Consider the following game:

		Player 2			
		Y1	Y2	Y3	Y4
Player 1	X1	(1,2)	(4,0)	(0,3)	(1,1)
	X2	(0,1)	(2,2)	(1,2)	(0,3)
	X3	(1,2)	(0,3)	(3,0)	(0,1)
	X4	(0.5,1)	(0,0)	(0,0)	(2,0)

What are all the Nash equilibria (pure and mixed) of this game?

Solution:

The game has no pure NE.

Strategy X2 is strictly dominated, since when player 1 plays X1 with probability $1/2+\varepsilon$ and X3 with probability $1/2-\varepsilon$ he gets a strictly greater expected payoff for any strategy of player 2. When we consider the game without the second row (which corresponds to strategy X2), we can see that strategy Y4 is strictly dominated by Y1. In the game that occurs after the elimination of the fourth column (strategy Y4), X4 is strictly dominated, by mixing X1 and X3. Thus in order to compute the mixed strategy Nash equilibria, it suffices to consider the game where player 1 only has two strategies (X1 and X3) and player 2 has three (Y1, Y2 and Y3).

Suppose player 1 plays X1 and X3 with probability p and $1-p$ respectively. The expected payoff from player 2's strategies are $\Pi_{Y1} = 2$, $\Pi_{Y2} = 3-3p$ and $\Pi_{Y3} = 3p$. So player 2 is indifferent between Y1 and Y2 for $p = 1/3$ and indifferent between Y1 and Y3 for $p = 2/3$. Suppose player 2 plays $Y1$ with probability q and Y2 with probability $1-q$. Then player 1 is indifferent between X1 and X3 for $q = 1$. Similarly, if player 2 plays Y1 with probability q and Y3 with probability $1-q$, we get $q = 1$. For $1/3 < p < 2/3$, Y1 is the best response of player 2. On the other hand, given that player 2 plays Y1, player 1 is indifferent between playing X1 and playing X3.

This shows that the game has the following mixed strategy Nash equilibria: Player 1 plays X1 with probability p and X3 with probability $1 - p$, where $1/3 \leq p \leq 2/3$ and player 2 plays Y1.

Comments on solution:

Note first that when we are checking whether a strategy is dominated, *it doesn't matter what strategy (pure or mixed) we use as the dominating strategy.* Thus, for example, we mix X1 and X3 with certain probabilities in the first step of iterated strict dominance above, but mix with different probabilities later. Some of you questioned this, since it would require player 1 to use *different* mixtures to dominate X2 and X4, respectively. But this is perfectly fine; said differently, if player 1 ever contemplating putting positive probability on X2 in a Nash equilibrium, he could do better by replacing that by a mixture over X1 and X3 with weights as described above; and similarly for X4. It doesn't matter that those weights are different.

Second, note that when we are computing best responses to a mixed strategy, we only compute *pure best responses*, and then we mix over them. This means that: (1) we first consider the payoffs at each pure strategy, given the mixed strategy of the other player; (2) select those pure strategies that maximize payoff; and (3) any mixture over those strategies is also a best response.