Speculative Attacks against a Strategic Agent with Incomplete Information

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Abstract

When faced with a speculative attack, banks and governments often hesitate, attempting to withstand the attack but giving up after some time, suggesting they have some ex-ante uncertainty about the magnitude of the attack they will face. I model that uncertainty as arising from incomplete information about speculators’ payoffs and find conditions such that unsuccessful partial defences are possible equilibrium outcomes. There exist priors over the distribution of speculators’ payoffs that can justify any possible partial defence strategy. With Normal uncertainty, partial resistance is more likely when there is more aggregate uncertainty regarding agents’ payoffs and less heterogeneity among them.

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1 Introduction

In August and September of 1992, the Bank of England sold billions of dollars of foreign reserves in an attempt to maintain the pound’s exchange rate within the bands of the European ERM. On September 16 it finally gave up and abandoned the ERM. The cost of having attempted to defend the parity was later estimated at approximately £3.3 billion.

This pattern, of first attempting to defend the existing regime and giving up after some time, is a common feature of speculative attacks and explaining it presents a theoretical challenge. So-called first-generation models based on Krugman (1979), such as Flood and Garber (1984) and Broner (2007) account for it in a very simple way: by assuming the government follows and attempts to defend an unsustainable policy for exogenous reasons and abandons it only when forced to do so. However, these models leave unanswered the question of why a government would behave this way.

As formalized by Obstfeld (1996) and others, speculative attacks often have a self-fulfilling aspect: if enough agents believe the government will abandon a regime, they will act in ways that make it optimal for the government to indeed abandon it. The unsatisfying conclusion of models of self-fulfilling equilibria is that, at least within some range of parameters, the outcome is arbitrary or depends on ad-hoc unmodeled factors. Following Morris and Shin (1998), many authors have argued that modifying the common-knowledge assumptions of games that have self-fulfilling equilibria may help to resolve this indeterminacy and provide more definite predictions.

The basic building block of models in this literature is a game played by many small agents (“speculators”) who are incompletely informed about the relevant parameters of the economy, and one large agent (“the bank”) who has complete information. Typically, the focus of the analysis is on the structure of actions and information of the game played by the speculators. In contrast, the bank’s information and objectives are usually described in very simple terms so that its strategy can be summarized, or even replaced, by a simple rule such as “defend the existing regime unless a mass of speculators larger than $A^*$ attacks it”.

This paradigm (and for that matter the multiple-equilibria paradigm too) fails to account for why the bank, acting rationally, would ever engage in an unsuccessful partial defence of the regime, as the Bank of England did in 1992. In these models, the bank knows the “fundamentals” of the economy and can therefore perfectly predict (or in some versions observe) what the size of the speculative attack is going to be. As long as defending the regime is costly, it would never be the case that it attempts to defend it but surrenders after some time, since this failure would have been foreseen.

However, in some contexts the possibility of a temporary, unsuccessful defence of the status quo makes an important difference. For example, if we wish to apply these methods to the
study of bank runs, as Goldstein and Pauzner (2005) do, the *only* reason why depositors would run is if they believe that the bank will pay some of them before falling or deciding to suspend convertibility.

How can the theory account for the phenomenon of unsuccessful defences? One possibility, implicitly assumed by Morris and Shin (1998), is that defending the regime is not costly at the margin; conditional on regime change, the bank does not have a preference for how far it held out. In many contexts this is not a reasonable assumption: the losses to the central bank’s balance sheet are greater the more reserves it has spent trying to defend a fixed exchange rate; the liquidation costs a banks incurs in are smaller the sooner it suspends convertibility; the retaliation against a dictator is likely to be harsher the longer it held on to power.

If unsuccessful defences are costly, a theory that accounts for them must somehow allow the bank to have uncertainty about the size of the attack it is going to face.\(^1\) With uncertainty and suitable timing assumptions, the bank’s decision may be viewed as an optimal stopping problem: as the attack escalates, it must decide whether to surrender or to continue to defend the regime in the hope that the attack will be over soon, using its appropriately updated beliefs about how large the attack is likely to be.

One way to introduce uncertainty is to abandon Nash equilibrium as a solution concept. In any Nash equilibrium, the bank knows the strategies of the speculators, and is thus able to predict the size of the speculative attack with no uncertainty.\(^2\) However, under appropriate conditions (although not in the Morris-Shin limit), both attacking a regime and not attacking it are rationalizable actions. If the requirement that the bank know the speculators’ strategies is dropped, it is possible to simply endow it with beliefs about the joint distribution of (rationalizable) actions the speculators might take, and under these beliefs a policy of partial defence may indeed be optimal. The trouble with explaining the phenomenon along these lines is that this explanation relies on arbitrary assumptions about the bank’s beliefs and simply picks one of the many rationalizable action profiles.

This paper introduces uncertainty into the bank’s decision problem in a different way. As in Diamond and Dybvig (1983), there is aggregate uncertainty about the distribution of (heterogeneous) preferences in the population. This distribution is governed by a single random parameter \(\theta\), and neither the bank nor the speculators know its realization. Although the bank knows the equilibrium strategies, the equilibrium size of the attack, conditional on the bank’s information, is a random variable, so it faces a nontrivial optimal stopping problem.

For this problem to have an interior solution, in which the bank surrenders after some time

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1. This is also noted by Goldstein, Ozdenoren and Yuan (2008). In their model the central bank has uncertainty about the value of maintaining a fixed exchange rate and may learn about this by observing the speculative attack. Their model does not, however, allow the central bank to surrender to the attack midway.
2. This is strictly true in pure strategies, though possibly not in correlated mixed strategies if the bank does not observe the correlating device.
Despite having incurred sunk costs of defence, it must be that as the attack progresses the bank becomes sufficiently more pessimistic about the magnitude of the attack that it will face. In particular, it must think that an attack that is not over by the stopping point is unlikely to be over soon after that, which is consistent with some ex-ante beliefs \( f(A) \) about the size of the attack \( A \) but not with others. However, I show that given any probability distribution \( f(A) \), it is possible to reverse-engineer a prior about \( \theta \) such that \( f(A) \) is indeed the endogenous probability distribution in an equilibrium of the game. Hence a basic finding is that the model is able to deliver the kinds of uncertainty that could justify partial defences.

This alone, however, provides no clear guidance as to what factors make unsuccessful defences likely to arise. To answer this question, I specialize the model to a simple example with linear payoffs and normal uncertainty. In this case, I show that partial defences will occur if heterogeneity in preferences is sufficiently small relative to aggregate uncertainty about average preferences. The model can be shown to have multiple equilibria in certain cases. The bank’s uncertainty and the possibility of partial unsuccessful defences, however, are not due to this but to uncertainty about outcomes within a given equilibrium.

The model introduces dynamics into the model in an extremely limited way. The speculators must still decide whether or not to attack at the beginning of the game and are not allowed to learn from one another’s actions. This abstracts from what is certainly a very important dimension in real-life speculative attacks. Furthermore, although I will informally describe the bank’s actions as “waiting” there is no real temporal dimension to the problem: the attack is assumed to build up continuously and the bank must choose a point in this continuum to stop defending the attack.

Section 2 introduces the model and defines equilibrium conditions. Section 3 explores, in the general case, how uncertainty about preferences translates into uncertainty about outcomes. Section 4 discusses the special case of normal uncertainty and linear payoffs. Section 5 briefly concludes. The Appendix contains the proofs omitted in the text.

2 The Model

The backbone of the model is a variant of the simple binary action game of Morris and Shin (2003). There is a measure-one continuum of speculators, indexed by \( i \in [0, 1] \) and a bank. At the beginning of the game, each speculator chooses one of two actions: either attack the current policy regime, \( a_i = 1 \), or not attack, \( a_i = 0 \). Agents who decide to attack form a queue and inform the bank of their decision one by one. The bank cannot observe the length of the queue. After each one of them announces his decision the bank has two options: either abandon the regime or to continue to support it. If the bank abandons the regime the game ends in Defeat for the bank. If it continues to support it, two things may happen. If the attacker was the last
in the queue, the game ends in Survival for the bank; otherwise, the game continues with the next attacker.

2.1 Speculators’ Payoffs

The payoff from not attacking is normalized to zero whereas attacking has an idiosyncratic cost $c_i$ and brings a benefit of 1 if the regime is defeated. Agent $i$’s payoff is therefore given by

$$u_i = a_i [\mathbb{1}\{\text{Defeat}\} - c_i]$$

Costs of attacking are distributed in the population according to the distribution $G(c_i|\theta)$ with density $g(c_i|\theta)$, where $\theta$ is a parameter. $G(c_i|\theta)$ is assumed to be decreasing in $\theta$, so $c_i$ is increasing in $\theta$ in a FOSD sense. $c_i$ is not constrained to lie in $[0, 1]$ so one of the actions may be dominant for some of the speculators. In the context of bank runs, where attacking means withdrawing a deposit, $G(0|\theta)$ can represent the fraction of speculators that receive a liquidity shock and need to attack, as in Diamond and Dybvig (1983), and $1 - G(1|\theta)$ may represent the fraction of speculators who have long-term deposits, or are out of town or are otherwise unable to participate in the run. In the context of currency attacks, $G(0|\theta)$ may represent the demand for foreign currency to pay for imports and $1 - G(1|\theta)$ may represent the post-devaluation domestic money demand, as in Krugman (1979).

2.2 Bank’s payoff

Let $A = \int_0^1 a_i d_i$ denote the total mass of speculators who attack. The bank’s payoff is given by a pair of functions: $S(A), D(\tau)$. $S(A)$ is the payoff the bank obtains if the game ends in Survival after $A$ attackers. $D(\tau)$ is the payoff the bank obtains if the game ends in Defeat after $\tau$ attackers, i.e. if the bank abandons the regime after supporting it against $\tau$ attackers. Assume $S$ and $D$ are weakly decreasing, so defending the regime is costly at the margin, and $S(0) > D(0)$, so if no one attacks the bank prefers to survive.

The four panels in Figure 1 illustrate different possible cases of payoffs the bank might have. Panel (i) shows the payoffs assumed by Morris and Shin (1998), which are the special case where $S(A) = v - A$ and $D(\tau) = 0$. As mentioned in the introduction, for these payoffs the bank’s problem is trivial since it would always defend the regime up to the point where $A = v$.

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3A more complete model of bank runs would require modeling the dependence of speculators’ payoff on the size of the attack and not just the survival of the bank. For instance, some speculators may join the queue but never get paid because the bank surrenders before they reach the front of the queue. This introduces strategic substitutability as well as strategic complementarity in speculators’ actions since, beyond the point where the run topples the bank, the incentive to attack diminishes with the size of the attack. Goldstein and Pauzner (2005) show how to adapt global games techniques to account for this.
Panel (ii) shows an interpretation where the bank has finite liquidity reserves. As proposed by Bagehot (1873), the bank does not mind using reserves as long as it succeeds in maintaining the regime, but it would rather not waste them on an unsuccessful defence. Panel (iii) shows a case where there is a fixed benefit of maintaining the regime and the cost of defending it is linear in the size of the attack, both for successful and failed defences. Panel (iv) shows a similar example but with increasing marginal cost of using reserves.

![Figure 1: Examples of payoffs for the bank](image)

2.3 Information

At the beginning of the game, nature draws the random variable $\theta$ from some prior density $p(\theta)$. No one observes the realized $\theta$, but speculators observe their own realized $c_i$. A speculator’s individual cost is informative about the distribution of costs in the population summarized by $\theta$; applying Bayes’ rule, a speculator’s posterior is:

$$p(\theta | c_i) = \frac{g(c_i | \theta) p(\theta)}{\int_{\theta} g(c_i | \theta) p(\theta) d\theta}$$  \hspace{1cm} (1)

**Definition 1.** $g(c_i | \theta)$ satisfies the monotone inference property if, for any $p(\theta)$, the posterior distribution $P(\theta | c_i)$ is decreasing in $c_i$.

If $g(c_i | \theta)$ satisfies this property then a speculator who observes a higher cost $c_i$ for himself will infer that the parameter $\theta$ is likely to be higher, which implies that the costs of the other speculators are also likely to be higher.
2.4 Equilibrium

I focus on monotone equilibria, defined as perfect Bayesian equilibria such that a speculator attacks if and only if \( c_i \) is less than some threshold \( c^* \), which is identical for all speculators. In such an equilibrium, the aggregate size of the attack is \( A(\theta) = \Pr[c_i \leq c^*|\theta] = G(c^*|\theta) \). Since the bank does not know \( \theta \), it does not know the realized value of \( A \). Instead, it has beliefs given by the density

\[
f(A) = p(G^{-1}(c^*; A)) \left| \frac{\partial G^{-1}(c^*; A)}{\partial A} \right|
\]  

(2)

where \( G^{-1} \) is the inverse of \( G \) with respect to its second argument, i.e. \( A \equiv G(c|G^{-1}(c; A)) \).

While the attack is taking place, the bank gradually learns about the realized value of \( A \) but in a limited way: it only learns from the fact that the attack is not over yet. Hence its optimal stopping problem can be formulated simply as the following one-dimensional optimization problem:

\[
\max_{\tau \in [0,1]} V(\tau) = \int_0^\tau S(A) f(A) dA + [1 - F(\tau)] D(\tau)
\]  

(3)

The first term of (3) is the value the bank obtains from the possibility of surviving if the size of the attack turns out to be less than \( \tau \). The second term is the value it will obtain if it surrenders after \( \tau \) attackers. Program (3) may or may not have an interior optimum. In case it does, the first order necessary condition is

\[
V'(\tau) = [S(\tau) - D(\tau)] f(\tau) + [1 - F(\tau)] D'(\tau) = 0
\]

\Rightarrow h(\tau) [S(\tau) - D(\tau)] = -D'(\tau)

(4)

where \( h(A) \equiv \frac{f(A)}{1 - F(A)} \) is the hazard function of the size of the attack. The marginal benefit of waiting, given by the difference between the value of survival and defeat, \( S(\tau) - D(\tau) \) times the instantaneous probability that the attack will be over, \( h(\tau) \), must equal the marginal cost of waiting, which is \( -D'(\tau) \). Since the function \( V(\tau) \) is not necessarily concave, condition (4) could also denote a local minimum. The second order condition for a local maximum is:

\[
V''(\tau) = [S'(\tau) - 2D'(\tau)] f(\tau) + [S(\tau) - D(\tau)] f'(\tau) + [1 - F(\tau)] D''(\tau) \leq 0
\]  

(5)

Even if (4) and (5) hold, they may identify a local but not global maximum, so in general it is only possible to say that the bank’s best response is given by \( \tau^* \in \arg\max_{\tau \in [0,1]} V(\tau) \).

\footnote{Denoting by \( W(a) \) the value function conditional on having resisted an attack of size \( a \), then the problem can be represented by the differential equation \( W'(a) = -h(a)[S(a) - W(a)] \), with value matching condition \( W(\tau) = D(\tau) \) and smooth pasting condition \( W'(\tau) = D'(\tau) \). These are equivalent to the first order conditions in the text.}
The distinction between interior and corner solutions to program (3) is important. If $\tau^* = 0$ then the bank does not attempt to resist at all and falls as soon as the attack begins. If $\tau^* = 1$ then the bank withstands the attack no matter how large it is (although this does not rule out that it may ex-post regret having done so if the realized value of $S(A(\theta))$ is less than $D(0)$). Unsuccessful partial defences only occur when $\tau^* \in (0, 1)$ and the realized $A(\theta)$ happens to be greater than $\tau^*$.

The regime will survive iff $A(\theta) = G(c^*|\theta) \leq \tau^*$. Higher values of $\theta$ are associated with FOSD higher costs of attacking and therefore, for given $c^*$, smaller attacks. If $\lim_{\theta \to -\infty} G(c^*|\theta) > \tau^*$, then the smallest possible attack is sufficiently large for the bank to fail; conversely if $\lim_{\theta \to -\infty} G(c^*|\theta) \leq \tau^*$, then the largest possible attack is too small to make the bank fail. Otherwise there exists a unique $\theta^*$ such that the bank fails whenever $\theta < \theta^*$. This critical value is defined by

$$G(c^*|\theta^*) = \tau^* \quad (6)$$

For an individual speculator, attacking is a best response if, given his information, the probability of the bank failing is greater than the cost of attacking, i.e. if

$$\Pr [\theta \leq \theta^*|c_i] = P(\theta^*|c_i) > c_i \quad (7)$$

If $g(c_i|\theta)$ satisfies the monotone inference property then the LHS of (7) is decreasing in $c_i$, so given $\theta^*$ there exists a unique $c^* \in [0, 1]$ such that the speculators attack iff $c_i \leq c^*$. This shows that the best response to threshold strategies are threshold strategies. The speculator who has cost $c_i = c^*$ must be indifferent between attacking and not attacking, which implies

$$P(\theta^*|c^*) = c^* \quad (8)$$

**Definition 2.** A monotone equilibrium consists of a threshold $\theta^* \in \mathbb{R} \cup \{-\infty, +\infty\}$, strategies $\tau^* \in [0, 1]$ and $c^* \in [0, 1]$ and beliefs $f(A)$ such that

1. Either (i) (6) holds, (ii) $G(c^*|\theta) > \tau^*, \forall \theta$ and $\theta^* = +\infty$ or (iii) $G(c^*|\theta) < \tau^*, \forall \theta$ and $\theta^* = -\infty$.

2. $\tau^*$ solves program (3) using $f(A)$, so the bank is best-responding given its beliefs.

3. (8) holds, so the speculators are best-responding.

4. $f(A)$ satisfies (2), so the bank’s beliefs are consistent with the speculators’ strategies.
3 Beliefs about the size of the attack

Suppose the probability distribution of the sizes of attacks $A$ were known to be given by some density function $f(A)$. In principle, $f(A)$ could be estimated empirically from the sizes of actual (successful or unsuccessful) speculative attacks. Could the model account for $f(A)$ as arising from the equilibrium of the game described above? The answer is that it is always possible to reverse-engineer some prior $p(\theta)$ such that the game has an equilibrium where the unconditional density of $A$ is $f(A)$.

Proposition 1. Let $f(A)$ be an arbitrary continuous pdf on $[0, 1]$ and let $g(c_i|\theta)$ be a continuous pdf on $\mathbb{R}$ such that

1. $g(c_i|\theta)$ satisfies the monotone inference property
2. $\lim_{\theta \to -\infty} G(c_i|\theta) = 0, \forall c_i$ and $\lim_{\theta \to \infty} G(c_i|\theta) = 1, \forall c_i$

Then there exists a prior $p(\theta)$ such that under primitives $g(c_i|\theta), p(\theta)$ there is an equilibrium where the unconditional distribution of $A$ is $f(A)$.

The way to construct such an equilibrium is as follows. Beliefs $f(A)$ immediately imply a best response $\tau^*$ for the bank. Given any strategy $c^*$ for the speculators, it is mechanically possible to find a prior about the value of $\theta$, $p_{c^*}(\theta)$, such that the posterior belief about the size of the attack is indeed $f(A)$. Under prior $p_{c^*}(\theta)$, the speculators’ best response to a cutoff strategy $c^*$ will be some other cutoff strategy $T(c^*, p_{c^*})$. An equilibrium consists of a fixed point such that $T(c^*, p_{c^*}) = c^*$ and continuity implies that such a fixed point exists.

The importance of Proposition 1 resides in that, depending on the functional forms of $S(A)$ and $D(\tau)$, it could be that program (3) has an interior solution only for some functions $f(A)$ and not for others. This means that some functions $f(A)$ will be consistent with the possibility of observing failed partial defences while others will not. Proposition 1 says that it is always possible to generate examples that produce $f(A)$ that are consistent with an equilibrium with interior $\tau^*$.

Table 1 shows example of what priors could justify different $f(A)$ functions. In all cases $f(A)$ is some Beta distribution, the bank’s payoffs are $S(A) = 0.6 - A$, $D(\tau) = -\tau$ and $g(c_i|\theta) = \phi(c_i - \theta)$, where $\phi$ is the density for a standard normal, which satisfies the conditions of Proposition 1, as shown in Lemma 1 below.

Lemma 1. $g(c_i|\theta) = \sqrt{\alpha} \phi(\sqrt{\alpha} (c_i - \theta))$ satisfies the monotone inference property

Proposition 1 also implies that making precise predictions requires imposing more structure on the problem, since under the general specification virtually anything could happen. The following section explores a simple special case of the model.
Table 1: Examples of priors that would lead to beliefs about A

<table>
<thead>
<tr>
<th>$f(A)$</th>
<th>$p(\theta)$</th>
<th>Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta(1, 1)$ (uniform)</td>
<td><img src="image1.png" alt="Beliefs about attack" /></td>
<td>$c^*$</td>
</tr>
<tr>
<td><img src="image2.png" alt="Prior about $\theta$" /></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\beta(.25, .25)$</td>
<td><img src="image3.png" alt="Beliefs about attack" /></td>
<td>0.95</td>
</tr>
<tr>
<td><img src="image4.png" alt="Prior about $\theta$" /></td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\beta(2, 1)$</td>
<td><img src="image5.png" alt="Beliefs about attack" /></td>
<td>1</td>
</tr>
</tbody>
</table>
4 Normal uncertainty and linear payoffs

Consider the following special case of the model. The bank’s payoffs are $S(A) = v - A$ and $D(\tau) = -\tau$, so there is a fixed value of survival $v \in (0, 1)$ and a constant marginal cost of defending the regime, as in panel (iii) of Figure 1. The costs of attacking are normally distributed in the population, with mean $\theta$ and variance $\frac{1}{\alpha_c}$ and the prior on $\theta$ is also normal, with mean $\mu$ and variance $\frac{1}{\alpha_\mu}$, i.e. $g(c_i|\theta) = \sqrt{\frac{1}{\alpha_c}} \phi \left( \sqrt{\frac{1}{\alpha_c}} (c_i - \theta) \right)$ and $p(\theta) = \sqrt{\frac{1}{\alpha_\mu}} \phi \left( \sqrt{\frac{1}{\alpha_\mu}} (\theta - \mu) \right)$. $\frac{1}{\alpha_c}$ and $\frac{1}{\alpha_\mu}$ are measures of heterogeneity and aggregate uncertainty respectively.

In order to highlight the role of the bank’s uncertainty and speculators’ incomplete information, I first analyze variants of the game with common knowledge and where the bank is informed.

4.1 Common knowledge benchmark

Assume first that $\theta$ is common knowledge for both the bank and the speculators. The bank’s decision becomes simpler because, in any Nash equilibrium, it knows what size of attack it will face. If the bank decides that it will not outlast the attack then it finds it optimal to choose $\tau^* = 0$. As in Morris and Shin (1998), the bank will abandon the regime iff $A(\theta) > v$.

A mass $\Phi \left( -\sqrt{\frac{1}{\alpha_c}} \theta \right)$ of speculators have $c_i < 0$ so attacking is dominant for them. Conversely, a mass $1 - \Phi \left( \sqrt{\frac{1}{\alpha_c}} (1 - \theta) \right)$ have $c_i > 1$ so not attacking is dominant for them. If $\theta$ is such that $v - \Phi \left( -\sqrt{\frac{1}{\alpha_c}} \theta \right) > 0 > v - \Phi \left( \sqrt{\frac{1}{\alpha_c}} (1 - \theta) \right)$, i.e. if $\theta \in \left[ -\Phi^{-1}(v) \sqrt{\frac{1}{\alpha_c}}, 1 - \Phi^{-1}(v) \sqrt{\frac{1}{\alpha_c}} \right]$ then the game will have multiple equilibria. If the bank expects all the speculators who have $c_i \in [0, 1]$ to attack, then $\tau = 0$ is a best response, which in turn justifies their decision to attack. Conversely, if the bank expects all the speculators who have $c_i \in [0, 1]$ not to attack, then any $\tau^* > \Phi \left( -\sqrt{\frac{1}{\alpha_c}} \theta \right)$ is a best response, which justifies the speculators’ decision.

4.2 Informed bank benchmark

Now assume instead that the bank knows the realized value of $\theta$ but the speculators do not. As with common knowledge, in any Nash equilibrium the bank will always choose either $\tau^* = 0$ or $\tau^* = 1$ because it will know the size of the attack. I will look for a monotone equilibrium such that the bank chooses $\tau^* = 1$ iff $\theta \geq \theta^*$.

From equation (1), a speculator’s Bayesian posterior about $\theta$ is a normal distribution, with mean $\frac{\alpha_\mu + \alpha_c \theta^*}{\alpha_\mu + \alpha_c}$ and variance $\frac{1}{\alpha_\mu + \alpha_c}$. Given a cutoff $\theta^*$ for the bank’s strategy, speculator’s best response cutoff $c^*$ is given by:

$$
\Phi \left( \sqrt{\frac{1}{\alpha_\mu + \alpha_c}} \left( \theta^* - \frac{\alpha_\mu + \alpha_c c^*}{\alpha_\mu + \alpha_c} \right) \right) = c^*
$$

(9)
which is just a special case of the indifference condition (8).

The size of the attack will be given by

\[ A(\theta) = \Phi(\sqrt{\alpha_c}(c^* - \theta)) \]

which is decreasing in \( \theta \). This implies that, as long as \( v \in (0, 1) \), there will be a unique cutoff \( \theta^* \) such that the bank prefers \( \tau = 1 \) to \( \tau = 0 \) iff \( \theta \geq \theta^* \). The cutoff is given by the indifference condition

\[ \Phi(\sqrt{\alpha_c}(c^* - \theta^*)) = v \tag{10} \]

Solving (9) and (10), the speculators’ cutoff must satisfy

\[ \Phi\left(\sqrt{\alpha_c}\left(\frac{\alpha_\mu (c^* - \mu)}{\alpha_\mu + \alpha_c} - \frac{\Phi^{-1}(c^*)}{\sqrt{\alpha_\mu + \alpha_c}}\right)\right) = v \tag{11} \]

The derivative of the left hand side of (11) is

\[ \phi(\cdot) \sqrt{\alpha_c} \left(\frac{\alpha_\mu}{\alpha_\mu + \alpha_c} - \frac{1}{\phi(\Phi^{-1}(c^*)) \sqrt{\alpha_\mu + \alpha_c}}\right) \leq \phi(\cdot) \sqrt{\alpha_c} \left(\frac{\alpha_\mu}{\alpha_\mu + \alpha_c} - \frac{\sqrt{2\pi}}{\sqrt{\alpha_\mu + \alpha_c}}\right) \]

Therefore if

\[ \frac{\alpha_\mu}{\sqrt{\alpha_\mu + \alpha_c}} \leq \sqrt{2\pi} \tag{12} \]

the left hand side is decreasing and (11) has a unique solution. Furthermore, if condition (12) does not hold, there exists values of \( \mu \) and \( v \) such that (11) has multiple solutions. Condition (12) says that in order to guarantee uniqueness in the game where the bank has complete information, heterogeneity \( \frac{1}{\alpha_c} \) must be small relative to aggregate uncertainty \( \frac{1}{\alpha_\mu} \). When heterogeneity is small, the idiosyncratic cost \( c_i \) is a very good signal about \( \theta \), which rules out multiple self-fulfilling equilibria.

The following proposition summarizes the above benchmark results.

**Proposition 2.**

1. Under common knowledge, there are multiple equilibria if \( \theta \in \left[-\frac{\Phi^{-1}(v)}{\sqrt{\alpha_c}}, 1 - \frac{\Phi^{-1}(v)}{\sqrt{\alpha_c}}\right] \).

2. If the bank knows the realized \( \theta \) but the speculators do not, there is a unique equilibrium for every \( \mu, v \) iff \( \frac{\alpha_\mu}{\sqrt{\alpha_\mu + \alpha_c}} \leq \sqrt{2\pi} \).

3. In both cases \( \tau^* \in \{0, 1\} \), so there are never unsuccessful defences.
4.3 Uninformed bank

Now consider the game as described in section 2, where the bank does not observe \( \theta \). Given a threshold \( \theta^* \) for the bank’s survival, speculators’ best response cutoff is still given by (9). The threshold \( \theta^* \) is related to the bank’s strategy by equation (6), which reduces to

\[
\theta^* = c^* - \frac{\Phi^{-1}(\tau^*)}{\sqrt{\alpha_c}}
\]  

(13)

if \( \tau \in (0, 1) \), \( \theta^* = +\infty \) if \( \tau^* = 0 \) and \( \theta^* = -\infty \) if \( \tau^* = 1 \).

With linear \( S(\cdot) \) and \( D(\cdot) \) payoff functions, the FOC and SOC for an interior solution to the bank’s optimal stopping problem (3) simplify to:

\[
v h(\tau) = 1 \quad (14)
\]

\[
h'(\tau) \leq 0 \quad (15)
\]

The SOC (15) says that for the bank to find it optimal to abandon its defence at some interior point \( \tau \), \( f(A) \) must be such that it is decreasingly likely that the attack will be over soon.

Conditions (14) and (15) only identify a local optimum. For a global optimum, the bank must compare the value it obtains from an interior \( \tau \) to the value of offering no resistance \( V(0) = 0 \) and the value of resisting any possible attack \( V(1) = v - E(A) \) (as well as comparing it to other local optima if there are any).

Finally, by equation (2), the bank’s beliefs about the attack it will face are given by:

\[
f(A) = \frac{\sqrt{\frac{\alpha_\mu}{\alpha_c}} \phi \left( \frac{\alpha_\mu}{\alpha_c} \Phi^{-1}(A) - \sqrt{\frac{\alpha_\mu}{\alpha_c}} (c^* - \mu) \right)}{\phi(\Phi^{-1}(A))}
\]

(16)

In what follows I make use of the following properties of \( f(A) \)

Lemma 2.

1. \( E(A) = \Phi \left( \frac{c^* - \mu}{\sqrt{\alpha_\mu + \alpha_c}} \right) \)

2. If \( \alpha_\mu < \alpha_c \), (or \( \alpha_\mu = \alpha_c \) and \( c^* < \mu \)), then \( \lim_{A \to 0} f(A) = \infty \)

3. If \( \alpha_\mu > \alpha_c \) and \( c^* > \mu \), then \( f(A) \) has increasing hazard

Part 1 of Lemma 16 simply computes the expected size of the attack from the point of view of the bank. If the cutoff \( c^* \) for not attacking is high compared to \( \mu \), the prior mean of \( c_i \), then attacks will tend to be larger. The expected attack will be more sensitive to this difference when the bank has less overall (aggregate plus idiosyncratic) uncertainty about any
speculator’s $c_i$. Part 2 of Lemma 16 states that, if aggregate uncertainty is large relative to heterogeneity, then the probability that attacks will be very small is high. The reason is that, for a given $c^*$, small heterogeneity makes the size of the attack more sensitive to the realization of $\theta$, while large aggregate uncertainty makes realizations of $\theta$ themselves more variable, which makes realized values of $A$ near the extremes (and in particular near $A = 0$) more likely. Part 3 of Lemma 16 says that if instead aggregate uncertainty is small relative to heterogeneity and in addition the cutoff $c^*$ is higher than the prior mean of $\theta$, then $f(A)$ has increasing hazard. The reason is that relatively small aggregate uncertainties shift $f(A)$ towards the center rather than the extremes, while $c^* > \mu$ shifts it to the right, which suffices for the hazard function to be increasing.

I distinguish between three different kinds of equilibria: “no resistance” equilibria, with $\tau^* = 0$; “full resistance” equilibria, with $\tau^* = 1$ and “waiting” equilibria, with $\tau \in (0, 1)$ and characterize necessary and sufficient conditions under which each may exist.

### 4.4 No resistance equilibria

In a no resistance equilibrium, the bank settles for obtaining $V(0) = 0$ by giving up immediately. Since the regime always falls, i.e. $\theta^* = \infty$, then any speculator for whom not attacking is not dominant will attack, i.e. $c^* = 1$.

A simple necessary condition for no resistance to be optimal is that $V'(0) = vf(0) - 1 \leq 0$. As long as the density at 0 is greater than $\frac{1}{v}$, i.e. as long as there is sufficient chance that the attack will be very small, at least a little resistance will be preferable to no resistance at all. Hence a necessary condition for a no resistance equilibrium is that the conditions of Lemma 2.2 not hold, i.e. $\alpha_\mu > \alpha_e$ (or $\alpha_\mu = \alpha_e$ and $\mu \leq 1$).

Under the conditions of Lemma 2.3, there can never be an interior solution to the bank’s problem since this would contradict (15). The bank will not value the option to wait and will simply choose between $\tau = 0$ and $\tau = 1$. By Lemma 2.1, the former is preferred if $\Phi\left(\frac{(1 - \mu)}{\sqrt{\frac{1}{\alpha_\mu} + \frac{1}{\alpha_e}}}\right) > v$. Hence the conditions $\alpha_\mu > \alpha_e$, $1 > \mu$ and $\Phi\left(\frac{(1 - \mu)}{\sqrt{\frac{1}{\alpha_\mu} + \frac{1}{\alpha_e}}}\right) > v$ are sufficient to ensure that there is a no resistance equilibrium.

### 4.5 Full resistance equilibria

In a full resistance equilibrium, the bank obtains a value $V(1) = v - E(A)$. The regime never falls, so $\theta^* = -\infty$ and $c^* = 0$.

Since the bank could always obtain zero by choosing $\tau = 0$, a necessary condition for this kind of equilibrium to exist is $v \geq E(A) = \Phi\left(-\mu/\sqrt{\frac{1}{\alpha_\mu} + \frac{1}{\alpha_e}}\right)$. Furthermore, under the

\[\text{Strictly speaking, the density at 0 may not be well defined. Let } f(0) = \lim_{A \to 0} f(A)\]
conditions for Lemma 2.3, there can never be an interior solution for the bank’s problem, so conditions \( \alpha_\mu > \alpha_c, \mu < 0 \) and \( v \geq \Phi \left( -\mu / \sqrt{1/\alpha_\mu + 1/\alpha_c} \right) \) are sufficient to ensure that there is a full resistance equilibrium.

### 4.6 Waiting equilibria

In a waiting equilibrium, the bank chooses some intermediate \( \tau^* \in (0, 1) \). By (15), this requires that the hazard function be decreasing at some point. By Lemma 2.3, this means that either \( \alpha_\mu \leq \alpha_c \) or \( \mu \geq 0 \) must hold. Furthermore, if \( V'(0) > 0 \) and \( V(0) > V(1) \) then any optimum must necessarily be interior. Therefore, if the conditions for Lemma 2.2 hold and \( v < \Phi \left( -\mu / \sqrt{1/\alpha_\mu + 1/\alpha_c} \right) \) we know that any equilibrium must be a waiting equilibrium.

### 4.7 Discussion

The conditions for existence of the various types of equilibria are summarized in the following proposition.

**Proposition 3.** The following are necessary and sufficient conditions for no resistance, full resistance and waiting equilibria respectively

<table>
<thead>
<tr>
<th></th>
<th>Necessary conditions</th>
<th>Sufficient conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No resistance</strong></td>
<td>( \Phi \left( \frac{1-\mu}{\sqrt{1/\alpha_\mu + 1/\alpha_c}} \right) \geq v )</td>
<td>( \Phi \left( \frac{1-\mu}{\sqrt{1/\alpha_\mu + 1/\alpha_c}} \right) \geq v )</td>
</tr>
<tr>
<td></td>
<td>( \alpha_\mu &gt; \alpha_c, ) (or ( \alpha_\mu = \alpha_c ) and ( \mu \leq 1 ))</td>
<td>( \alpha_\mu &gt; \alpha_c )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \mu &lt; 1 )</td>
</tr>
<tr>
<td><strong>Full resistance</strong></td>
<td>( \Phi \left( \frac{-\mu}{\sqrt{1/\alpha_\mu + 1/\alpha_c}} \right) \leq v )</td>
<td>( \Phi \left( \frac{-\mu}{\sqrt{1/\alpha_\mu + 1/\alpha_c}} \right) \leq v )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \alpha_\mu &gt; \alpha_c )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \mu &lt; 0 )</td>
</tr>
<tr>
<td><strong>Waiting</strong></td>
<td>( \alpha_\mu \leq \alpha_c ) (or ( \mu \geq 0 ))</td>
<td>( \alpha_\mu &lt; \alpha_c )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Phi \left( \frac{-\mu}{\sqrt{1/\alpha_\mu + 1/\alpha_c}} \right) &gt; v )</td>
</tr>
</tbody>
</table>

Notice that Proposition 3 does not rule out the possibility of multiple equilibria. Indeed, if

\[
v \in \left[ \Phi \left( \frac{-\mu}{\sqrt{1/\alpha_\mu + 1/\alpha_c}} \right), \Phi \left( \frac{1-\mu}{\sqrt{1/\alpha_\mu + 1/\alpha_c}} \right) \right]
\]  

then the necessary conditions for both full resistance equilibria and no resistance equilibria hold. This interval always exists, although it becomes smaller as either \( \alpha_\mu \) or \( \alpha_c \) become large. If,
in addition, \( \mu < 0 \), the sufficient conditions for both full resistance and no resistance equilibria hold as well and we are certain to have multiple equilibria.

The logic behind self-fulfilling equilibria is not exactly the same as in the case where the bank knows \( \theta \). In the game where the bank is informed, multiplicity arises when information is sufficiently common across speculators to allow for coordination on different equilibria. This requires small aggregate uncertainty, so that speculators’ common prior is highly informative and large heterogeneity so that speculator’s idiosyncratic cost is not very informative about the aggregate state. In the game where the bank is uninformed, multiplicity arises if the expected mass of speculators who do not have a dominant strategy is sufficiently large to justify each of the bank’s possible decision rules. This requires that the bank’s desire to survive \( v \) take intermediate values and that the unconditional distribution of \( c_i \) be sufficiently concentrated, which requires small heterogeneity and small aggregate uncertainty.

The timing assumptions matter for the types of equilibria that may arise. Under the usual assumption that the bank moves after observing \( A \), its decision is always trivial. If it could, the bank would want to commit to full resistance in order to steer the equilibrium towards its desired outcome, where the attack is small and the bank survives. However, its commitment is not credible: if the attack turns out to be large, the bank will choose to fail, making the attack self-fulfilling. Instead, when the bank has uncertainty and makes decisions continuously, a form of commitment may arise. In equilibria where \( \tau^* = 1 \), during the course of play the bank will always (correctly) believe that an ongoing attack is likely to be over soon. This justifies its persistence in defending the regime even when ex-post it would prefer not to have done so.

Interestingly, the same forces that lead to uniqueness in the informed-bank case lead to equilibria with waiting when the bank is uninformed. When there is little heterogeneity and large aggregate uncertainty, private sources of information (i.e. the idiosyncratic cost \( c_i \)), which the bank does not have access to, are relatively more informative. This force eliminates multiplicity in the informed-bank case, as in Morris and Shin (2000), but heightens the informational disadvantage of the bank in the uninformed-bank case. For the uninformed bank, an interior \( \tau \) is justified when the attack is likely to be very small or very large: it is willing to wait up to \( \tau \) because of the chance that the attack might be very small but if after \( \tau \) attackers the attack is not over then it realizes that the attack will not be over soon and it abandons its defence of the regime. As discussed above, probability distributions of this kind, which have most of the mass in the extremes, arise when there is little heterogeneity relative to the amount of aggregate uncertainty.

One prediction of the model is that one should expect to see failed partial defences against speculative attacks in the same kinds of environments when one also observes successful defences against small attacks. For instance, in a fixed-exchange regime where money demand is highly variable, the central bank will often experience what the model describes as small speculative
attacks, simply from shifts in \( G(0|\theta) \). It will therefore be more willing to engage in a partial defence than a central bank in a country where money demand is very stable and a speculative attack is not easily mistaken for day-to-day variation in money demand.

Proposition 3 does not fully characterize the possible equilibria that will arise for each combination of parameters, but it is possible to compute the equilibria numerically in order to find sharper boundaries for the regions in the parameter space where each equilibrium occurs. Figure 2 shows the regions of \( \alpha_\mu, \alpha_c \) where each of the types of equilibrium occurs, fixing \( \mu = 0.3 \) and setting \( v = 0.4 \) or \( v = 0.6 \) in each case.

\[
\mu = 0.3, v = 0.4 \quad \text{and} \quad \mu = 0.3, v = 0.6
\]

Figure 2: Regions of no resistance, full resistance and waiting equilibria

In the case where \( v = 0.4 \), Proposition 3 implies that no resistance equilibria will exist iff \( \alpha_\mu > \alpha_c \), i.e. in the northwest half of the graph. When this condition does not hold, \( V'(0) > 0 \) so some resistance is preferable, and since \( v \) is not too high there exists a waiting equilibrium. Moreover, when the unconditional distribution of \( c_i \) is concentrated (the northeast of the graph), there also exists a full-resistance equilibrium. In it, speculators’ knowledge that \( \tau^* = 1 \) shifts enough mass towards a no-attack strategy that full resistance is desirable. Hence there is a region of multiplicity with waiting and full resistance equilibria and a region of multiplicity with no resistance and full resistance equilibria.

In the case where \( v = 0.6 \), when \( \alpha_c \) is sufficiently greater than \( \alpha_\mu \) (the southeast of the graph), then the distribution \( f(A) \) is shifted towards the extremes and \( E(A) \) approaches \( \frac{1}{2} \). This means that, although \( \tau = 1 \) is preferable to \( \tau = 0 \) (since \( v > \frac{1}{2} \)), the bank can always do even better by choosing \( \tau \) positive but small, since the likelihood of a very small attack is high and it is not very costly to wait to see if it indeed takes place. Hence, only waiting equilibria exist sufficiently southeast in the graph. However, as in the case where \( v = 0.4 \), there is a region where waiting and full resistance equilibria both exist. When \( \alpha_\mu > \alpha_c \), waiting equilibria do not exist since \( f(A) \) has more mass in the center, which makes waiting less desirable. If both
\(\alpha_c\) and \(\alpha_\mu\) are sufficiently high, no resistance equilibria may exist, as the bank’s pessimism becomes self-justifying. Otherwise, only full resistance equilibria exist.

Within the waiting-equilibrium region in the right panel of Figure 2, Figure 3 shows how the equilibrium stopping point \(\tau^*\) is affected by various parameters.

(i) Ratio of heterogeneity to aggregate uncertainty

(ii) Total variance

(iii) Value of surviving

(iv) Prior mean

Figure 3: Equilibrium \(\tau^*\) for different parameters

Panel (i) shows that the high ratios of heterogeneity to aggregate uncertainty lead to lower resistance in waiting equilibria (as well as ruling out waiting equilibria entirely if they are sufficiently high). This is because higher heterogeneity leads to a \(f(A)\) with less mass at the extremes, which means that \(h(\tau)\) will be decreasing only for very low values of \(\tau\). Panel (ii) shows that higher overall uncertainty for the bank (measured by \(\frac{1}{\alpha_\mu} + \frac{1}{\alpha_c}\)) leads to more waiting. This is because in these examples it happens that \(c^* > \mu\), so by (16), this makes small attacks more likely, which justifies waiting more. Panel (iii) shows that the more the bank values survival the longer it is willing to wait. This is due to two reinforcing effects: firstly, given a function \(f(A)\), (14) implies that higher \(v\) requires a lower hazard for the bank not to wish
to continue defending; since the hazard must be decreasing at an optimum, higher \( v \) implies waiting more; secondly, a higher \( \tau^* \) leads to a lower equilibrium \( c^* \) for the speculators, which makes a small attack more likely and justifies waiting more. Finally, panel (iv) shows that the higher the bank’s prior belief about speculators’ costs of attacking, the more it will be willing to wait, simply because this will lead it to believe that attacks are likely to be small.

5 Final Remarks

If banks and governments undertake costly defence measures when faced with speculative attacks and after some time decide to abandon them, then (assuming they are acting rationally) it must be that in the meantime they learned something about the environment or the actions of the speculators that they did not know at first. This paper provides one possible explanation of where their original uncertainty may stem from: uncertainty about (some aspect of) the distribution of speculators’ payoffs from attacking or not attacking the regime.

Under fairly general conditions, virtually any beliefs about the attack could be consistent with the equilibrium of a simple coordination game. In a special case with normal uncertainty and linear payoffs, beliefs that would justify some, but not complete, defence of the status quo arise when aggregate uncertainty is great compared to the degree of heterogeneity in the population, so that very small or very large attacks are likely.

Of course, this is not the only possible source of uncertainty that banks or governments may face in these episodes. They could be unsure, as in Goldstein, Ozdenoren and Yuan (2008), about the costs of regime change (\( v \) in this model) or about what information the public has. Part of the analysis of the present model is likely to extend to these settings, such as the nature of the optimal stopping problem and the key role of the (endogenous) hazard function. Other aspects, such as the role of heterogeneity, are more special to the exact way uncertainty is introduced in the model.

One important feature of real speculative attacks that the model does not include is the possibility that speculators may learn as the attack progresses. An extension of the framework along those lines is left for future work.

Appendix

Proof of Proposition 1. Given \( f(A) \), let \( \tau^* \) solve program (3) and define the operator \( T(c^*) \) by the following series of steps:

1. Given \( c^* \), let \( \theta^* \) satisfy (6) if a solution exists, \( \theta = \infty \) if \( G(c^*|\theta) > \tau^*, \forall \theta \) and \( \theta = -\infty \) if \( G(c^*|\theta) < \tau^*, \forall \theta \)
2. Using (2), let

\[ p(\theta) = \frac{f(A)}{\partial G^{-1}(c^*; A)} \bigg|_{A=G(c^*|\theta)} \]  

(18)

Property 2 in the statement of the proposition ensures that \( G^{-1}(c^*; A) \) exists so \( p(\theta) \) is well defined.

3. Let \( T(c^*) \) be the solution to (8) where \( \theta^* \) is the value obtained in step 1 and the function \( p(\theta) \) derived in step 2 is used in (1) to compute \( P(\theta^*|c) \). Since \( g(c_i|\theta) \) satisfies the monotone inference property, this equation always has a unique solution.

Since \( f(A) \) and \( g(c_i|\theta) \) are continuous, then the operator \( T(c^*) : \mathbb{R} \rightarrow [0, 1] \) is a continuous function, so it must have a fixed point in \([0, 1]\). If \( c^* \) is such a fixed point, then under prior \( p(\theta) \) given by (18), \( \{\tau^*, c^*, \theta^*, f(A)\} \) is an equilibrium of the game.

\[ \square \]

Proof of Lemma 1. Using (1)

\[ p(\hat{\theta}|c) = \frac{\sqrt{\alpha} \phi(\sqrt{\alpha} (c - \theta)) p(\theta)}{\int_{\theta} \sqrt{\alpha} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta} \]

\[ P(\hat{\theta}|c) = \frac{\int_{\theta \leq \hat{\theta}} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta}{\int_{\theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta} \]

Taking derivatives and rearranging:

\[ \frac{\partial P(\hat{\theta}|c)}{\partial c} = \frac{\alpha}{\left[ \int_{\theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right]^2} \left[ \left( \int_{\theta \leq \hat{\theta}} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \left( \int_{\theta \geq \hat{\theta}} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) - \left( \int_{\theta \geq \hat{\theta}} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \left( \int_{\theta \leq \hat{\theta}} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \right] \]

\[ \leq \frac{\alpha}{\left[ \int_{\theta} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right]^2} \left[ \hat{\theta} \left( \int_{\theta \leq \hat{\theta}} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \left( \int_{\theta \geq \hat{\theta}} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) - \hat{\theta} \left( \int_{\theta \geq \hat{\theta}} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \left( \int_{\theta \leq \hat{\theta}} \phi(\sqrt{\alpha} (c - \theta)) p(\theta) d\theta \right) \right] \]

\[ = 0 \]

\[ \square \]

Proof of Lemma 2.

1. By direct computation
2. 

\[ f(A) = \frac{\sqrt{\frac{\alpha u}{\alpha c}} \phi \left( \sqrt{\frac{\alpha u}{\alpha c}} \Phi^{-1}(A) - \sqrt{\alpha u} (c^* - \mu) \right)}{\phi(\Phi^{-1}(A))} = \sqrt{\frac{\alpha u}{\alpha c}} \exp \left[ \frac{1}{2} \left( 1 - \frac{\alpha u}{\alpha c} \right) \left[ \Phi^{-1}(A) \right]^2 + 2 \frac{\alpha u}{\sqrt{\alpha c}} (c^* - \mu) \Phi^{-1}(A) - \alpha u (c^* - \mu)^2 \right] \]

Taking the limit as \( A \to 0 \) gives the result.

3. Let \( \gamma = \sqrt{\frac{\alpha u}{\alpha c}} \) and \( C = -\sqrt{\alpha \mu} (c^* - \mu) \)

\[ h(A) = \frac{\phi(\gamma \Phi^{-1}(A) + C)}{\phi(\Phi^{-1}(A))} \int_A^1 \frac{\phi(\gamma \Phi^{-1}(a) + C)}{\phi(\Phi^{-1}(a))} da \]

Define

\[ u(A) \equiv \phi(\gamma \Phi^{-1}(A) + C) \]
\[ v(A) \equiv \phi(\Phi^{-1}(A)) \]
\[ w(A) \equiv \int_A^1 \frac{\phi(\gamma \Phi^{-1}(a) + C)}{\phi(\Phi^{-1}(a))} da \]

Taking derivatives,

\[ h'(A) = \frac{u'(A) v(A) w(A) - u(A) v'(A) w(A) - u'(A) v(A) w'(A)}{[w(A)]^2} \]

\[ = \frac{u(A)}{[w(A)]^2} \left\{ \left[ 1 - \gamma^2 \right] \Phi^{-1}(A) - \gamma C \right\} \int_A^1 \frac{\phi(\gamma \Phi^{-1}(a) + C)}{\phi(\Phi^{-1}(a))} da + \phi(\gamma \Phi^{-1}(A) + C) \}
\]

\[ = \frac{u(A)}{[w(A)]^2} \left\{ \left[ 1 - \frac{\gamma^2}{\gamma} \right] \Phi^{-1}(A) - \gamma C \right\} \int_{\gamma \Phi^{-1}(A) + C}^1 \phi(x) dx + \phi(\gamma \Phi^{-1}(A) + C) \}
\]

\[ = \frac{u(A)}{[w(A)]^2} \left[ 1 - \Phi(\gamma \Phi^{-1}(A) + C) \right] \left\{ \left[ 1 - \frac{\gamma^2}{\gamma} \right] \Phi^{-1}(A) - \gamma C \right\} + H(\gamma \Phi^{-1}(A) + C) \]

where I have used the change of variable \( x = \gamma \Phi^{-1}(a) + C \) and \( H \) is the hazard function of the standard normal distribution. The assumptions of the Lemma can be restated as \( \gamma > 1, C < 0 \). For \( A < \frac{1}{2} \), they imply that all the terms in brackets are positive, so the
hazard must be increasing. For $A \geq \frac{1}{2}$, use the fact that $H(y) > y$ so

$$\frac{(1 - \gamma^2) \Phi^{-1}(A) - \gamma C}{\gamma} + H(\gamma \Phi^{-1}(A) + C) \geq -\gamma \Phi^{-1}(A) - C + H(\gamma \Phi^{-1}(A) + C) > 0$$

References


