In this lecture, we develop the theory of supermodular games; key references are the papers of Topkis [7], Vives [8], and Milgrom and Roberts [3]. Our development closely follows that of Milgrom and Roberts, though we will also note other references where necessary.

1 Lattices and Tarski's Theorem

We start with some basic definitions and facts about lattices. Given a set X, a binary relation $\succeq$ is a *partial ordering* on X if it is reflexive (i.e., $x \succeq x$ for all $x \in X$); transitive (i.e., $x \succeq y$ and $y \succeq z$ implies $x \succeq z$); and antisymmetric (i.e., $x \succeq y$ and $y \succeq x$ implies $x = y$). The relation $\succeq$ is a *total ordering* if $x \succeq y$ or $y \succeq x$ for all x, y.

Given any set $S \subset X$, an element x is called an *upper bound* of S if $x \succeq y$ for all $y \in S$; similarly, x is called a lower bound of S if $y \succeq x$ for all $y \in S$. We say that x is a *supremum* or *least upper bound* of S in X if x is an upper bound of S, and for any other upper bound x' of S, we have $x' \succeq x$; note that the supremum is unique if it exists. In this case we write $x = \sup S$. We similarly define infimum (or greatest lower bound), and denote it by $\inf S$. We will occasionally need to be explicit about the underlying set in which we are computing the supremum or infimum; in such situations, we will write $\sup_X S$ or $\inf_X S$ for the supremum of S in X, and the infimum of S in X, respectively.

The partially ordered set $(X, \succeq)$ is a *lattice* if for all pairs $x, y \in X$, the elements $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in X. The lattice $(X, \succeq)$ is a *complete lattice* if in addition, for all nonempty subsets $S \subset X$, the elements $\sup S$ and $\inf S$ exist in X. A set S is a *sublattice* of $(X, \succeq)$ if for any two $x, y \in S$, the elements $\sup_X\{x, y\}$ and $\inf_X\{x, y\}$ lie in S. Note that $(S, \succeq)$ can be a lattice without being a sublattice; i.e., $\sup_S\{x, y\}$ and $\inf_S\{x, y\}$ may exist in S, but $\sup_X\{x, y\}$ and $\inf_X\{x, y\}$ may not lie in S.

The following theorem is a basic result in theory of lattices. Note that a function $f : X \to X$ is increasing if $x \succeq y$ implies $f(x) \succeq f(y)$.

Theorem 1 (Tarski) *Suppose that $(X, \succeq)$, and f is an increasing function from $X \to X$. Define:*

$$E = \{x \in X : f(x) = x\},$$

the set of fixed points *of f. Then E is nonempty, and $(E, \succeq)$ is a complete lattice. In particular, $\sup_X E \in E$, and $\inf_X E \in E$.*

The last claim of the theorem is straightforward to establish, by observing that $\sup_X E = \sup_E E$, and $\inf_X E = \inf_E E$.

The following example shows that the set of fixed points need not be a sublattice.

Example 1 (Vives, [8]) Consider the lattice $X = \{0, 1, 2\} \times \{0, 1, 2\}$, with the usual vector ordering: $\boldsymbol{x} \geq \boldsymbol{y}$ if and only if $x_i \geq y_i$ for all i. Define $f(\boldsymbol{x}) = \boldsymbol{x}$, except for $\boldsymbol{x} = (1, 2)$, $(2, 1)$, or

$(1, 1)$, which are all mapped to $(2, 2)$. Then the set of fixed points is:

$$E = \{(0, 0), (0, 1), (0, 2), (1, 0), (2, 0), (2, 2)\}.$$

This is a complete lattice, but it is not a sublattice, since $\sup_X\{(0, 1), (1, 0)\} = (1, 1) \notin E$.

Zhou [9] has generalized Tarski's theorem to increasing correspondences (i.e., point-to-set mappings); we do not state it formally here, but simply note that conclusions similar to Tarski's theorem hold: namely, an increasing correspondence has a fixed point, and the set of fixed points is a complete lattice.

For the remainder of the notes, *we restrict attention to lattices that are subsets of Euclidean space, $X \subset \mathbb{R}^n$, with the usual vector ordering* (as in the example): for $\boldsymbol{x}, \boldsymbol{y} \in X$, $\boldsymbol{x} \geq \boldsymbol{y}$ if and only if $x_i \geq y_i$ for all i. It is easy to see in this case that the lattice is complete if and only if it is compact.

2 Supermodularity and Increasing Differences

Let $(X, \succeq)$ be a lattice. We say that $f : X \to \mathbb{R}$ is *supermodular* if for all $x, y \in X$, there holds:

$$f(x) + f(y) \leq f(\inf\{x, y\}) + f(\sup\{x, y\}).$$

We say f is strictly supermodular if the preceding inequality is strict for all x and y. Note that if $\succeq$ is a total ordering, then every function f on X is supermodular. In the case of Euclidean lattices, if X is one-dimensional, then every function on X is supermodular.

Supermodularity is closely related to increasing differences. Let $(X, \succeq_X)$ and $(T, \succeq_T)$ be complete lattices. Then $f : X \times T \to \mathbb{R}$ has *increasing differences* in x and t if for all $x' \succeq x$ and $t' \succeq t$, there holds:

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

Again, we say f has strictly increasing differences in x and t if the preceding inequality is strict. Note that we can view $X \times T$ as a lattice, with $(x', t') \succeq (x, t)$ if and only if $x' \succeq_X x$ and $t' \succeq_T t$. Note that if f is supermodular on this lattice, then f has increasing differences in x and t. If the sets X and T are totally ordered, then supermodularity on $X \times T$ and increasing differences in x and t coincide.

Checking supermodularity and increasing differences is simplified when the function f is twice differentiable. Given a lattice X in Euclidean space, a twice differentiable function $f : X \to \mathbb{R}$ is supermodular if and only if:

$$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j} \geq 0, \quad \text{for all } i \neq j.$$

Similarly, given lattices X and T in Euclidean space, a twice differentiable function $f : X \times T \to \mathbb{R}$ has increasing differences in x and t if and only if:

$$\frac{\partial^2 f(\boldsymbol{x}, \boldsymbol{t})}{\partial x_i \partial t_j} \geq 0, \quad \text{for all } i, j.$$

The proofs of both these claims are immediate.

The following result of Topkis is central to our analysis. Note that a function $f : A \to \mathbb{R}$ on $A \subset \mathbb{R}^n$ is *upper semicontinuous* if $\limsup_{x \to x_0} f(x) \leq f(x_0)$.

Theorem 2 (Topkis) *Suppose that X and T are complete lattices in Euclidean space. Let $f : X \times T \to \mathbb{R}$ be a function that is supermodular on X, has increasing differences in $\boldsymbol{x}$ and $\boldsymbol{t}$, and is upper semicontinuous in $\boldsymbol{x}$ (for fixed $\boldsymbol{t}$). Then for each $\boldsymbol{t} \in T$,*

$$\phi(\boldsymbol{t}) = \arg\max_{\boldsymbol{x} \in X} f(\boldsymbol{x}, \boldsymbol{t})$$

is a nonempty complete sublattice of X.

Further, if $\boldsymbol{x}' \in \phi(\boldsymbol{t}')$ and $\boldsymbol{x} \in \phi(\boldsymbol{t})$ with $\boldsymbol{t}' \geq \boldsymbol{t}$, then $\sup\{\boldsymbol{x}, \boldsymbol{x}'\} \in \phi(\boldsymbol{t}')$, and $\inf\{\boldsymbol{x}, \boldsymbol{x}'\} \in \phi(\boldsymbol{t})$. Thus $\overline{\boldsymbol{x}}(\boldsymbol{t}) = \sup \phi(\boldsymbol{t})$ and $\underline{\boldsymbol{x}}(t) = \inf \phi(\boldsymbol{t})$ are both increasing functions from T to X.

Additional insight is gained with stronger assumptions on f. If f is strictly supermodular, then $\phi(\boldsymbol{t})$ is totally ordered. If f has strictly increasing differences, then ϕ itself is "strictly increasing": if $\boldsymbol{t}' \geq \boldsymbol{t}$, and $\boldsymbol{x}' \in \phi(\boldsymbol{t}')$, $\boldsymbol{x} \in \phi(\boldsymbol{t})$, then $\boldsymbol{x}' \geq \boldsymbol{x}$.

We also note that these types of results are called *monotone comparative statics* results: they give results on the monotonicity of optimal solutions, as a function of parameters in the optimization problem. Milgrom and Shannon [5] derive monotone comparative statics results under conditions that generalize supermodularity and increasing differences, called *single crossing conditions.* These conditions have found wide application in economics; for example, single crossing conditions can be used to characterize equilibrium bid functions in auctions. We direct the reader to [5] for further details on this theory.

3 Supermodular Games

We consider a finite N-player game, where each player i has action set A_i, and payoff function Π_i; we let $A = \prod_i A_i$ denote the space of composite strategy vectors. The resulting game is *supermodular* if for each i:

1. A_i is a complete (and thus compact) lattice in Euclidean space $\mathbb{R}^{n_i}$.
2. $\Pi_i(a_i, \boldsymbol{a}_{-i})$ is upper semicontinuous in a_i for fixed $\boldsymbol{a}_{-i})$, and continuous in $\boldsymbol{a}_{-i}$ for fixed a_i, with a finite uniform upper bound in $\boldsymbol{a}$.
3. $\Pi_i(a_i, \boldsymbol{a}_{-i})$ is supermodular in a_i.
4. $\Pi_i(a_i, \boldsymbol{a}_{-i})$ has increasing differences in a_i and $\boldsymbol{a}_{-i}$.

The game is *strictly supermodular* if conditions 3 and 4 are strict.

Let $BR_i(\boldsymbol{a}_{-i})$ denote the best response mapping of player i. Let $\overline{BR}_i(\boldsymbol{a}_{-i}) = \sup BR_i(\boldsymbol{a}_{-i})$ (the "upper" best response), and $\underline{BR}_i(\boldsymbol{a}_{-i}) = \inf BR_i(\boldsymbol{a}_{-i})$ (the "lower" best response). We define $BR : A \to A$ as the mapping where the i'th coordinate is BR_i; and similarly define $\overline{BR}$ and $\underline{BR}$.

Topkis' theorem then immediately implies that $\overline{BR}_i$ and $\underline{BR}_i$ always exist, and are increasing functions. Applying Tarski's theorem to $\overline{BR}$ and $\underline{BR}$ then establishes that both have fixed points; note that this immediately implies that a pure Nash equilibrium exists for any supermodular game. (Using the generalization of Tarski's theorem to correspondences actually establishes that the entire set of Nash equilibria is a complete lattice, though again it need not be a sublattice; see [9].)

We will pursue a different approach to understanding the set of Nash equilibria; as in [3], we focus our attention on the set of actions surviving iterated strict dominance in pure actions (ISD-P) in a supermodular game. We will need some additional notation involving ISD-P. Given $T \subset A$, we define $U_i(T)$ as follows:

$$U_i(T) = \{a_i \in A_i : \text{for all } a_i' \in A_i, \text{ there exists } \boldsymbol{a} \in T \text{ s.t. } \Pi_i(a_i, \boldsymbol{a}_{-i}) \geq \Pi_i(a_i', \boldsymbol{a}_{-i})\}.$$

In other words, $U_i(T)$ is the set of pure actions of player i that are *not dominated by any pure action*, given that all other players play using action vectors in T. We let $U(T) = \prod_i U_i(T)$. We also use $U^k(T)$ to denote the set of pure strategies remaining after k applications of U to the set T, with U^0 equal to the identity map.

It is straightforward to check the following claims:

1. *Monotonicity*: If $T \subset T'$, then $U(T) \subset U(T')$.
2. *Decreasing sequence property*: $U^{k+1}(A) \subset U^k(A)$ for all k. (Note that this need not be true if we iterate U starting from a set *strictly smaller* than the entire strategy space A, since for an arbitrary set T we need not have $U(T) \subset T$.)

In light of the second claim, we let $U^\infty(A) = \bigcap_{k \geq 0} U^k(A)$. Note that this is the set of *strategies surviving ISD-P*.

We emphasize that typically, strict dominance includes *mixed* actions; i.e., given $T \subset A$, we can define $U_i^M(T)$ as follows:

$$U_i(T) = \{a_i \in A_i : \text{for all } s_i \in \Delta(A_i), \text{ there exists } \boldsymbol{a} \in T \text{ s.t. } \Pi_i(a_i, \boldsymbol{a}_{-i}) \geq \Pi_i(s_i, \boldsymbol{a}_{-i})\};$$

In other words, $U_i(T)$ is the set of pure actions of player i that are *not dominated by any mixed action*, given that all other players play using action vectors in T. We define $U^{M\infty}(A) = \bigcap_{k \geq 0} (U^M)^k(A)$; this is the set of *strategies surviving iterated strict dominance in mixed actions* (ISD-M). The game is *dominance solvable* if $U^{M\infty}(A)$ is a singleton. It is clear that $U^{M\infty}(A) \subset U^\infty(A)$; in particular, if $U^\infty(A)$ is a singleton, then the game is dominance solvable.

Our main result is the following theorem.

Theorem 3 (Milgrom and Roberts [3]) *Given a supermodular game with composite strategy space $A = \prod_i A_i$, let $\overline{\boldsymbol{a}}^* = \sup U^\infty(A)$, and let $\underline{\boldsymbol{a}}^* = \inf U^\infty(A)$. Then both $\overline{\boldsymbol{a}}^*$ and $\underline{\boldsymbol{a}}^*$ are pure Nash equilibria.*

Proof. Given a pair of pure action vectors $\boldsymbol{a}, \boldsymbol{a}'$, we write $[\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}] = \{\hat{\boldsymbol{a}} \in A : \underline{\boldsymbol{a}} \leq \hat{\boldsymbol{a}}, \hat{\boldsymbol{a}} \leq \overline{\boldsymbol{a}}\}$. (In Euclidean space, this is set of all strategy vectors that lie in the "box" with lower corner $\underline{\boldsymbol{a}}$ and upper corner $\overline{\boldsymbol{a}}$.)

We start with the following lemma.

Lemma 4 *For any vectors $\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}} \in A$, there holds:*

$$\sup U([\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}]) = \overline{BR}(\overline{\boldsymbol{a}}); \;\; \inf U([\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}]) = \underline{BR}(\underline{\boldsymbol{a}}).$$

Proof of Lemma. Suppose there exists $\boldsymbol{a} \in [\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}]$ and a player i such that $a_i \not\leq \overline{BR}_i(\overline{\boldsymbol{a}}_{-i})$. Let $x_i = \inf\{a_i, \overline{BR}_i(\overline{\boldsymbol{a}}_{-i})\}$. Then for any $\boldsymbol{x} \in A$:

$$\begin{aligned}\Pi_i(a_i, \boldsymbol{x}_{-i}) - \Pi_i(\inf\{a_i, \overline{BR}_i(\overline{\boldsymbol{a}}_{-i})\}, \boldsymbol{x}_{-i}) &\leq \Pi_i(a_i, \overline{\boldsymbol{a}}_{-i}) - \Pi_i(\inf\{a_i, \overline{BR}_i(\overline{\boldsymbol{a}}_{-i})\}, \overline{\boldsymbol{a}}_{-i}) \\ &\leq \Pi_i(\sup\{a_i, \overline{BR}_i(\overline{\boldsymbol{a}}_{-i})\}, \overline{\boldsymbol{a}}_{-i}) - \Pi_i(\overline{BR}_i(\overline{\boldsymbol{a}}_{-i}), \overline{\boldsymbol{a}}_{-i}) < 0.\end{aligned}$$

The first inequality follows by increasing differences, and the second by supermodularity. The third uses the fact that $a_i \not\leq \overline{BR}_i(\overline{\boldsymbol{a}}_{-i})$, so that we must have $\sup\{a_i, \overline{BR}_i(\overline{\boldsymbol{a}}_{-i})\} > \overline{BR}_i(\overline{\boldsymbol{a}}_{-i})$; as a result, $\sup\{a_i, \overline{BR}_i(\overline{\boldsymbol{a}}_{-i})\}$ cannot be a best response to $\overline{\boldsymbol{a}}_{-i}$. The calculation shows that a_i is strictly dominated. Thus if $\boldsymbol{a} \in U([\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}])$, then $\boldsymbol{a} \leq \overline{BR}(\overline{\boldsymbol{a}})$; the claim that $\boldsymbol{a} \geq \underline{BR}(\underline{\boldsymbol{a}})$ is similar. To conclude the proof it suffices to note that $\overline{BR}(\overline{\boldsymbol{a}}), \underline{BR}(\underline{\boldsymbol{a}}) \in U([\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}])$, by definition of best response. □

To conclude the proof, let $\overline{\boldsymbol{a}}^0 = \sup A$, $\underline{\boldsymbol{a}}^0 = \inf A$. Inductively, define $\overline{\boldsymbol{a}}^{k+1} = \overline{BR}(\overline{\boldsymbol{a}}^k)$, $\underline{\boldsymbol{a}}^{k+1} = \underline{BR}(\overline{\boldsymbol{a}}^k)$. We prove by induction that $U^k(A) \subset [\underline{\boldsymbol{a}}^k, \overline{\boldsymbol{a}}^k]$. The claim is trivially true at $k = 0$ (we define $U^0(A) = A$); so assume it holds for k. Then:

$$U^{k+1}(A) \subset U([\underline{\boldsymbol{a}}^k, \overline{\boldsymbol{a}}^k]) \subset [\underline{BR}(\underline{\boldsymbol{a}}^k), \overline{BR}(\overline{\boldsymbol{a}}^k)] = [\underline{\boldsymbol{a}}^{k+1}, \overline{\boldsymbol{a}}^{k+1}].$$

Thus the induction is complete. This observation also shows that $[\underline{\boldsymbol{a}}^{k+1}, \overline{\boldsymbol{a}}^{k+1}] \subset [\underline{\boldsymbol{a}}^k, \overline{\boldsymbol{a}}^k]$ for all k, so that $\underline{\boldsymbol{a}}^k$ and $\overline{\boldsymbol{a}}^k$ are both monotonic sequences. By compactness they must have limits $\underline{\boldsymbol{a}}$ and $\overline{\boldsymbol{a}}$, respectively, and our induction yields $U^\infty \subset [\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}]$.

To complete the proof, it suffices to show that $\underline{\boldsymbol{a}}$ and $\overline{\boldsymbol{a}}$ are both Nash equilibria. By definition of best response, for fixed i and k and arbitrary a_i we have:

$$\Pi_i(a_i, \overline{\boldsymbol{a}}^k_{-i}) \leq \Pi_i(\overline{a}_i^{k+1}, \overline{\boldsymbol{a}}^k_{-i}).$$

As $k \to \infty$, by continuity in $\boldsymbol{a}_{-i}$, the left hand side approaches $\Pi_i(a_i, \overline{\boldsymbol{a}}_{-i})$. On the right hand side, by upper semicontinuity of Π_i, we have:

$$\limsup_{k\to\infty} \Pi_i(\overline{a}_i^{k+1}, \overline{\boldsymbol{a}}^k_{-i}) \leq \Pi_i(\overline{a}_i, \overline{\boldsymbol{a}}_{-i}).$$

(Note that $\Pi_i(\boldsymbol{a})$ is upper semicontinuous in $\boldsymbol{a}$ because it is upper semicontinuous in a_i for fixed $\boldsymbol{a}_{-i}$, and continuous in $\boldsymbol{a}_{-i}$ for fixed a_i.) We conclude that:

$$\Pi_i(a_i, \overline{\boldsymbol{a}}_{-i}) \leq \Pi_i(\overline{a}_i, \overline{\boldsymbol{a}}_{-i}),$$

which establishes that $\overline{\boldsymbol{a}}$ is a Nash equilibrium, so that $\overline{\boldsymbol{a}} = \sup U^\infty(A)$. The argument that $\underline{\boldsymbol{a}}$ is a Nash equilibrium, so that $\underline{\boldsymbol{a}} = \inf U^\infty(A)$, is similar. □

The proof has several strong implications. First, we can find Nash equilibria by application of strict dominance in pure or mixed strategies. Second, for any Nash equilibrium $\boldsymbol{a}$, we have $\boldsymbol{a} \le \overline{\boldsymbol{a}}^*$, and $\underline{\boldsymbol{a}}^* \le \boldsymbol{a}$. Thus the infimum and supremum of the set of strategies surviving ISD-P or ISD-M are also the largest and smallest Nash equilibrium. Further, if $U^\infty(A)$ is a singleton the game is dominance solvable; the theorem then implies that if the game has a unique Nash equilibrium, the game is dominance solvable.

We conclude with a some additional important observations about supermodular games, presented in the following subsections.

3.1 Best Response Dynamics

Let $\boldsymbol{a}^0 \in A$ be an initial action vector, and let $\boldsymbol{a}^t \in BR(\boldsymbol{a}^{t-1})$; these are the discrete best response dynamics. Note that if we start with $\boldsymbol{a}^0 \ge \overline{\boldsymbol{a}}^*$, then in the notation of the proof of Theorem 3, we have $\overline{\boldsymbol{a}}^0 \ge \boldsymbol{a}^0 \ge \overline{\boldsymbol{a}}^*$. If we now inductively define $\boldsymbol{a}^k = \overline{BR}(\boldsymbol{a}^{k-1})$, then the proof of the theorem and the fact that $\overline{BR}$ is an increasing function gives $\overline{\boldsymbol{a}}^k \ge \boldsymbol{a}^k \ge \overline{\boldsymbol{a}}^*$. Taking the limit as $k \to \infty$ yields that $\boldsymbol{a}^k \to \overline{\boldsymbol{a}}^*$. In other words, if in the best response dynamic we always apply the upper best response, then starting from any strategy vector above $\overline{\boldsymbol{a}}^*$, the dynamics converge to $\overline{\boldsymbol{a}}^*$. Similarly, starting from any strategy vector below $\underline{\boldsymbol{a}}^*$, if we always apply the lower best response, the dynamics will converge to $\underline{\boldsymbol{a}}^*$. See [8] for a slightly more refined version of these statements; in particular, Vives shows that best response dynamics always converge to a Nash equilibrium starting from above $\overline{\boldsymbol{a}}^*$ or below $\underline{\boldsymbol{a}}^*$. In Lecture 9 we show in a far more general setting that best response dynamics, as well as fictitious play, for supermodular games always converge to the set $[\underline{\boldsymbol{a}}^*, \overline{\boldsymbol{a}}^*]$; see also [4].

3.2 Comparative Statics

Suppose that for each player i, the payoff function $\Pi_i(\boldsymbol{a}, \tau)$ is indexed by a parameter τ, and that Π_i has increasing differences in $\boldsymbol{a}$ and τ; this is a *family of supermodular games indexed by* τ. Let $\overline{\boldsymbol{a}}^*(\tau)$ and $\underline{\boldsymbol{a}}^*(\tau)$ be the largest and smallest Nash equilibria, respectively, in the supermodular game with fixed τ. It is straightforward to show that the functions $\overline{\boldsymbol{a}}^*$ and $\underline{\boldsymbol{a}}^*$ are both increasing in τ. This property can be used to study the behavior of Nash equilibria as an external parameter of the game is changed.

3.3 Pareto Efficiency

Simple results on Pareto efficiency follow easily for supermodular games under some additional conditions. For example, suppose that for all i, $\Pi_i(a_i, \boldsymbol{a}_{-i})$ is increasing in $\boldsymbol{a}_{-i}$; then it follows that for all i, and for any other Nash equilibrium $\boldsymbol{a}$, $\Pi_i(\overline{\boldsymbol{a}}^*) \ge \Pi_i(\boldsymbol{a})$. Thus the largest Nash equilibrium is Pareto efficient in the set of Nash equilibria in this case. Milgrom and Roberts discuss some other (minor) refinements of this observation [3].

4 Examples

In this section we briefly present two examples of supermodular games: games with network effects, and wireless interference games.

4.1 Network Effects

Supermodular games capture network effects and positive externalities very well, because network effects are inherently about complementarity: typically, the payoff to a player increases with increasing effort from his opponents when there are positive externalities.

We start with an example of a game studied by Farrell and Saloner in an influential paper [1]. Suppose that a set N of users can use one of two technologies, X or Y. We let $B_i(S, X)$ denote the payoff to player i when the subset S of users is using technology X, and $i \in S$; implicitly, the subset $N \setminus S$ is using technology Y. We similarly define $B_i(S, Y)$ as the payoff to i when the subset of users S is using technology Y, and $i \in S$. Intuitively, a network effect exists if player i is better off when more users use the same technology as him; i.e.:

$$B_i(S, k) \leq B_i(S', k), \quad \text{if } S \subset S'.$$

We now show that this assumption leads to a natural supermodular game.

Consider a simultaneous move game where the action of a player is the technology they choose to use, either X or Y . We impose a lattice structure on the action space by assuming $Y \succeq X$. Given an action vector $\boldsymbol{a}$, let $X(\boldsymbol{a}) = \{i \in N : a_i = X\}$, and let $Y(\boldsymbol{a}) = \{i \in N : a_i = Y\}$. We define the payoff to a player as follows:

$$\Pi_i(a_i, \boldsymbol{a}_{-i}) = \begin{cases} B_i(X(\boldsymbol{a}), X), & \text{if } a_i = X; \\ B_i(Y(\boldsymbol{a}), Y), & \text{if } a_i = Y. \end{cases}$$

It is then straightforawrd to check that under our assumption on S, this is a supermodular game.

Another network effects model can be provided with continuous action spaces as follows. Suppose each player i chooses an "effort" $e_i \in [0, B_i]$. This was initially motivated in terms of the search for trading partners in a market: e_i is the effort expended by a player to try to find a match for a trade (see [3]). We assume the payoff to player i is as follows:

$$\Pi_i(e_i, \boldsymbol{e}_{-i}) = \alpha_i e_i \sum_{j \neq i} e_j - c_i(e_i),$$

where $\alpha_i > 0$ and $c_i(e_i)$ is increasing and continuous. It is easy to see that $\partial^2 \Pi_i / \partial e_i \partial e_j = \alpha_i > 0$, so this is a supermodular game. Note that here $\Pi_i(e_i, \boldsymbol{e}_{-i})$ is increasing in $\boldsymbol{e}_{-i}$ for all i, so (not surprisingly) the Pareto preferred Nash equilibrium is the equilibrium where agents exert the most effort. We also observe that $\Pi_i(e_i, \boldsymbol{e}_{-i}; \alpha_i)$ has increasing differences in α_i; thus if the vector of externality coefficients $\boldsymbol{\alpha}$ increases (in the usual Euclidean lattice), then our comparative statics result shows that the effort exerted in the largest Nash equilibrium will increase as well.

These basic models are the foundation for far more sophisticated analysis in the literature. For some recent papers that include network structure in models of network effects, see [2] and [6].

4.2 Wireless Interference Games

Consider a collection of N wireless devices, where device i can choose a transmission power level $p_i \in [0, P_i]$. The *signal to interference-plus-noise ratio* (SINR) of device i when power levels $\boldsymbol{p}$ are chosen is:

$$\gamma_i(\boldsymbol{p}) = \frac{p_i}{\sum_{j \neq i} p_j + N_i},$$

where $N_i > 0$ is the noise level seen by device i. (Note that we are ignoring channel gains here.) We assume that each node incurs a cost for power usage, given by $c_i(p_i)$; we assume c_i is increasing and continuous. We assume that the utility to a node is a function of its SINR; thus:

$$\Pi_i(\boldsymbol{p}) = U_i(\gamma_i(\boldsymbol{p})) - c_i(p_i).$$

Here U_i is an increasing, twice differentiable function. To show that this game is supermodular, it suffices to check that $\partial^2 \Pi_i / \partial p_i \partial p_j \geq 0$. This will hold as long as:

$$\varepsilon_i(x) = -\frac{x U_i''(x)}{U_i'(x)} \geq 1, \text{ for all } x \geq 0.$$

The constant $\varepsilon_i(x)$ is the *elasticity* of the marginal utility function U_i' at x; it gives the percentage decrease in marginal utility for a 1% change in SINR. Since $U_i' > 0$, the condition also requires that $U_i'' < 0$; i.e., the utility function must be strictly concave.

It is worth noting that if $U_i(x) = \log(1 + x)$, which is the expression for Shannon capacity in a Gaussian interference channel with SINR x, the elasticity condition does not hold. Indeed, one can show that for this utility function the resulting game exhibits *decreasing* differences, i.e., $\partial^2 \Pi_i / \partial p_i \partial p_j < 0$. One possible solution that works with two players is to change the ordering on one of the player's strategy spaces. In particular, define:

$$\hat{\Pi}_i(p_1, p_2) = \Pi_i(p_1, -p_2), \quad i = 1, 2.$$

Then it is straightforward to check that when $U_i(x) = \log(1 + x)$, the payoffs $\hat{\Pi}_i$ exhibit increasing differences. However, we also observe this "trick" only works with two players: with more than two players, no consistent order reversal can change decreasing differences to increasing differences.

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