psi = \varnothing$, then

$$U_i([\Phi; q]) = U_i([\Phi; p]) \leq U_i([\Xi; p]) = U_i([\Xi; q]),$$

where the first step follows from $q_{\Omega \backslash \Psi} = p_{\Omega \backslash \Psi}$, the second step follows from the optimality of Ξ_i at prices p, and the third step follows from $q_{\Omega \backslash \Psi} = p_{\Omega \backslash \Psi}$; hence, $\Xi \in D_i(q)$. Alternatively, if $\Phi_i \cap \Psi = \{\varphi\}$, then

$$\begin{aligned} U_i([\Phi; q]) = U_i([(\Phi \backslash \{\varphi\}) \cup \{\xi\}; q]) &= U_i([(\Phi \backslash \{\varphi\}) \cup \{\xi\}; p]) \\ &\leq U_i([\Xi; p]) = U_i([\Xi; q]), \end{aligned}$$

where the first step follows from perfect substitutability of the trades in Ψ, the second step follows from the facts that $p_\xi = q_\xi$ and $q_{\Omega\setminus\Psi} = p_{\Omega\setminus\Psi}$, the third step follows from the optimality of Ξ_i at prices p, and the fourth step follows from the facts that $p_\xi = q_\xi$ and $q_{\Omega\setminus\Psi} = p_{\Omega\setminus\Psi}$; hence, $\Xi \in D_i(q)$.

If $\Xi_i \cap \Psi \neq \varnothing$, let $\{\xi\} = \Xi_i \cap \Psi$; note that $\xi \in \arg\max_{\psi\in\Psi} p_\psi$ as the trades in the set Ψ are perfectly substitutable and Ξ_i is optimal given prices p. Now consider $\Phi \in D_i(q)$. As $|\Phi_i \cap \Psi| \in \{0, 1\}$ by mutual incompatibility of the trades in Ψ, there are two subcases to consider: If $\Phi_i \cap \Psi = \varnothing$, then

$$U_i([\Phi; q]) = U_i([\Phi; p]) \leq U_i([\Xi; p]) = U_i([\Xi; q]),$$

where the first step follows from $q_{\Omega\setminus\Psi} = p_{\Omega\setminus\Psi}$, the second step follows from the optimality of Ξ_i at prices p, and the third step follows from the facts that $p_\xi = q_\xi$ and $q_{\Omega\setminus\Psi} = p_{\Omega\setminus\Psi}$; hence, $\Xi \in D_i(q)$. Alternatively, if $\Phi_i \cap \Psi = \{\varphi\}$, then

$$\begin{aligned} U_i([\Phi; q]) = U_i([(\Phi\setminus\{\varphi\}) \cup \{\xi\}; q]) &= U_i([(\Phi\setminus\{\varphi\}) \cup \{\xi\}; p]) \\ &\leq U_i([\Xi; p]) = U_i([\Xi; q]), \end{aligned}$$

where the first step follows from perfect substitutability of the trades in Ψ, the second step follows from the facts that $p_\xi = q_\xi$ and $q_{\Omega\setminus\Psi} = p_{\Omega\setminus\Psi}$, the third step follows from the optimality of Ξ_i at prices p, and the fourth step follows from the facts that $p_\xi = q_\xi$ and $q_{\Omega\setminus\Psi} = p_{\Omega\setminus\Psi}$; hence, $\Xi \in D_i(q)$.

Second, we show that for an arbitrary agent $j \neq i$, $\Xi \in D_j(q)$. If $\Xi_j \cap \Psi = \varnothing$, then $p_{\Xi_j} = q_{\Xi_j}$ while $p_{\Omega_{j\to}} = q_{\Omega_{j\to}}$ and $p_{\Omega_{\to j}} \leq q_{\Omega_{\to j}}$; hence $\Xi \in D_j(q)$. If $\Xi_j \cap \Psi \neq \varnothing$, then $\Xi_j \cap \Psi = \{\xi\}$ for some ξ because of mutual incompatibility of the trades in Ψ for i; note that $\xi \in \arg\max_{\psi\in\Psi} p_\psi$ as the trades in the set Ψ are perfectly substitutable and Ξ_i is optimal for i given prices p. Therefore, $p_{\Xi_j} = q_{\Xi_j}$ while $p_{\Omega_{j\to}} = q_{\Omega_{j\to}}$ and $p_{\Omega_{\to j}} \leq q_{\Omega_{\to j}}$; hence $\Xi \in D_j(q)$.

References

Ausubel, Larry M., and Paul Milgrom. 2002. "Ascending Auctions with Package Bidding." *Frontiers Theoretical Econ.* 1:1–42.

Baake, Pio, Jörg Oechssler, and Cristoph Schenk. 1999. "Explaining Cross-Supplies." *J. Econ.* 70:37–60.

Baldwin, Elizabeth, and Paul Klemperer. 2013. "Tropical Geometry to Analyse Demand." Working paper, Oxford Univ.

Bikhchandani, Sushil, and John W. Mamer. 1997. "Competitive Equilibrium in an Exchange Economy with Indivisibilities." *J. Econ. Theory* 74:385–413.

Blair, Charles. 1988. "The Lattice Structure of the Set of Stable Matchings with Multiple Partners." *Math. Operations Res.* 13:619–28.

Crawford, Vincent P., and Elsie Marie Knoer. 1981. "Job Matching with Heterogeneous Firms and Workers." *Econometrica* 49:437–50.

Echenique, Federico, and Jorge Oviedo. 2006. "A Theory of Stability in Many-to-Many Matching Markets." *Theoretical Econ.* 1:233–73.

Gale, David. 1960. *The Theory of Linear Economic Models.* Chicago: Univ. Chicago Press.

Gale, David, and Lloyd S. Shapley. 1962. "College Admissions and the Stability of Marriage." *American Math. Monthly* 69:9–15.

Gale, Ian L., Donald B. Hausch, and Mark Stegeman. 2000. "Sequential Procurement with Subcontracting." *Internat. Econ. Rev.* 41:989–1020.

Gul, Faruk, and Ennio Stacchetti. 1999. "Walrasian Equilibrium with Gross Substitutes." *J. Econ. Theory* 87:95–124.

———. 2000. "The English Auction with Differentiated Commodities." *J. Econ. Theory* 92:66–95.

Hatfield, John William, and Fuhito Kojima. 2008. "Matching with Contracts: Comment." *A.E.R.* 98:1189–94.

———. 2010. "Substitutes and Stability for Matching with Contracts." *J. Econ. Theory* 145:1704–23.

Hatfield, John William, and Scott Duke Kominers. 2012. "Matching in Networks with Bilateral Contracts." *American Econ. J.: Microeconomics* 4:176–208.

———. 2013a. "Contract Design and Stability in Many-to-Many Matching." Working paper, Harvard Univ.

———. 2013b. "Hidden Substitutes." Working paper, Harvard Univ.

Hatfield, John William, Scott Duke Kominers, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp. 2013. "Full Substitutability in Trading Networks." Working paper, Grad. School Bus., Stanford Univ.

Hatfield, John William, and Paul R. Milgrom. 2005. "Matching with Contracts." *A.E.R.* 95:913–35.

Hatfield, John William, Charles R. Plott, and Tomomi Tanaka. 2012a. "Price Controls, Non-price Quality Competition, and Stable Outcomes." Working paper, Grad. School Bus., Stanford Univ.

———. 2012b. "Understanding Price Controls and Non-price Competition through Matching Theory." *A.E.R. Papers and Proc.* 102:371–75.

Kamien, Morton I., Lode Li, and Dov Samet. 1989. "Bertrand Competition with Subcontracting." *RAND J. Econ.* 20:553–67.

Kelso, Alexander S., and Vincent P. Crawford. 1982. "Job Matching, Coalition Formation, and Gross Substitutes." *Econometrica* 50:1483–1504.

Klaus, Bettina, and Markus Walzl. 2009. "Stable Many-to-Many Matchings with Contracts." *J. Math. Econ.* 45 (7–8): 422–34.

Konishi, Hideo, and M. Utku Ünver. 2006. "Credible Group-Stability in Many-to-Many Matching Problems." *J. Econ. Theory* 129:57–80.

Koopmans, Tjalling C., and Martin Beckmann. 1957. "Assignment Problems and the Location of Economic Activities." *Econometrica* 25:53–76.

Larsen, Bradley. 2013. "The Efficiency of Dynamic, Post-auction Bargaining: Evidence from Wholesale Used-Auto Auctions." Working paper, Massachusetts Inst. Tech.

Marion, Justin. 2013. "Sourcing from the Enemy: Horizontal Subcontracting in Highway Procurement." Working paper, Univ. California, Santa Cruz.

NADA (National Automobile Dealers Assoc.). 2012. "NADA DATA 2012: State-of-the-Industry Report." http://www.nada.org/Publications/NADADATA/2012/default.htm.

NIADA (National Independent Automobile Dealers Assoc.). 2011. "NIADA's Used Car Industry Report." http://issuu.com/niada/docs/2011ucr.

Ostrovsky, Michael. 2008. "Stability in Supply Chain Networks." *A.E.R.* 98:897–923.

Patrik, Gary S. 2001. "Reinsurance." In *Foundations of Casualty Actuarial Science*, 4th ed., 343–484. Arlington, VA: Casualty Actuarial Soc.

Powers, Michael R., and Martin Shubik. 2001. "Toward a Theory of Reinsurance and Retrocession." *Insurance: Math. and Econ.* 29:271–90.

Roth, Alvin E. 1984. "Stability and Polarization of Interests in Job Matching." *Econometrica* 52:47–57.

Roth, Alvin E., and Marilda A. O. Sotomayor. 1990. *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge: Cambridge Univ. Press.

Shapley, Lloyd S., and Martin Shubik. 1971. "The Assignment Game I: The Core." *Internat. J. Game Theory* 1:111–30.

Sotomayor, Marilda A. O. 1999. "Three Remarks on the Many-to-Many Stable Matching Problem." *Math. Soc. Sci.* 38:55–70.

———. 2007. "Connecting the Cooperative and Competitive Structures of the Multiple-Partners Assignment Game." *J. Econ. Theory* 134:155–74.

Spiegel, Yossef. 1993. "Horizontal Subcontracting." *RAND J. Econ.* 24:570–90.

Sun, Ning, and Zaifu Yang. 2006. "Equilibria and Indivisibilities: Gross Substitutes and Complements." *Econometrica* 74:1385–1402.

———. 2009. "A Double-Track Adjustment Process for Discrete Markets with Substitutes and Complements." *Econometrica* 77:933–52.

Tadelis, Steven, and Florian Zettelmeyer. 2011. "Information Disclosure as a Matching Mechanism: Theory and Evidence from a Field Experiment." Working paper, Univ. California, Berkeley.

Westkamp, Alexander. 2010. "Market Structure and Matching with Contracts." *J. Econ. Theory* 145:1724–38.