finding your inner ancova

Stat 209

[redacted]

Linear Mixed Models

Appendix to *An R and S-PLUS Companion to Applied Regression*

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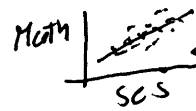
3.2 Fitting a Hierarchical Linear Model with lme

(HLM)

160 schools
7185 students

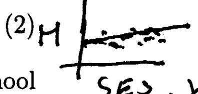
Following Bryk and Raudenbush (1992) and Singer (1998), I will fit a hierarchical linear model to the math-achievement data. This model consists of two equations: First, within schools, we have the regression of math achievement on the individual-level covariate SES; it aids interpretability of the regression coefficients to center SES at the school average; then the intercept for each school estimates the average level of math achievement in the school.

Using centered SES, the individual-level equation for individual j in school i is



elitist?

$$\text{mathach}_{ij} = \alpha_{0i} + \alpha_{1i}\text{cses}_{ij} + \varepsilon_{ij} \quad (2)$$



egalitarian

ancova $\bar{Y}$

$\hat{\delta}_{01} = 5.3$

$\hat{\delta}_{02} = 1.22$

At the school level, also following Bryk and Raudenbush, I will entertain the possibility that the school intercepts and slopes depend upon sector and upon the average level of SES in the schools:

1) ancova on class means

$\bar{Y}$ on S $\overline{SES}$

$$\begin{aligned} \alpha_{0i} &= \gamma_{00} + \gamma_{01}\text{meanses}_i + \gamma_{02}\text{sector}_i + u_{0i} \quad (3)\\ \alpha_{1i} &= \gamma_{10} + \gamma_{11}\text{meanses}_i + \gamma_{12}\text{sector}_i + u_{1i} \end{aligned}$$

ancova $\hat{\beta}$ slope

2) ancova on slopes, Math/SES

public
catholic

This kind of formulation is sometimes called a *coefficients-as-outcomes* model.

Substituting the school-level equation 3 into the individual-level equation 2 produces

$\hat{\delta}_{11} = 1.04$

$\hat{\delta}_{12} = -1.64$

$$\begin{aligned} \text{mathach}_{ij} &= \gamma_{00} + \gamma_{01}\text{meanses}_i + \gamma_{02}\text{sector}_i + u_{0i}\\ &\quad +(\gamma_{10} + \gamma_{11}\text{meanses}_i + \gamma_{12}\text{sector}_i + u_{1j})\text{cses}_{ij} + \varepsilon_{ij} \end{aligned}$$

Rearranging terms,

Cath schools higher achievement more egalatarian (compensatory)

$$\begin{aligned} \text{mathach}_{ij} &= \gamma_{00} + \gamma_{01}\text{meanses}_i + \gamma_{02}\text{sector}_i + \gamma_{10}\text{cses}_{ij}\\ &\quad +\gamma_{11}\text{meanses}_i\text{cses}_{ij} + \gamma_{12}\text{sector}_i\text{cses}_{ij}\\ &\quad +u_{0i} + u_{1i}\text{cses}_{ij} + \varepsilon_{ij} \end{aligned}$$

combined model eq. Berk sec 10.3 (10.15)

Here, the γ's are fixed effects, while the u's (and the individual-level errors ε_{ij}) are random effects.

Finally, rewriting the model in the notation of the linear mixed model (equation 1),

$$\begin{aligned} \text{mathach}_{ij} &= \beta_1 + \beta_2\text{meanses}_i + \beta_3\text{sector}_i + \beta_4\text{cses}_{ij} \quad (4)\\ &\quad +\beta_5\text{meanses}_i\text{cses}_{ij} + \beta_6\text{sector}_i\text{cses}_{ij}\\ &\quad +b_{i1} + b_{i2}\text{cses}_{ij} + \varepsilon_{ij} \end{aligned}$$

even though there are interactions galore in combined model, none in the Level 2 ancova models

lme (linear mixed effects) function in the nlme library, however, employs the *Laird-Ware form* of the linear mixed model (after a seminal paper on the topic published by Laird and Ware, 1982):

mixed effects
$y = X\beta + Zb + \varepsilon$

$$\begin{aligned} y_{ij} &= \beta_1 x_{1ij} + \cdots + \beta_p x_{pij} \quad (1)\\ &\quad +b_{i1}z_{1ij} + \cdots + b_{iq}z_{qij} + \varepsilon_{ij}\\ b_{ik} &\sim \text{N}(0, \psi_k^2), \text{Cov}(b_k, b_{k'}) = \psi_{kk'}\\ \varepsilon_{ij} &\sim \text{N}(0, \sigma^2\lambda_{ijj}), \text{Cov}(\varepsilon_{ij}, \varepsilon_{ij'}) = \sigma^2\lambda_{ijj'} \end{aligned}$$

where

- y_{ij} is the value of the response variable for the jth of n_i observations in the ith of M groups or clusters.
- $\beta_1, \ldots, \beta_p$ are the fixed-effect coefficients, which are identical for all groups.
- $x_{1ij}, \ldots, x_{pij}$ are the fixed-effect regressors for observation j in group i; the first regressor is usually for the constant, $x_{1ij} = 1$.
- $b_{i1}, \ldots, b_{iq}$ are the random-effect coefficients for group i, assumed to be multivariately normally distributed. The random effects, therefore, vary by group. The b_{ik} are thought of as random variables, not as parameters, and are similar in this respect to the errors ε_{ij}.

HSB by ancova

Week 5 Stat 209

p.2

```
> #let's do the hsb ancova
> hsbancdat = read.table(file="D:\\drr09\\stat209\\hsbancova", header = T)
> summary(hsbancdat)
```

colums of dataset, 160 rows

```
     school        Intercept        csesslp        sector     meanses
> attach(hsbancdat)
```

Intercept: $\hat{\alpha}_0$; csesslp: $\hat{\alpha}_1$; sector: P/C

```
> tapply(Intercept, sector, summary)
$C    Min.  1st Qu.  Median    Mean 3rd Qu.    Max.
   7.336  13.200  14.470  14.200  15.900  19.720
$P  Min.  1st Qu.  Median    Mean 3rd Qu.    Max.
   4.240    9.719  11.710  11.390  13.200  18.110
```

Cath higher on $\bar{Y}$

```
> tapply(csesslp, sector, summary)
$C     Min.  1st Qu.  Median    Mean 3rd Qu.    Max.
  -2.0150  0.5698  1.5230  1.4680  2.4600  5.2580
$P    Min.  1st Qu.  Median    Mean 3rd Qu.    Max.
  -1.014   1.695   2.922   2.772   3.824   6.266
```

Cath lower on $\hat{\alpha}_1$ school slope

```
> #initial differences on covariate?
> tapply(meanses, sector, summary)
$C    Min.  1st Qu.  Median    Mean 3rd Qu.    Max.
-0.7619 -0.1039  0.2388  0.1601  0.4346  0.8250
$P    Min.   1st Qu.    Median      Mean   3rd Qu.     Max.
-1.19400 -0.40090 -0.09058 -0.13550  0.11370  0.68200
> #Cath higher on school-level SES
```

so ancova will adjust a little bit

```
> # do the ancova on school-level outcomes (level, slope)
> hsbancdat$gr = 2 - as.numeric(sector) # code Cath = 1, Pub = 0 on sector
> intancova = lm(Intercept ~ gr + meanses)
> summary(intancova)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  12.1195     0.2026  59.807  < 2e-16 ***
gr            1.2219     0.3169   3.855 0.000168 ***
meanses       5.3874     0.3810  14.140  < 2e-16 ***
---
Residual standard error: 1.859 on 157 degrees of freedom
Multiple R-squared: 0.6489,     Adjusted R-squared: 0.6444
F-statistic: 145.1 on 2 and 157 DF,  p-value: < 2.2e-16
```

make 0,1 group var

school level

$\hat{\delta}_{02}$, $\hat{\delta}_{01}$ close

alternative: individual level ancova? Y_{ij}, G, sesij, csesij

```
> # compare with hlm/lme coeffs gr vs 1.226 (.306) df 157 t = 4.00
> # compare with hlm/lme coeffs meanses vs 5.33 (.369) df 157 t = 14.4
```

```
> slpancova = lm(csesslp ~ gr + meanses)
> summary(slpancova)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   2.8886     0.1600  18.049  < 2e-16 ***
gr           -1.5580     0.2503  -6.224 4.22e-09 ***
meanses       0.8612     0.3009   2.862  0.00478 **
---
Residual standard error: 1.468 on 157 degrees of freedom
Multiple R-squared: 0.1999,     Adjusted R-squared: 0.1897
F-statistic: 19.61 on 2 and 157 DF,  p-value: 2.492e-08
```

$\hat{\delta}_{12}$, $\hat{\delta}_{11}$

school slope

```
> # compare with hlm/lme coeffs gr vs -1.64 (.239) t = -6.85
> # compare with hlm/lme coeffs meanses vs 1.03 (.299) df 157 t = 3.48
```

a little different, due to weighting.

```
> # look at comparing regressions; neither outcome refutes parallel regressions
> intcnrl = lm(Intercept ~ gr + meanses + I(gr*meanses))
Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)       12.1825     0.2075  58.714  < 2e-16 ***
gr                 1.2487     0.3168   3.942 0.000122 ***
meanses            5.8524     0.5139  11.388  < 2e-16 ***
I(gr * meanses)   -1.0261     0.7634  -1.344 0.180855
---
> slpcnrl = lm(csesslp ~ gr + meanses + I(gr*meanses))
Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)        2.9122     0.1646  17.692  < 2e-16 ***
gr                -1.5480     0.2513  -6.160 5.92e-09 ***
meanses            1.0351     0.4077   2.539   0.0121 *
I(gr * meanses)   -0.3837     0.6056  -0.634   0.5272
---
```

dependence of sector effect on SES