1 Note on exponential-affine stock prices

The idea is to specify the dividend yield $\delta$ and short rate $r$ to be affine in $X$ where $X$ is an affine diffusion under the risk-neutral measure. Then the stock price is guessed to be exponential affine. To show that the guess works, I have to show that the guess satisfies

$$P (t) = E^*_t \left[ \int_t^\infty e^{-\int_t^r r(u)du} \delta (s) P (s) ds \right],$$

(1)

where $E^*$ denotes expectation under the risk-neutral measure. The following simple example with a normally distributed dividend yield, zero short rate and zero market prices of risk illustrates that the functional form result applies. Backshi and Chen (1997, JFE) compute exponential-affine stock prices for the case where $X$ is a square-root process.

PROPOSITION: Assume that the short rate is zero, $r = 0$ (in the notation of the paper $r_0 = 0$ and $r_X = 0$). The dividend yield is $\delta = X$ (in the notation of the paper, $\delta_0 = 0$ and $\delta_X = 1$) where $X$ is an OU process which solves

$$dX (t) = k(\theta - X (t))dt + \sigma dW (t).$$

Suppose $P$ is of the form

$$P (t) = \exp (at + \frac{X (t)}{k}),$$

(2)

with

$$a = -\theta - \frac{1}{2} \frac{\sigma^2}{k} < 0.$$  

(3)

then $P$ satisfies the pricing equation (1).

PROOF OF PROPOSITION (the proof refers to a series of facts stated in section 2 of this note): Assume that the price satisfies the guess (2). We need to show that

$$P (t) = E_t \int_t^\infty \delta (s) P (s) ds$$

$$= E_t \left[ \lim_{T \to \infty} \int_t^T \delta (s) P (s) ds \right].$$

(4)
Define

\[ P^T (t) = E_t \int_t^T \delta (s) P (s) \, ds \]

I will show (4) in two steps. Step (i) is

\[ P (t) = \lim_{T \to \infty} P^T (t) \] (5)

Step (ii) is

\[ \lim_{T \to \infty} P^T (t) = E_t \left[ \lim_{T \to \infty} \int_t^T \delta (s) P (s) \, ds \right] \] (6)

Together, these two steps yield (4).

**STEP (i) :** I want to interchange expectation and integral,

\[ P^T (t) = E_t \left[ \int_t^T P (s) \delta (s) \, ds \right] = \int_t^T E_t [P (s) \delta (s)] \, ds \] (7)

For Fubini to apply, I need that

\[ E_t \int_t^T |P (s) \delta (s)| \, ds < \infty. \] (8)

Tonelli’s theorem says

\[ E_t \int_t^T |P (s) \delta (s)| \, ds = \int_t^T E_t |P (s) \delta (s)| \, ds \] (9)

The RHS is finite, because by FACT 3:

\[ E_t [\|P (s) \delta (s)\|] = E_t [\|\delta (s)\| P (s)] \]

\[ = E_t [\|Z (s)\|] \exp \left( as + m_t (s) / k + \frac{1}{2} \nu_t (s) / k^2 \right) \]

where \( Z (s) \sim N (m_t (s) + \nu_t (s) / k, \nu_t (s)) \). This expression is continuous in \( s \).
The term beneath the integral in (7) is given by FACT 2:

\[ E_t [\delta(s) P(s)] = E_t [X(s) \exp (as + X(s) /k)] = (m_t(s) + v_t(s) /k) \exp \left( as + m_t(s) /k + \frac{1}{2} v_t(s) /k^2 \right) \]

Using FACT 4,

\[ P^T(t) = \int_t^T E_t [P(s) \delta(s)] \, ds \]

\[ = \int_t^T (m_t(s) + v_t(s) /k) \exp \left( as + m_t(s) /k + \frac{1}{2} v_t(s) /k^2 \right) \, ds \]

\[ = - \exp \left( aT + m_t(T) /k + \frac{1}{2} v_t(T) /k^2 \right) + \exp \left( at + m_t(t) /k + \frac{1}{2} v_t(t) /k^2 \right). \]

Since \( \lim_{T \to \infty} m_t(T) = \theta \) and \( \lim_{T \to \infty} v_t(T) = \frac{a^2}{2k} \), I have

\[ \lim_{T \to \infty} \exp \left( aT + m_t(T) /k + \frac{1}{2} v_t(T) /k^2 \right) = 0 \]

as long as \( a < 0 \), which I assumed in (3). This leaves

\[ \lim_{T \to \infty} P^T(t) = \exp \left( at + m_t(t) /k + \frac{1}{2} v_t(t) /k^2 \right). \]

where I can note that \( m_t(t) /k = X(t) /k \) ad \( v_t(t) /k^2 = 0 \), so that I indeed get equation (5) for our guess (2).

STEP (ii) : From step (i), I know that

\[ \lim_{T \to \infty} P^T(t) = \lim_{T \to \infty} \int_t^T E_t [P(s) \delta(s)] \, ds \]

I want to use Fubini to argue that the RHS of the last equation is equal to the RHS of (6). For Fubini to apply, I need condition (8) for \( T = \infty \). The same arguments go through as before, and I know that \( m_t(s) \) and \( v_t(s) \) go to constants for \( s \to \infty \), which means that the expression in (10) goes to zero because \( a < 0 \). This completes the proof that (6) holds.
2 Useful facts

FACT 1. Suppose $X$ solves

$$dX(t) = k(\theta - X(t))dt + \sigma dW(t).$$  \hspace{1cm} (11)

starting at $X(0) = x_0$ and for constants $k$, $\theta$ and $\sigma$. Then the solution to (11) is

$$X_s = \exp(-k(s-t))X_t + \theta (1 - \exp(-k(s-t))) + \int_t^s \exp(-k(s-u))\sigma dW(u).$$

which is normal with mean

$$m_t(s) \equiv \exp(-k(s-t))X_t + \theta (1 - \exp(-k(s-t))),$$

and variance

$$v_t(s) \equiv \frac{\sigma^2}{2k} (1 - \exp(-2k(s-t))).$$

FACT 2: Suppose $X \sim N(m, v)$. Then I have for any constant $c$

$$E \left[ X e^{cX} \right] = (m + cv) \exp \left( cm + \frac{1}{2} c^2 v \right).$$

This can be verified by direct computation

$$E \left[ X e^{cX} \right] = \int X \exp(cX) \exp \left( \frac{- (X - m)^2}{0.5v} \right) \frac{1}{\sqrt{2\pi v}} dX$$

$$= \int X \exp \left( \frac{-X^2 - m^2 + 2X(m + cv)}{2v} \right) \frac{1}{\sqrt{2\pi v}} dX$$

$$= \int X \exp \left( \frac{- (X - (m + cv))^2 + 2mcv + c^2v^2}{2v} \right) \frac{1}{\sqrt{2\pi v}} dX$$

$$= \int X \exp \left( mc + \frac{1}{2} c^2 v \right) \exp \left( \frac{- (X - (m + cv))^2}{2v} \right) \frac{1}{\sqrt{2\pi v}} dX$$
FACT 3: Suppose $X \sim N(m, v)$. Then we have for any constant $c$

$$E[|X|e^{cX}] = E[|Y|] \exp \left( cm + \frac{1}{2}c^2v \right).$$

where $Y \sim N(m + cv, v)$

FACT 4:

$$\frac{d}{ds} \exp \left( \frac{as + m_t(s)}{k} + \frac{1}{2}v_t(s) / k^2 \right)$$

$$= - \left( m_t(s) + v_t(s) / k \right) \exp \left( \frac{as + m_t(s)}{k} + \frac{1}{2}v_t(s) / k^2 \right)$$

as long as

$$a = -\theta - \frac{1}{2} \frac{\sigma^2}{k^2}$$

PROOF OF FACT 4: Taking derivatives:

$$\frac{d}{ds} \exp \left( \frac{as + m_t(s)}{k} + \frac{1}{2}v_t(s) / k^2 \right)$$

$$= \left( a + \frac{\partial m_t(s)}{\partial s} \frac{1}{k} + \frac{\partial v_t(s)}{\partial s} \frac{1}{2k^2} \right) \exp \left( \frac{as + m_t(s)}{k} + \frac{1}{2}v_t(s) / k^2 \right)$$

$$\frac{\partial m_t(s)}{\partial s} = -k \exp (-k(s-t)) (X_t - \theta)$$

$$\frac{\partial v_t(s)}{\partial s} = \sigma^2 \exp (-2k(s-t))$$

$$a - \exp (-k(s-t)) (X_t - \theta) + \sigma^2 \exp (-2k(s-t)) \frac{1}{2k^2}$$

$$= -\theta - \frac{1}{2} \frac{\sigma^2}{k^2} - \exp (-k(s-t)) (X_t - \theta) + \sigma^2 \exp (-2k(s-t)) \frac{1}{2k^2}$$

$$= - (\theta + \exp (-k(s-t)) (X_t - \theta)) - \left( \frac{1 - \exp (-2k(s-t))}{2k^2} \right) \sigma^2$$

$$= - (m_t(s) + v_t(s) / k)$$
3 Remarks

Theorem 1 of the paper states a solution of the form

\[ P(t) = \exp(A(t) - B(t)X(t)) \]

with coefficients (10)-(12)

\[
\begin{align*}
0 &= A'(t) - \theta kB(t) + \frac{1}{2}\sigma^2 B(t)^2 \\
0 &= 1 - B'(t) + kB
\end{align*}
\]

for \( \delta_0 = 0 \) and \( \delta_X = 1 \) in \( \delta = \delta_0 + \delta_X X \) and \( r_0 = 0 \) (because \( r = 0 \)). Now use Restriction 2 from the paper, which sets \( B'(t) = 0 \). This implies

\[ B(t) = -\frac{1}{k} \]

and therefore

\[ 0 = A'(t) + \theta + \frac{1}{2}\sigma^2 \]

This equation is solved for \( A(t) = at \) where

\[ a = -\theta - \frac{1}{2}\sigma^2 \frac{1}{k^2}. \]