In the last lecture, we defined several variants of *fictitious play*. In this lecture we will discuss some examples that reveal the behavior of fictitious play, and then recap the key results on convergence. We will also prove convergence in two cases: two-player zero sum games under the PCTFP dynamic, and N-player games of identical interest under the CTFP dynamic. We use the same notation and terminology as Lecture 6.

1 Examples

1.1 A Coordination Game

Consider the following two player game:

		Player 2: a	Player 2: b
Player 1	A	(1,1)	(0,0)
Player 1	B	(0,0)	(1,1)

This is a coordination game with a unique fully mixed Nash equilibrium, where both players put probability $1/2$ on each action. Now consider DTFP, where we start with $p_1^0(A) = 1 - p_1^0(B) = 3/4$, and $p_2^0(a) = 1 - p_2^0(b) = 1/4$. Then the empirical distributions and play at each time period evolve as follows:

t	p_1^t	p_2^t	a_1^t	a_2^t
0	(3/4, 1/4)	(1/4, 3/4)	B	a
1	(3/4, 5/4)	(5/4, 3/4)	A	b
2	(7/4, 5/4)	(5/4, 7/4)	B	a
3	(7/4, 9/4)	(9/4, 7/4)	A	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

We make two observations here. First, notice that the marginal empirical distribution of each player is converging to $(1/2, 1/2)$ as $t \to \infty$. This is an example where DTFP converges. However, notice also that *the joint empirical distribution has no weight on the diagonal entries of the payoff matrix*; i.e., the payoff to each player is zero at every time period. This is an example that highlights the flaws inherent in standard fictitious play: when the players play pure actions at each time period, the marginal empirical distributions can converge to a Nash equilibrium, but the actual play may resemble nothing like a Nash equilibrium. Examples similar to this have been presented by [2] and [5], among others.

1.2 The Shapley Game

Shapley presented an example where DTFP does not converge, in the sense that the empirical distributions never converge [13]. His game is as follows:

		Player 2		
		L	M	R
Player 1	T	(0,0)	(1,0)	(0,1)
	M	(0,1)	(0,0)	(1,0)
	B	(1,0)	(0,1)	(0,0)

Note that the game has a unique Nash equilibrium, where the players both uniformly randomize over available actions. In this example, Shapley showed that if play begins with (T, M), then the sequence of action pairs visited by DTFP is $(T, M) \to (T, R) \to (M, R) \to (M, L) \to (B, L) \to (B, M) \to (T, M) \to \cdots$. (Note this is the same sequence exhibited by the standard best response dynamics if started in (T, M).) The key observation made by Shapley is that DTFP spends an exponentially increasing amount of time in each action pair; as a result, the empirical distributions never converge.

1.3 Three-Player "Matching Pennies"

While compelling, Shapley's example only establishes that there may exist initial conditions for which DTFP diverges; by itself, it does not contradict, for example, the conjecture that Nash equilibria may be locally stable under DTFP. (Informally, "local stability" refers to the fact that for all initial conditions that are a sufficiently small distance from the Nash equilibrium, DTFP converges asymptotically to the NE.)

Jordan [5] provides a counterexample to this assertion. In particular, Jordan considered a three-player "matching pennies" game. Each player has two actions, H or T. Player 1 wants to match the action of player 2; player 2 wants to match the action of player 3; and player 3 wants to match the *opposite* of the action of player 1. Each player receives a payoff of 1 if they match as desired, and -1 otherwise. It is straightforward to check that this game has a unique Nash equilibrium, where all players uniformly randomize. Jordan shows that this Nash equilibrium is locally unstable in a strong sense: for any $\varepsilon > 0$, and for almost all initial empirical distributions that are within (Euclidean) distance ε of the unique Nash equilibrium, DTFP does not converge to the NE; instead, it enters a limit cycle asymptotically as $t \to \infty$.

2 Convergence Results: A Survey

In this section we survey the key convergence results on fictitious play for zero-sum games, identical interest games (i.e., games where all players have the same payoff functions), and two-player 2×2 games. We also briefly discuss a collection of other related results.

2.1 DTFP

All the earliest results on convergence of fictitious play concerned the discrete-time variant. Robinson proved convergence of DTFP for *two-player zero-sum games* in 1951 [11]; this was the first rigorous result on convergence. More formally, she proved that $\max_{s_1\in\Delta(A_1)} s_1\boldsymbol{M}p_2^t$ and $\min_{s_2\in\Delta(A_2)} p_1^t\boldsymbol{M}s_2$ both converge to the value of the game, where $\boldsymbol{M}$ is the matrix for the game. Note that this does not necessarily establish that the empirical distributions converge to optimal strategies as well; however, such a result follows from work on CTFP (see below).

Monderer and Shapley established that DTFP converges for *games of identical interest* in 1996 [8]. Miyasawa proved convergence of DTFP for *two-player 2×2 games* in 1961 [7]. However, it is worth noting that the two-player 2×2 case follows from earlier results: it is not difficult to show that any two-player 2×2 game that is "nondegenerate" in an appropriate sense is best response equivalent to either a zero-sum game, or a game of identical interest. Thus Miyasawa's result follows from those of Robinson and Monderer and Shapley. ("Best response equivalent" means that two games with the same action spaces give rise to the same best response mappings; see Problem Set 1 for more details on the 2×2 game result.)

2.2 CTFP

Harris' 1998 paper [3] establishes convergence of CTFP for *two-player zero-sum games* as well as *games of identical interest*; further, he shows that convergence of CTFP can be used to demonstrate convergence of DTFP. The previous insight is used to prove that the rate of convergence of DTFP in the zero-sum case is $1/t$: i.e., after t stages, the difference between $\max_{s_1\in\Delta(A_1)} s_1\boldsymbol{M}p_2^t$ or $\min_{s_2\in\Delta(A_2)} p_1^t\boldsymbol{M}s_2$ and the value of the game is bounded by $\Theta(1/t)$. He also establishes that the empirical distributions converge to the set of optimal strategies, which in fact shows that $p_1^t\boldsymbol{M}p_2^t$ approaches the value of the game as well. Combining his results establishes that CTFP converges for *two-player 2×2 games* as well.

We note in passing that Hofbauer has written an unpublished manuscript entitled "Stability for the Best Response Dynamic" (1995), that also contains results similar to Harris' 1998 paper; indeed, Harris cites the manuscript in his paper. However, it appears that Hofbauer's manuscript has never been published in a journal.

2.3 SFP and PCTFP

Stochastic fictitious play was introduced by Fudenberg and Kreps in 1993 [2], who also proved convergence for *two-player 2×2 games* that have a unique mixed Nash equilibrium. Their approach involved elements of stochastic approximations, but did not explicitly employ the connection between the differential equation of PCTFP and the stochastic evolution of SFP.

The most general convergence results for both SFP and PCTFP are provided by Hofbauer and Sandholm in 2002 [4]. They establish that PCTFP converges in both *perturbed two-player zero-sum games*, as well as *perturbed games of identical interest*. They then use these convergence results to establish convergence of SFP in the same settings.

Critical to the analysis carried out by Hofbauer and Sandholm was the use of Lyapunov methods to study the PCTFP differential equation. Shamma and Arslan [12] observed that the same basic underlying Lyapunov structure can be used to study both zero-sum games and identical interest games; they use this approach to unify proofs of convergence for several cases, including 2×2 games. We note here that the structure exploited by Shamma and Arslan was initially employed by Harris for zero-sum games and identical interest games in his 1998 paper on CTFP [3].

2.4 Comments

We conclude with some comments on the results above.

1. A (finite) *weighted potential game* [9] is a finite simultaneous-move game together with a *potential* function $\mathcal{V} : \prod_i \Delta(A_i) \to \mathbb{R}$ and a vector of weights $\boldsymbol{w}$, with the following property:
$$\Pi_i(s_i, \boldsymbol{s}_{-i}) - \Pi_i(s_i', \boldsymbol{s}_{-i}) = w_i \left(\mathcal{V}(s_i, \boldsymbol{s}_{-i}) - \mathcal{V}(s_i', \boldsymbol{s}_{-i})\right),$$
with weights $w_i > 0$. When $w_i = 1$ for all i, we call this a *exact potential game*. It follows from this relationship that every weighted potential game is best response equivalent to a game of identical interest, and conversely every game of identical interest is a potential game with weights $w_i = 1$ for all i. Thus all the results on convergence of fictitious play for identical interest games also translate to results on convergence of fictitious play for weighted potential games.

2. Some other specialized results are known. For example, SFP converges in two-player games where one player has two actions and the other has an arbitrary finite number of actions (called $2\times n$ games) [1]. Fictitious play also converges for dominance solvable games [6, 10]. In subsequent lectures on supermodular games we will also discuss convergence of fictitious play in those settings.

3 Proofs of Convergence

In this section we give two simple proofs of convergence. We first provide a proof of convergence of PCTFP in two-player zero-sum games; the proof we give is presented by Shamma and Arslan [12], but is very closely related to work of both Harris [3] and Hofbauer and Sandholm [4]. We then give a proof of convergence of CTFP in games of identical interest due to Harris [3]; the proof is analogous to convergence proofs for DTFP [8] and PCTFP [12].

Central to our result will be the following function for each player i:

$$U_i(s_i, \boldsymbol{s}_{-i}) = \max_{s_i' \in \Delta(A_i)} \Pi_i(s_i', \boldsymbol{s}_{-i}) - \Pi_i(s_i, \boldsymbol{s}_{-i}).$$

The function U_i gives the maximum possible payoff improvement player i can achieve by a unilateral deviation in his own (mixed) action. We will see below in our analysis that the function $\mathcal{W}(t) = \sum_i U_i(\boldsymbol{p}^t)$ will play a major role in proving convergence in all cases. Observe that $U_i(\boldsymbol{s}) \geq 0$ for all $\boldsymbol{s}$. Further, if $U_i(\boldsymbol{s}) = 0$ for all i, then $\boldsymbol{s}$ must be a Nash equilibrium.

3.1 Two-Player Zero-Sum Games and PCTFP

We consider a two-player zero-sum matrix game with payoff matrix $\boldsymbol{M}$, where the payoffs of both players are perturbed by i.i.d. noise. We can represent the payoffs of the two players in the following form:

$$\Pi_1(s_1, s_2) = s_1^\top \boldsymbol{M} s_2 - V_1(s_1);$$
$$\Pi_2(s_1, s_2) = -s_1^\top \boldsymbol{M} s_2 - V_2(s_2).$$

Here the function V_i is convex, and is associated to the perturbed best response function C_i of player i, as discussed in Lecture 6; recall C_i is also called the choice probability function. In particular, we have:

$$C_i(s_i) = \arg\max_{s_i' \in \Delta(A_i)} \Pi_i(s_i', s_{-i}).$$

Note that the perturbed game is not necessarily zero-sum.

PCTFP corresponds to the following pair of differential equations:

$$\frac{dp_i^t}{dt} = C_i(p_{-i}^t) - p_i^t, \quad i = 1, 2.$$

The advantage of proving convergence in this setting is that it is a differential equation, instead of a differential inclusion as in standard CTFP.

We use a Lyapunov function approach to prove convergence. We consider a function $\mathcal{W}(t) = U_1(\boldsymbol{p}^t) + U_2(\boldsymbol{p}^t)$. Explicitly, we have:

$$\mathcal{W}(t) = \max_{s_1' \in \Delta(A_1)} \Pi_1(s_1', p_2^t) + \max_{s_2' \in \Delta(A_2)} \Pi_2(p_1^t, s_2') + V_1(p_1^t) + V_2(p_2^t).$$

We have the following theorem; the proof is inspired by the presentation of Shamma and Arslan [12].

Theorem 1 *Regardless of the initial condition $\boldsymbol{p}^0$, the following limits hold:*

$$\lim_{t \to \infty} \left(p_i^t - C_i(p_{-i}^t) \right) = 0, \quad i = 1, 2.$$

Thus asymptotically, each player's empirical distribution is a best response to that of his opponent. Note that if there are multiple Nash equilibria, this does not necessarily imply convergence of the empirical distributions; however, if the Nash equilibria are isolated, then we can conclude that PCTFP will converge to a Nash equilibrium.

Proof. We prove the theorem by showing that $d\mathcal{W}(t)/dt \leq 0$, with equality if and only if $p_i^t \in C_i(p_{-i}^t)$ for $i = 1, 2$; i.e., we establish that $\mathcal{W}$ is a Lyapunov function.

We will need the following lemma.

Lemma 2 (Envelope Theorem) *Let $F(\boldsymbol{x}, \boldsymbol{u})$ be a continuously differentiable function of $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$. Let $U \subset \mathbb{R}^m$ be an open convex subset, and suppose $\boldsymbol{\mu}^*(\boldsymbol{x})$ is a continuously differentiable function such that:*

$$F(\boldsymbol{x}, \boldsymbol{\mu}^*(\boldsymbol{x})) = \min_{\boldsymbol{u} \in U} F(\boldsymbol{x}, \boldsymbol{u}).$$

Define $H(\boldsymbol{x}) = \min_{\boldsymbol{u} \in U} F(\boldsymbol{x}, \boldsymbol{u})$. Then:

$$\nabla_{\boldsymbol{x}} H(\boldsymbol{x}) = \nabla_{\boldsymbol{x}} F(\boldsymbol{x}, \boldsymbol{\mu}^*(\boldsymbol{x})).$$

Proof. Differentiate:

$$\begin{aligned} \nabla_{\boldsymbol{x}} H(\boldsymbol{x}) &= \nabla_{\boldsymbol{x}} F(\boldsymbol{x}, \boldsymbol{\mu}^*(\boldsymbol{x})) + \nabla_{\boldsymbol{u}} F(\boldsymbol{x}, \boldsymbol{\mu}^*(\boldsymbol{x})) \nabla_{\boldsymbol{x}} \boldsymbol{\mu}^*(\boldsymbol{x}) \\ &= \nabla_{\boldsymbol{x}} F(\boldsymbol{x}, \boldsymbol{\mu}^*(\boldsymbol{x})), \end{aligned}$$

where we use the fact that $\nabla_{\boldsymbol{u}} F(\boldsymbol{x}, \boldsymbol{\mu}^*(\boldsymbol{x})) = 0$ since $\boldsymbol{\mu}^*(\boldsymbol{x})$ minimizes $F(\boldsymbol{x}, \boldsymbol{u})$. □

Using the preceding lemma, we see that:

$$\begin{aligned} \frac{d}{dt}\left[\max_{s_1' \in \Delta(A_1)} \Pi_1(s_1', p_2^t)\right] &= \nabla_{s_2} \Pi_1(C_1(p_2^t), p_2^t)^\top \frac{dp_2^t}{dt} \\ &= C_1(p_2^t)^\top \boldsymbol{M} \frac{dp_2^t}{dt} \\ &= C_1(p_2^t)^\top \boldsymbol{M} (C_2(p_1^t) - p_2^t). \end{aligned}$$

Similarly,

$$\frac{d}{dt}\left[\max_{s_2' \in \Delta(A_2)} \Pi_2(p_1^t, s_2')\right] = -(C_1(p_2^t) - p_1^t)^\top \boldsymbol{M} C_2(p_1^t).$$

Combining, we have:

$$\frac{d\mathcal{W}(t)}{dt} = -C_1(p_2^t)^\top \boldsymbol{M} p_2^t + (p_1^t)^\top \boldsymbol{M} C_2(p_1^t) + \nabla V_1(p_1^t)^\top \frac{dp_1^t}{dt} + \nabla V_2(p_2^t)^\top \frac{dp_2^t}{dt}. \tag{1}$$

We now observe that since C_i is the perturbed best response function, we have:

$$C_1(p_2^t)^\top \boldsymbol{M} p_2^t - V_1(C_1(p_2^t)) \geq (p_1^t)^\top \boldsymbol{M} p_2^t - V_1(p_1^t),$$

and

$$-(p_1^t)^\top \boldsymbol{M} C_2(p_1^t) - V_2(C_2(p_1^t)) \geq -(p_1^t)^\top \boldsymbol{M} p_2^t - V_2(p_2^t),$$

with equality in both cases if and only if $C_i(p_{-i}^t) = p_i^t$. (The latter claim follows by uniqueness of the perturbed best response.)

Adding and rearranging terms, we have:

$$\begin{aligned}
-C_1(p_2^t)^\top \boldsymbol{M} p_2^t &+ (p_1^t)^\top \boldsymbol{M} C_2(p_1^t) \\
&\leq \sum_i V_i(p_i^t) - V_i(C_i(p_{-i}^t)) \\
&\leq -\sum_i \nabla V_i(p_i^t)^\top (C_i(p_i^t) - p_i^t) \\
&= -\sum_i \nabla V_i(p_i^t)^\top \frac{dp_i^t}{dt},
\end{aligned}$$

where to establish the second inequality we use the fact that V_i is convex. The preceding relationship and (1) immediately imply that $d\mathcal{W}(t)/dt \leq 0$ for all t, with equality if and only if $C_i(p_{-i}^t) = p_i^t$ for both players. The theorem follows. □

To develop some intuition, suppose that we could *ignore* the perturbation terms involving V_i, but continue to assume the best response is unique. Since the resulting game is exactly a zero-sum game, then the best response for player 1 guarantees $C_1(p_2^t)^\top \boldsymbol{M} p_2^t \geq \mathrm{val}(\boldsymbol{M})$, and similarly the best response for player 2 guarantees $(p_1^t)^\top \boldsymbol{M} C_2(p_1^t) \leq \mathrm{val}(\boldsymbol{M})$. If we thus consider (1), ignoring the perturbation terms and assuming the game is exactly zero-sum yields $d\mathcal{W}(t)/dt \leq 0$, with equality if and only if player 1's payoff is $\mathrm{val}(\boldsymbol{M})$ and player 2's payoff is $-\mathrm{val}\,\boldsymbol{M}$. This heuristic argument suggests that $\mathcal{W}$ is a Lyapunov function, and that $\Pi_i(p_1^t, p_2^t)$ both converge to the value of the game.

Indeed, if the argument of the preceding paragraph were valid, we can actually establish more: if we remove the perturbation terms, the expression (1) reduces to:

$$\frac{d\mathcal{W}(t)}{dt} = -\mathcal{W}(t).$$

This differential equation has a straightforward solution: $\mathcal{W}(t) = e^{-t} W(-\infty)$ (where $W(-\infty)$ is the initial condition). Thus CTFP should converge exponentially fast; further, using the logarithmic time transformation, i.e., letting $\hat{W}(\hat{t}) = W(\log \hat{t})$ (see Lecture 6), this also suggests that DTFP converges in $O(1/t)$ time; i.e., $\hat{W}(\hat{t}) = W(-\infty)/\hat{t}$. Indeed, Harris uses exactly such an insight to establish the convergence rate of DTFP [3].

Of course, the argument in the preceding paragraphs are heuristic: if we indeed chose to remove the perturbation functions V_i, then the best response for each player would *no longer be unique*, and the resulting continuous-time fictitious play is a differential inclusion rather than a differential equation. Nevertheless, the approach we described here can be used to prove convergence for PCTFP, as demonstrated in the theorem.

3.2 Games of Identical Interest and CTFP

We now consider an N-player game of identical interest, i.e., a game where all players share the same payoff function Π. In this case we consider the CTFP dynamic:

$$\frac{dp_i^t}{dt} \in BR_i(\boldsymbol{p}_{-i}^t) - p_i^t.$$

Let $\{p_i^t\}$ be a CTFP process, and let $s_i^t = p_i^t + dp_i^t/dt$. Note that $s_i^t \in BR_i(\boldsymbol{p}_{-i}^t)$.

We have the following theorem, taken from Harris [3].

Theorem 3 *For all players i, and regardless of the initial condition $\boldsymbol{p}^0$:*

$$\lim_{t\to\infty}\left[\max_{s_i'\in\Delta(A_i)} \Pi(s_i', \boldsymbol{p}_{-i}^t) - \Pi(p_i^t, \boldsymbol{p}_{-i}^t)\right] = 0.$$

The result means that p_i^t is asymptotically a (mixed) best response to $\boldsymbol{p}_{-i}^t$.

Proof. We define $\mathcal{W}(t) = \sum_i U_i(\boldsymbol{p}^t)$. Observe that:

$$\begin{aligned}
\frac{d}{dt}(\Pi(\boldsymbol{p}^t)) &= \frac{d}{dt}\left[\sum_{a_i\in A_i}\cdots\sum_{a_N\in A_N} p_1^t(a_1)\cdots p_N^t(a_N)\Pi(\boldsymbol{a})\right] \\
&= \sum_i \sum_{a_i\in A_i}\cdots\sum_{a_N\in A_N} \frac{dp_i^t}{dt}(a_i)\left(\prod_{j\neq i} p_j^t(a_j)\right)\Pi(\boldsymbol{a}) \\
&= \sum_i \Pi\left(\frac{dp_i^t}{dt}, \boldsymbol{p}_{-i}^t\right).
\end{aligned}$$

The preceding explicit derivation is equivalent to asserting that when viewed as a function on Euclidean space, Π is multilinear in its arguments—the mixed strategies of the players—so the time derivative can be directly applied to the arguments. In other words, given s_i and s_i' and $\alpha, \alpha' \in \mathbb{R}$, we have $\Pi(\alpha s_i + \alpha' s_i', \boldsymbol{s}_{-i}) = \alpha\Pi(s_i, \boldsymbol{s}_{-i}) + \alpha'\Pi(s_i', \boldsymbol{s}_{-i})$.

Now observe that:

$$\Pi\left(\frac{dp_i^t}{dt}, \boldsymbol{p}_{-i}^t\right) = \Pi(s_i^t - p_i^t, \boldsymbol{p}_{-i}^t) = \Pi(s_i^t, p_i^t) - \Pi(\boldsymbol{p}^t) = U_i(\boldsymbol{p}^t).$$

The second equality again follows by multilinearity of Π, while the last equality uses the fact that $s_i^t \in BR_i(\boldsymbol{p}_{-i}^t)$.

Combining our calculations, we conclude that:

$$\frac{d}{dt}(\Pi(\boldsymbol{p}^t)) = \sum_i U_i(\boldsymbol{p}^t) = \mathcal{W}(t).$$

Since $\mathcal{W}$ is nonnegative everywhere, we conclude $\Pi(\boldsymbol{p}^t)$ is nondecreasing as t increases; thus $\Pi^* = \lim_{t\to\infty}\Pi(\boldsymbol{p}^t)$ exists. Since Π is bounded above, we must have $\Pi^* < \infty$.

To conclude the proof, it suffices to note that $\mathcal{W}$ is a Lipschitz continuous function in t, so there exists a constant K such that:

$$\mathcal{W}(t) \leq \mathcal{W}(t+\Delta) + K\Delta, \tag{$*$}$$

for all $\Delta \geq 0$. From our calculation above, we conclude:

$$\Pi^* - \Pi(\boldsymbol{p}^t) \geq \Pi(\boldsymbol{p}^{t+\Delta}) - \Pi(\boldsymbol{p}^t) = \int_0^\Delta \mathcal{W}(t+\tau)\, d\tau.$$

Observe from $(*)$ that:

$$\int_0^\Delta \mathcal{W}(t+\tau)\, d\tau \geq \Delta\mathcal{W}(t) - \frac{K\Delta^2}{2}.$$

If we choose $\Delta = \mathcal{W}(t)/K$, then we have:

$$\Pi^* - \Pi(\boldsymbol{p}^t) \geq \frac{\mathcal{W}(t)^2}{2K} \geq 0.$$

Since the left hand side converges to zero, we conclude $\mathcal{W}(t) \to 0$ as $t \to \infty$, as required. □

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