Appendix

To derive the affine bond pricing formulas and yield curve equations, consider the case with prices of risk $\lambda_t = [\lambda^1_t, \lambda^2_t]^\top$. (Note that equation (9) can be obtained from (10) by setting the prices of risk to zero.) There are two ways to derive these formulas. First, we can construct a risk-neutral probability measure under which the risk-neutral pricing formula (7) holds. Second, we can start from the Euler equation $E [d (m_tF_t)] = 0$.

Risk-neutral probability

Under the risk-neutral probability measure, the process $B^*$ which solves $dB^*_t = dB_t + \lambda_t dt$ is a Brownian motion. By solving for $dB_t$ and inserting this expression into the AR(1) dynamics of the factors (6), we get

$$dx^*_t = \kappa_i (\theta^*_i - x^*_i) dt + \sigma_i (dB^*_t - \lambda^i_t dt) \tag{11}$$

$$= (\kappa_i \theta^*_i - \kappa_i x^*_i - \sigma_i \lambda^i_0 - \sigma_i \lambda^1_i x^*_i) dt + \sigma_i dB^*_t \tag{12}$$

$$= (\kappa_i \theta^*_i - \sigma_i \lambda^i_0 - (\kappa_i + \sigma_i \lambda^1_i) x^*_i) dt + \sigma_i dB^*_t \tag{13}$$

$$= (\kappa_i + \sigma_i \lambda^1_i) \left( \frac{\kappa_i \theta^*_i - \sigma_i \lambda^i_0}{\kappa_i + \sigma_i \lambda^1_i} - x^*_i \right) dt + \sigma_i dB^*_t \tag{14}$$

$$= \kappa^*_i (\theta^*_i - x^*_i) dt + \sigma_i dB^*_t, \tag{15}$$

where

$$\kappa^*_i = \kappa_i + \sigma_i \lambda^1_i$$

$$\theta^*_i = \frac{\kappa_i \theta^*_i - \sigma_i \lambda^i_0}{\kappa_i + \sigma_i \lambda^1_i}$$

The price of the $\tau$-period bond is equal to

$$P_t^{(\tau)} = E^*_t \left( \exp \left( - \int_t^{t+\tau} r_s ds \right) \right),$$

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where the expectation operator $E^*$ uses the risk-neutral probability measure. Since the vector $x = (x_1, x_2)^T$ is Markov, this expectation is a function of the state today $x_t$. Thus, the bond price is a function

$$P_t^{(\tau)} = F(x_t, \tau)$$

of the state vector $x_t$ and time-to-maturity $\tau$. The Feynman-Kac formula says that $F$ solves the partial differential equation

$$F_t + r_t F = -\frac{\partial F}{\partial \tau} + \sum_{i=1}^{2} \left[ \frac{\partial F}{\partial x_i} \kappa_i^* (\theta_i^* - x_i^t) + \frac{1}{2} \frac{\partial^2 F}{\partial x_i^2 \sigma_i^2} \right]$$

with terminal condition $F(x, 0) = 1$.

We guess the solution

$$F(x_t, \tau) = \exp (A(\tau) + B(\tau) \cdot x_t)$$

which means that

$$\frac{\partial F}{\partial x_i} = B_i(\tau) F$$

$$\frac{\partial^2 F}{\partial x_i^2} = B_i(\tau)^2 F$$

$$\frac{\partial F}{\partial \tau} = (A'(\tau) + B'(\tau) \cdot x_t) F.$$ 

Insert these expressions into the partial differential equation and get

$$x_t^1 + x_t^2 = -A'(\tau) - B_1'(\tau) x_t^1 - B_2'(\tau) x_t^2$$

$$+ \sum_{i=1}^{2} \left[ B_i(\tau) \kappa_i^* (\theta_i^* - x_i^t) + \frac{1}{2} B_i(\tau)^2 \sigma_i^2 \right].$$
Matching coefficients results in

\[ A' (\tau) = \sum_{i=1}^{2} B_i (\tau) \kappa_i^* \theta_i^* + \frac{1}{2} B_i (\tau)^2 \sigma_i^2 \]

\[ 1 = -B_1' (\tau) - B_1 (\tau) \kappa_1^* \]

\[ 1 = -B_2' (\tau) - B_2 (\tau) \kappa_2^*. \]

The boundary conditions are

\[ A (0) = 0 \]

\[ B (0) = 0_{2\times1}. \]

The solution to these ODE’s are

\[ B_1 (\tau) = \frac{(\exp (-\kappa_1^* \tau) - 1)}{\kappa_1^*} \]

\[ B_2 (\tau) = \frac{(\exp (-\kappa_2^* \tau) - 1)}{\kappa_2^*}. \]

We can plug these solutions into the yield equation

\[ y_{t}^{(\tau)} = -\frac{A (\tau)}{\tau} - \frac{B_1 (\tau)}{\tau} x_1^2 - \frac{B_2 (\tau)}{\tau} x_2^2 \]

\[ = a^{NA} (\tau) + b_1^{NA} (\tau) x_1^1 + b_2^{NA} (\tau) x_2^2 \]

and get equations (9).

**Euler equation approach**

The Euler equation is

\[ P_{t}^{(\tau)} = E_t \left[ \frac{m_{t+\tau}}{m_t} \right] \]

and the instantaneous equation is

\[ E [d (m_t F_t)] = 0. \]
The bond price is a function $F(x, \tau)$ and we can apply Ito’s Lemma

$$dF = \mu_F dt + \sigma_F dB_t,$$

where the drift and volatility of $F$ are given by

$$\mu_F = -\frac{\partial F}{\partial \tau} + \sum_{i=1}^{2} \left[ \frac{\partial F}{\partial x_i} \kappa_i \left( \theta_i - x_i \right) + \frac{1}{2} \frac{\partial^2 F}{\partial x_i^2} \sigma_i^2 \right],$$

$$\sigma_F = \sum_{i=1}^{2} \frac{\partial F}{\partial x_i} \sigma_i.$$

Both $m_t$ and $F_t$ are Ito processes, so their product solves

$$d(m_t F_t) = -r_t m_t F_t dt + m_t \mu_F^F dt - m_t \lambda t \sigma_t^F dt$$

$$- F_t m_t \lambda t dB_t + m_t \sigma_t^F dB_t.$$

We use the Euler equation (19) and get

$$0 = -r_t m_t F_t + m_t \mu_t^F - m_t \lambda t \sigma_t^F,$$  \hspace{1cm} (20)

$$F_t r_t = \left( -\frac{\partial F}{\partial \tau} + \sum_{i=1}^{2} \left[ \frac{\partial F}{\partial x_i} \kappa_i \left( \theta_i - x_i \right) + \frac{1}{2} \frac{\partial^2 F}{\partial x_i^2} \sigma_i^2 \right] \right) - \sum_{i=1}^{2} \frac{\partial F}{\partial x_i} \sigma_i \lambda_t.$$

Again, guess the exponential-affine solution (16) and insert the expressions into (20), we get

$$x_t^1 + x_t^2 = -A' (\tau) - B_1' (\tau) x_t^1 - B_2' (\tau) x_t^2$$

$$+ \sum_{i=1}^{2} \left[ B_i (\tau) \kappa_i \left( \theta_i - x_i \right) + \frac{1}{2} B_i (\tau)^2 \sigma_i^2 \right]$$

$$- \sum_{i=1}^{2} B_i (\tau) \sigma_i \left( \lambda_0 + \lambda_1 x_i \right).$$

Matching coefficients, we get the ordinary differential equations:

$$A' (\tau) = \sum_{i=1}^{2} B_i (\tau) \left( \kappa_i \theta_i - \sigma_i \lambda_0 \right) + \frac{1}{2} B_i (\tau)^2 \sigma_i^2$$

$$1 = -B_1' (\tau) - B_1 (\tau) (\kappa_1 + \sigma_1 \lambda_1)$$

$$1 = -B_2' (\tau) - B_2 (\tau) (\kappa_2 + \sigma_2 \lambda_1).$$
From this expression, we can see that we get the coefficients (17a) with risk neutral parameters

\[
\begin{align*}
\kappa_i^* &= \kappa_i + \sigma_i \lambda_1^i \\
\kappa_i^* \theta_i^* &= \kappa_i \theta_i - \sigma_i \lambda_0^i \implies \theta_i^* = \frac{\kappa_i \theta_i - \sigma_i \lambda_0^i}{\kappa_i + \sigma_i \lambda_1^i}.
\end{align*}
\]