Estimating Rational Expectations Models

Monika Piazzesi*

May 7, 2007

1 Overview

Most dynamic models in economics assume that agents form expectations rationally. An equilibrium of a dynamic model can typically be described by a probability distribution over sequences of data. The rational expectations assumption says that every agent's subjective belief about the data is a conditional of this equilibrium probability distribution, where the conditioning is on the agent's information set. Expectations are thus consistent with outcomes generated by the model. They are also optimal, in the sense that they correctly use all information available to the agent.

The rational expectations assumption was first proposed by John F. Muth in the early 1960s in his analysis of linear macroeconomic models. Prior to Muth's work, expectations in those models had been parametrized distributed lags. In the early 1970s, Robert E. Lucas Jr. studied the rational expectations equilibrium of a model with optimizing agents who have different information sets. It was recognized early on that taking rational expectations models to data required new techniques. Building on the early work on tests of the natural rate hypothesis by Sargent (1971), there has been much progress in rational expectations econometrics over the last three decades (for example, see Hansen and Sargent 1980, Lucas and Sargent 1981, and Hansen and Sargent 1991). In the meantime, the rational expectations assumption has come to be used in many fields of economics, including finance, labor economics, and industrial organization.

Rational expectations impose cross-equation restrictions that have important implications for the estimation of models which I will describe below. These implications have lead to the development of new estimation and testing techniques. More recently, this development has generated techniques that handle models that cannot be solved analytically. Together with the rapid increase in computing power, these methods offer insights into the working of these models and thereby enable their refinement.

*University of Chicago and NBER. Graduate School of Business, University of Chicago, 5807 S Woodlawn, IL 60637. piazzesi@uchicago.edu

2 Cross-equation restrictions

The rational expectations assumption implies cross-equation restrictions that constrain parameters and shocks in different places of the model. There are (at least) three reasons for why these restrictions have important implications for estimation. First, cross-equation restrictions constrain the parameters associated with agents' expectations to be consistent with the parameters from the equilibrium probability distribution. These restrictions reduce the overall number of parameters that have to be estimated. In particular, they eliminate any free parameters associated with expectations. To see why, consider a dynamic model with an agent who maximizes some objective function subject to constraints. To solve this optimization program, the agent needs to form expectations about future variables such as growth rates. In a model without rational expectations, these expectations might be based on some subjective belief about the future. This belief introduces free parameters that need to be estimated in addition to other model parameters, such as preference parameters.

Take, for example, an endowment economy populated by a representative agent with time-separable power-utility. The agent may be optimistic and believe in high mean growth rates for the endowment. This optimistic belief will have an affect on equilibrium outcomes. For example, the agent's Euler equation will only hold for a high short real rate, because the high mean growth rate implies a strong consumption-smoothing motive. However, the actual mean growth rate in this economy may be lower than what the agent believes (so that the agent will be disappointed by the endowment realizations.)

The estimation of the model with an optimistic agent involves two parameters, the subjective mean of endowment growth and its true mean, which is the mean of the data-generating process of endowment growth. The assumption of rational expectations reduces the number of parameters to estimate, because the two mean parameters collapse: the agent's subjective belief is equal to the true data-generating process. In this simple example, the cross-equation restrictions only eliminated one parameter. In more realistic examples, the agent's subjective belief may involve many parameters (for example, because it is described by a vector autoregression in many variables and with many lags), so that the restrictions are important for keeping the estimation tractable.

The second important implication of cross-equation restrictions are that the processes for different endogenous variables often involve the same parameters and shocks. As a consequence, different data series are informative about the same set of parameters. This implication can be used to increase the efficiency of the estimation. Going back to the example of a representative-agent endowment economy, the equation describing the equilibrium process of an interest rate on a bond with m-period maturity is intimately related to the equation describing the process of an n-period interest rate for some $m \neq n$. The relationship between different interest-rate equations, or restrictions across equations, consists of parameters that enter both equations (e.g., expected growth) and also shock processes that affect both equations (e.g., surprises in growth). These restrictions help in the estimation and can be tested empirically with data on interest rates with different maturities.

Some of the earliest tests of cross-equation restrictions were indeed tests of the implica-

tions of rational expectations for the term-structure of interest rates. Sargent (1979) specifies a vector autoregression for short and long rates. Assuming Gaussian disturbances, Sargent estimates this VAR using maximum likelihood and performs likelihood ratio tests to see whether the restrictions imposed by the expectations hypothesis are satisfied. Subsequently, these tests were further refined, and the expectations hypothesis (which is a stronger assumption than rational expectations) was rejected in many empirical studies. The lessons from these statistical rejections have resulted in refined models with rational expectations but time-varying risk premia (e.g., Ang and Piazzesi 2003).

The third important implication of rational expectations is that the data-generating process that underlies agent beliefs is equal to the true data-generating process. This enables the estimation of rational expectations models using GMM based on moment conditions derived from Euler equations (see Hansen 1982, Hansen and Singleton 1982, and Lars Peter Hansen's contribution to the Palgrave.) Using the law of iterated expectations, such a GMM estimation also allows for the case that agents in the model have more information than the econometrician.

3 Estimation Methods

Estimation methods for rational expectations models can be distinguished by the amount of information they require. Generally speaking, there are full information methods and limited information methods. The goal of full information methods is to estimate the entire model by exploiting all its cross-equation restrictions. This estimation method is efficient and produces estimates for all the parameters in the model. These methods are maximum likelihood and its Bayesian counterparts (see the Palgrave contribution by Frank Schorfheide). To apply these methods, the econometrician needs to specify the entire structure of the model, including the distribution of shocks.

Limited information methods require less structure. The goal of these methods is to exploit only some of the restrictions imposed by the model and to obtain estimates for only some of the model parameters. These methods lose some of the efficiency of the full information methods, but they help the researcher to avoid contaminating the estimation results by model misspecification in parts of the model that are not of interest. For example, Hall (1978) and Hansen and Singleton (1982) use the Euler equations from a single-agent model as moment conditions for GMM and measure the empirical counterparts of these moments using data on consumption and financial returns. This procedure gives estimates for preference parameters and does not depend on any specific assumption on the distribution of shocks in the model.

Faced with the difficulty that many models do not have analytical solutions and have to be solved numerically, there has been progress regarding simulation-based estimation methods. These methods compare moments of data simulated from the model using some parameter values with their empirical counterparts. For a textbook treatment of these methods, see Gourieroux and Monfort (1996), Gourieroux and Jasiak (2001), and Singleton (2006).

References

[1] Ang, Andrew and Monika Piazzesi 2003. "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables" *Journal of Monetary Economics* 50(4), pp. 745-787.

[2] Gourieroux, Christian and Joann Jasiak 2001. "*Financial Econometrics.*" Princeton University Press, Princeton.

[3] Gourieroux, Christian and Alain Monfort 1996. "*Simulation-Based Econometric Methods.*" Oxford University Press, Oxford.

[4] Hall, Robert E. 1978. "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence." *Journal of Political Economy* 86, pp. 971-987.

[5] Hansen, Lars Peter 1982. "Large Sample Properties of Generalized Method of Moments Estimators." *Econometrica* 50, pp. 1029-1054.

[6] Hansen, Lars Peter and Thomas J. Sargent 1980. "Formulating and Estimating Dynamic Linear Rational Expectations Models." *Journal of Economic Dynamics and Control* 2(1), pp. 7-46.

[7] Hansen, Lars Peter and Thomas J. Sargent 1991. "*Rational Expectations Econometrics*" Westview Press.

[8] Hansen, Lars Peter and Kenneth Singleton 1982. "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns." *Journal of Political Economy* 91, pp. 249-265.

[9] Lucas, Robert E. and Thomas J. Sargent 1981. "*Rational Expectations and Econometric Practice.*" University of Minnesota Press, Minneapolis.

[10] Sargent, Thomas J. 1971. "A Note on the Accelerationist Controversy." *Journal of Money, Credit, and Banking* 3(3), pp. 721-725.

[11] Singleton, Kenneth 2006. *Empirical Dynamic Asset Pricing.* Princeton University Press.