MS&E 246: Lecture 4
Mixed strategies

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January 18, 2007

Outline

- Mixed strategies
- Mixed strategy Nash equilibrium
- Existence of Nash equilibrium
- Examples
- Discussion of Nash equilibrium

Mixed strategies

Notation:

Given a set X, we let $\Delta(X)$ denote the set of all *probability distributions* on X.

Given a strategy space S_i for player i, the *mixed strategies* for player i are $\Delta(S_i)$.

Idea: a player can randomize over *pure strategies*.

Mixed strategies

How do we interpret mixed strategies?

Note that players only play *once*; so mixed strategies reflect *uncertainty* about what the other player might play.

Payoffs

Suppose for each player i, $\mathbf{p}_i$ is a mixed strategy for player i; i.e., it is a distribution on S_i.

We extend Π_i by taking the *expectation*:

$$\Pi_i(\mathbf{p}_1, \ldots, \mathbf{p}_N) = \sum_{s_1 \in S_1} \cdots \sum_{s_N \in S_N} p_1(s_1) \cdots p_N(s_N) \Pi_i(s_1, \ldots, s_N)$$

Mixed strategy Nash equilibrium

Given a game (N, S_1, ..., S_N, Π_1, ..., Π_N):

Create a new game with N players,
strategy spaces $\Delta(S_1)$, ..., $\Delta(S_N)$,
and expected payoffs Π_1, ..., Π_N.

A *mixed strategy Nash equilibrium*
is a Nash equilibrium of this new game.

Mixed strategy Nash equilibrium

Informally:

All players can randomize over available strategies.

In a mixed NE, player i's mixed strategy must maximize his *expected payoff*, given all other player's mixed strategies.

Mixed strategy Nash equilibrium

Key observations:

(1) All our definitions -- dominated strategies, iterated strict dominance, rationalizability -- extend to mixed strategies.

Note: any *dominant* strategy must be a *pure strategy.*

Mixed strategy Nash equilibrium

(2) We can extend the definition of *best response set* identically:

$R_i(\mathbf{p}_{-i})$ is the set of mixed strategies for player i that maximize the expected payoff $\Pi_i(\mathbf{p}_i, \mathbf{p}_{-i})$.

Mixed strategy Nash equilibrium

(2) Suppose $\mathbf{p}_i \in R_i(\mathbf{p}_{-i})$, and $p_i(s_i) > 0$. Then $s_i \in R_i(\mathbf{p}_{-i})$.

(If not, player i could improve his payoff by not placing any weight on s_i at all.)

Mixed strategy Nash equilibrium

(3) It follows that $R_i(\mathbf{p}_{-i})$ can be constructed as follows:

(a) First find all *pure strategy* best responses to $\mathbf{p}_{-i}$; call this set $T_i(\mathbf{p}_{-i}) \subset S_i$.

(b) Then $R_i(\mathbf{p}_{-i})$ is the set of all probability distributions over T_i, i.e.:

$$R_i(\mathbf{p}_{-i}) = \Delta(T_i(\mathbf{p}_{-i}))$$

Mixed strategy Nash equilibrium

Moral:

A mixed strategy $\mathbf{p}_i$ *is a best response to* $\mathbf{p}_{-i}$

if and only if

every s_i *with* $p_i(s_i) > 0$ *is a best response to* $\mathbf{p}_{-i}$

Example: coordination game

We'll now apply this insight to the coordination game.

		Player 2	
		L	*R*
Player 1	*l*	(2,1)	(0,0)
	r	(0,0)	(1,2)

Example: coordination game

Suppose player 1 puts probability p_1 on *l* and probability 1 - p_1 on *r*.

Suppose player 2 puts probability p_2 on *L* and probability 1 - p_2 on *R*.

We want to find *all* Nash equilibria (pure and mixed).

Example: coordination game

- Step 1: Find best response mapping of player 1.

  Given p_2:

  $\Pi_1(l, \mathbf{p}_2) = 2\,p_2$

  $\Pi_1(r, \mathbf{p}_2) = 1 - p_2$

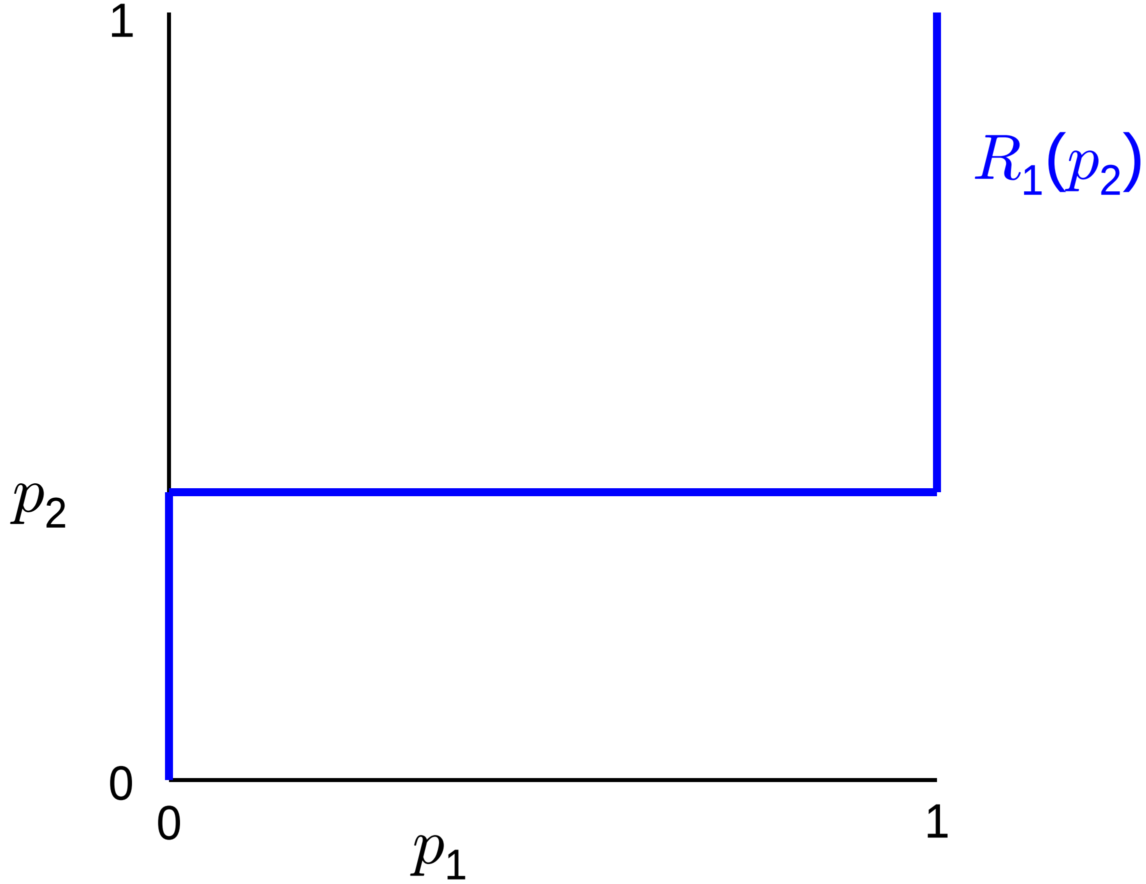
Example: coordination game

- Step 1: Find best response mapping of player 1.

If p_2 is:	Then best response is:
< 1/3	*r* (p_1 = 0)
> 1/3	*l* (p_1 = 1)
= 1/3	anything ($0 \leq p_1 \leq 1$)

Example: coordination game

Best response of player 1:



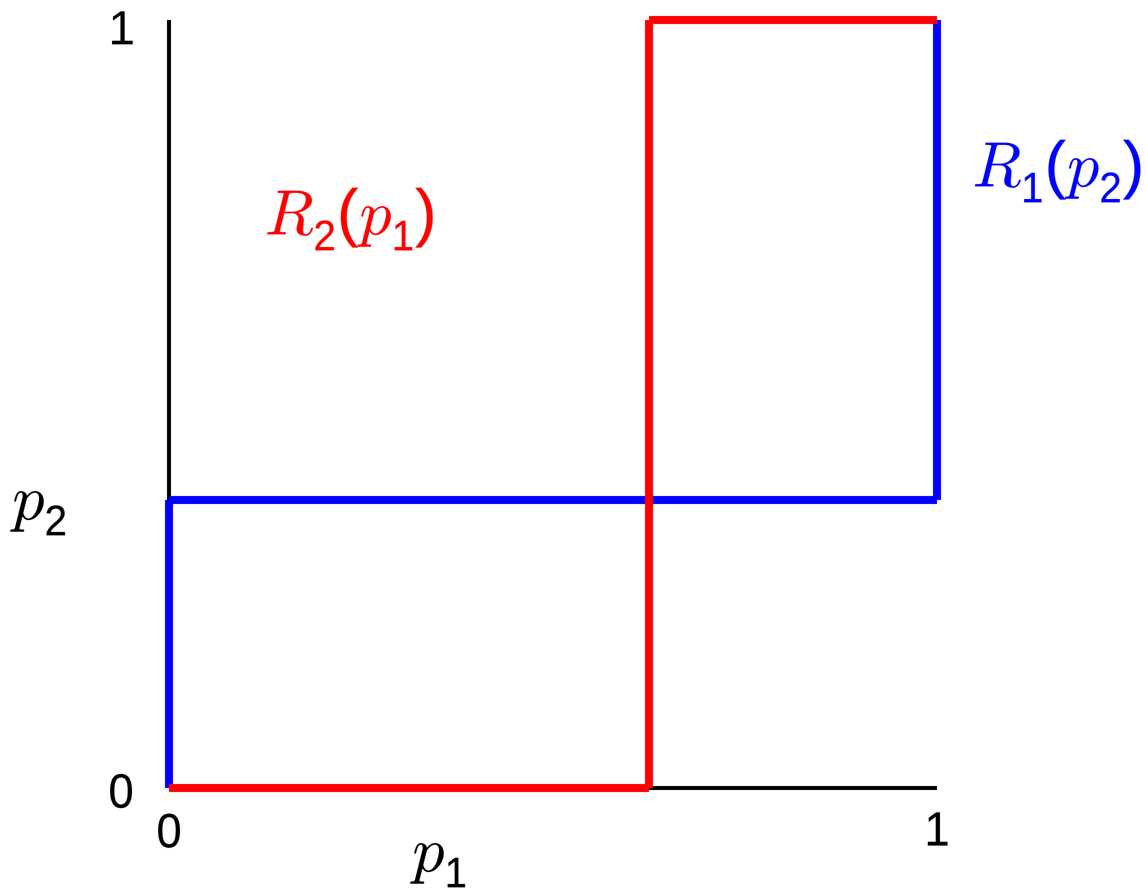
Example: coordination game

- Step 2: Find best response mapping of player 2.

If p_1 is:	Then best response is:
$< 2/3$	*R* ($p_2 = 0$)
$> 2/3$	*L* ($p_2 = 1$)
$= 2/3$	anything ($0 \leq p_1 \leq 1$)

Example: coordination game

Best response of player 2:



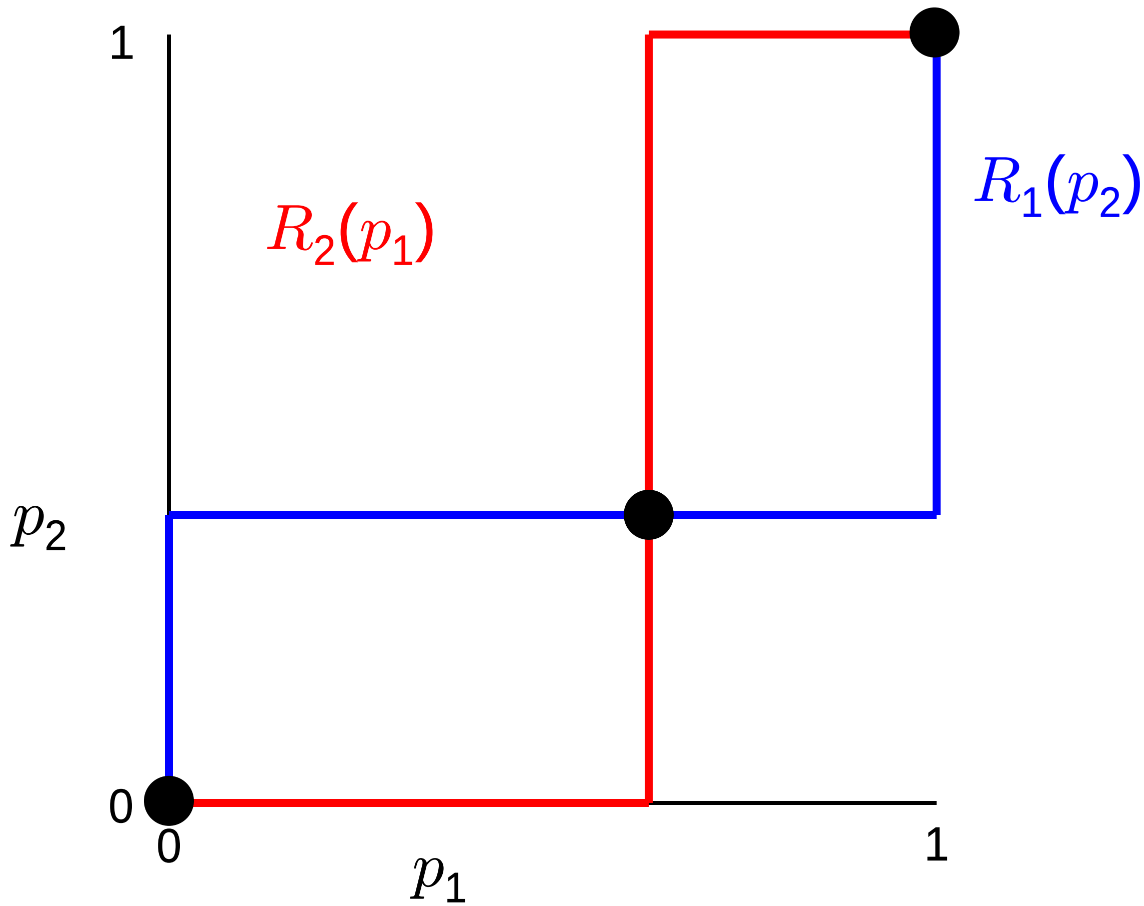
Example: coordination game

- Step 3: Find Nash equilibria.

  As before, NE occur wherever the best response mappings cross.

Example: coordination game

Nash equilibria:



Example: coordination game

Nash equilibria:

There are 3 NE:

$p_1 = 0$, $p_2 = 0 \Rightarrow$ (*r*, *R*)

$p_1 = 1$, $p_2 = 1 \Rightarrow$ (*l*, *L*)

$p_1 = 2/3$, $p_2 = 1/3$

Note: In last NE, both players get expected payoff:

2/3 x 1/3 x 2 + 1/3 x 2/3 x 1 = 2/3.

The existence theorem

Theorem:

Any N-player game where all strategy spaces are *finite* has at least one Nash equilibrium.

Notes:

- The equilibrium may be mixed.
- There is a generalization if strategy spaces are not finite.

The existence theorem: proof

Let $X = \Delta(S_1) \times \cdots \times \Delta(S_N)$ be the product of all mixed strategy spaces.

Define BR : $X \to X$ by:

BR$_i(\mathbf{p}_1, \ldots, \mathbf{p}_N) = R_i(\mathbf{p}_{-i})$

The existence theorem: proof

Key observations:

- $\Delta(S_i)$ is a closed and bounded subset of $\mathbb{R}^{|S_i|}$
- Thus X is a closed and bounded subset of Euclidean space
- Also, X is *convex*:

  If $\mathbf{p}$, $\mathbf{p}$' are in X, then so is any point on the line segment between them.

The existence theorem: proof

Key observations (continued):

-BR is "continuous"
(*i.e.*, best responses don't change suddenly as we move through X)

(*Formal statement:*

BR has a closed graph, with
convex and nonempty images)

The existence theorem: proof

By *Kakutani's fixed point theorem,* there exists $(\mathbf{p}_1, \ldots, \mathbf{p}_N)$ such that:

$(\mathbf{p}_1, \ldots, \mathbf{p}_N) \in \mathrm{BR}(\mathbf{p}_1, \ldots, \mathbf{p}_N)$

From definition of BR, this implies:

$\mathbf{p}_i \in R_i(\mathbf{p}_{-i})$ for all i

Thus $(\mathbf{p}_1, \ldots, \mathbf{p}_N)$ is a NE.

The existence theorem

Notice that the existence theorem is not *constructive*:

It tells you *nothing* about how players reach a Nash equilibrium, or an easy process to find one.

Finding Nash equilibria in general can be computationally difficult.

Discussion of Nash equilibrium

Nash equilibrium works best when
it is unique:

In this case, it is the only stable prediction of how rational players would play,

assuming common knowledge of rationality and the structure of the game.

Discussion of Nash equilibrium

How do we make predictions about play when there are multiple Nash equilibria?

1) Unilateral stability

Any Nash equilibrium is *unilaterally stable*:

If a regulator told players to play a given Nash equilibrium,
they have no reason to deviate.

2) Focal equilibria

In some settings, players may have prior preferences that "focus" attention on one equilibrium.

Schelling's example (see MWG text):

Coordination game to decide where to meet in New York City.

3) Focusing by prior agreement

If players agree ahead of time on a given equilibrium, they have no reason to deviate in practice.

This is a common justification, but can break down easily in practice:
when a game is played only once,
true enforcement is not possible.

4) Long run learning

Another common defense is that if players play the game many (independent) times, they will naturally "converge" to some Nash equilibrium as a stable convention.

Again, this is dangerous reasoning: it ignores a rationality model for dynamic play.

Problems with NE

Nash equilibrium makes very strong assumptions:

-complete information

-rationality

-common knowledge of rationality

-"focusing" (if multiple NE exist)

Example

Find *all* NE (pure and mixed) of the following game:

Player 2

Player 1	a	b	c	d
A	(1,2)	(4,0)	(0,3)	(1,1)
B	(0,1)	(2,2)	(1,2)	(0,3)
C	(1,2)	(0,3)	(3,0)	(0,1)
D	(0.5,1)	(0,0)	(0,0)	(2,0)