The equivalence theorem and gauge invariance in renormalizable theories

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A formal proof of the equivalence theorem is given which reveals its more precise meaning: the renormalized theories are equivalent. The same approach is used to prove the gauge invariance of renormalizable gauge theories which admit a gauge-invariant regularization: the massless Yang-Mills field and the Yang-Mills field with spontaneous symmetry breakdown. In these models the dependence of the Green's functions and of the renormalization constants on the gauge condition is investigated and it is proved that the physical masses, the renormalized coupling constants and the physical, renormalized S-matrix do not depend on the selection of gauge.

1. INTRODUCTION

In this paper we investigate the equivalence theorem and the gauge invariance in Lagrangian field theories. These questions have a long history[1], but in spite of this have not been solved until now. A rigorous result, namely the Borchers theorem, has so far found no applicability in the framework of Lagrangian field theory.

The basic defects of the known formal "proofs" of the equivalence theorem are that in all of them one compares and asserts the equivalence of unrenormalized quantities, whereas one should compare renormalized quantities. In addition to the physical considerations there are also mathematical reasons for requiring this; owing to the divergences in the Lagrangian field theory the unrenormalized quantities simply do not exist.

The same objections are applicable to the proofs of gauge invariance of models of gauge vector fields[2], in particular, of quantum electrodynamics. Of particular importance is the problem of proving gauge invariance in connection with the construction of renormalizable models of massive vector particles[3], in which the mass of the vector particles appears as a result of spontaneous breakdown of gauge invariance of models of the Yang-Mills type.

Such models have more chances to be physical than models which describe massless particles, and for these models one should give as rigorous as possible proofs of the gauge invariance.

The apparently most correct proof of the gauge invariance of quantum electrodynamics[1)] is contained in the paper of Bialynicki-Birula[4], where it is shown that the renormalized scattering matrix elements do not depend on the gauge. We would like to note that in that paper it is in fact assumed without verification that the physical mass of the electron is gauge-independent.

In the case of a neutral gauge vector meson with spontaneous symmetry breakdown, where it is necessary to verify the gauge-independence of the masses of all physical particles and of the renormalized coupling constants, the proof (which is of course applicable to ordinary quantum electrodynamics) is given in the paper of Fradkin and Tyutin[5].

In the present paper we propose a method of proof of the equivalence theorem (and of gauge invariance as a special case of that theorem), which will allow us to verify the equality of all the physical quantities (masses, renormalized coupling constants, matrix elements). Essential use is made of the hypothesis that the structure of the Green's function in a theory is determined by the diagram series of perturbation theory.

As an illustration of the method, Section 2 considers the proof of the independence of physical quantities on the selection of the field variables φ or ψ, related by $\varphi = F(\psi)$, in a theory of a scalar field described by an arbitrary Lagrangian $L(\partial_\mu\varphi, \varphi)$. The proof is formal in the sense that it is carried through in the unregularized theory for fixed unrenormalized constants. Nevertheless the discussion allows to demonstrate clearly (with the exception of the scheme of the proof) that only renormalized theories can be equivalent, and that the proof will be rigorous if in both formulations the theory will not contain any divergences.

The last condition is satisfied for model gauge theories when they are formulated in different (at least covariant) gauges. The regularization procedure proposed by Slavnov[6] is gauge-invariant and gauge-independent. This allows one to compare the formulations of the theory in various gauges, before removing the regularization.

In Sec. 3 we discuss the theory of a massless Yang-Mills field for a class of gauges which comprises all the known ones. It is shown that the mass of physical states (defined in that section) does not depend on the gauge and vanishes, and that the scattering matrix elements of the renormalized theory do not depend on the gauge. We also prove that the renormalized coupling constants may be considered independent of the gauge, so that the theory remains gauge-independent even after removing the regularization. We also derive a useful relation between the renormalization constants of the fermion and meson fields. In all the discussions we ignored the difficulties related to infrared divergences, assuming that these will be overcome in the future.

A completely analogous discussion is carried through in Sec. 4 for the case of the model of a Yang-Mills field with spontaneous symmetry breakdown[3] in covariant gauges. The gauge-independence of the masses of physical states, of renormalized coupling constants and of the renormalized physical S-matrix are verified.

In carrying out the proof in Secs. 3 and 4 we have only used the property of gauge invariance of the Lagrangian of the theory. Therefore we have not written out explicitly the part of the Lagrangian which corresponds to the regularization. The regularization scheme as well as the construction of the generating functional by means of the canonical quantization procedure in the presence of regularization are given in Appendix 2.

Finally, we note that the considerations of Secs. 3 and 4 show directly that the parameters of unphysical states (e.g., their masses) depend explicitly on the gauge.

2. THE EQUIVALENCE THEOREM

This section contains a formal proof of the equivalence theorem, the content of which, as is well-known, consists in asserting that a theory does not change under a substitution $\varphi \to F(\varphi)$ of the fields. This example will demonstrate the basic idea of the proof, which will be used in the two following sections. In addition it will be seen that quantum field theory leads naturally to the equivalence of renormalized models[2)].

We consider a model described by the Lagrangian

$$L\left\{\frac{Z}{2}(\partial_\mu\varphi)^2,\ Z^{1/2}\varphi\right\}. \tag{1}$$

The constant Z is chosen in such a manner that the propagator of the φ-field has a residue of one at its pole. Thus, φ is a renormalized field. We construct in the usual manner the Hamiltonian

$$\pi = Z\dot{\varphi}\frac{\partial}{\partial x}L(x, y), \qquad x = \frac{Z}{2}(\partial_\mu\varphi)^2, \qquad y = Z^{1/2}\varphi, \tag{2}$$

$$H = \pi\dot{\varphi} - L. \tag{3}$$

We assume that the theory has the usual structure in the sense that the asymptotic fields are described by the free Lagrangian or Hamiltonian of the usual form

$$L_0 = \frac{1}{2}\partial_\mu\varphi^{(0)}\partial_\mu\varphi^{(0)} - \frac{m^2}{2}\varphi^{(0)2}, \quad H_0 = \frac{1}{2}\pi^{(0)2} + \frac{1}{2}(\partial_k\varphi^{(0)})^2 + \frac{m^2}{2}\varphi^{(0)2}, \tag{4}$$

where m is the physical mass of the φ-field. We transform to the interaction picture

$$H_{int}(\pi^{(0)}, \varphi^{(0)}) = H(\pi^{(0)}, \varphi^{(0)}) - H_0(\pi^{(0)}, \varphi^{(0)}); \tag{5}$$

in (5) $H(\pi^{(0)}, \varphi^{(0)})$ depends functionally on $\pi^{(0)}$ and $\varphi^{(0)}$ in the same manner as the Hamiltonian in (3) depends on π, φ. We further determine the expression[3)] for the generating functional W of the Green's functions:

$$\begin{aligned} W &= \left\langle 0 \left| T_D \exp\left\{-i\int dx(H_{int} - \varphi^{(0)}J)\right\}\right| 0\right\rangle \\ &= \exp\left\{-i\int dx\, H_{int}\left\{\frac{\delta}{i\delta\xi}, \frac{\delta}{i\delta J}\right\}\right\}\left\langle 0 \left| T_D \exp\left\{i\int dx(\pi^{(0)}\xi + \varphi^{(0)}J)\right\}\right| 0\right\rangle\Big|_{\xi=0} \\ &= \exp\left\{-i\int dx\, H_{int}\left\{\frac{\delta}{i\delta\xi}, \frac{\delta}{i\delta J}\right\}\right\}\int d\varphi\, d\pi \exp\left\{i\int dx(\pi\varphi - H_0(\pi, \varphi) + \pi\xi \right. \\ &\quad \left. + \varphi J)\right\}\Big|_{\xi=0} = \int d\pi\, d\varphi \exp\left\{i\int dx(\pi\dot{\varphi} - H(\pi, \varphi) + \varphi J)\right\}. \end{aligned} \tag{6}$$

The transition from the second row to the third one in Eq. (6) is easily verified by direct calculation. We would also like to point out that in the present paper we consider the functional expression for W as an abbreviation of the results obtained in perturbation theory. The validity of all the transformations carried out below on the functional expression can be justified in perturbation theory. It is however important that the use of the functional expression allows one to trivialize clumsy and often obscure algebraic manipulations which one would have to perform directly on the diagrams of the perturbation series.

We perform in (6) the substitution

$$Z^{-1/2}\pi \to \pi', \quad Z^{1/2}\varphi \to \varphi'. \tag{7}$$

Here π' and φ' are the unrenormalized canonical momentum and field, $H(\pi', \varphi')$ is the unrenormalized Hamiltonian, constructed from the unrenormalized Lagrangian, and the expression for the generating functional goes over into the expression constructed for the unrenormalized theory, but with the source introduced for the renormalized field. It will be convenient, although not required as a matter of principle, to work with just such an expression for the generating functional:

$$W = \int d\pi\, d\varphi \exp\left\{i\int dx(\pi\dot{\varphi} - H(\pi, \varphi) + Z^{-1/2}\varphi J)\right\}, \tag{8}$$

where in (8) and everywhere below we shall not indicate explicitly that the Hamiltonian (or Lagrangian) is constructed in the framework of the unrenormalized theory.

We now consider the model described by the Lagrangian

$$L_1({}^1\!/_2\partial_\mu\psi\partial_\mu\psi, \psi) = L\{{}^1\!/_2\partial_\mu\varphi(\psi)\partial_\mu\varphi(\psi), \varphi(\psi)\}. \tag{9}$$

The Lagrangian L_1 is obtained from L by means of the substitution

$$\varphi = \varphi(\psi) = \psi + f(\psi), \tag{10}$$

where the expansion of the function f starts with its quadratic term. The corresponding generating functional has the form

$$W_1 = \int d\pi_\psi\, d\psi \exp\left\{i\int dx(\pi_\psi\dot{\psi} - H_1(\pi_\psi, \psi) + Z_1^{-1/2}\psi J)\right\}, \tag{11}$$

where the renormalization constant of the wave function, Z_1, is in general different from the renormalization constant Z.

We carry out in (6) the change of integration variables

$$\varphi \to \varphi(\psi), \qquad \pi \to \pi(\pi_\psi, \psi) = \pi_\psi\frac{\partial\psi}{\partial\varphi}. \tag{12}$$

Eq. (12) is a canonical transformation from the theory (1) (with $Z^{1/2}\varphi \to \varphi$) to the theory (9). Under this transformation $H(\pi, \varphi) = H_1(\pi_\psi, \psi)$ the Jacobian equals one, and, as is easily verified, $\pi\dot{\varphi} = \pi_\psi\dot{\psi}$. Thus,

$$W = \int d\pi\, d\psi \exp\left\{i\int dx(\pi\dot{\psi} - H_1(\pi, \psi) + Z^{-1/2}\varphi(\psi)J)\right\}. \tag{13}$$

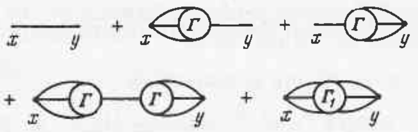
We see that (13) differs from (11) only in the source term. The functional derivatives of (13) with respect to J are, as before, the Green's functions of the theory (1).

We consider the structure of the propagator of the field φ, which is a consequence of (13):

$$\begin{aligned} \langle\varphi(x)\varphi(y)\rangle &= \langle\psi(x)\psi(y)\rangle_1 + \langle f(\psi(x))\psi(y)\rangle_1 \\ &\quad + \langle\psi(x)f(\psi(y))\rangle_1 + \langle f(\psi(x))f(\psi(y))\rangle_1. \end{aligned} \tag{14}$$

$\langle \ldots \rangle$ denotes the vacuum expectation value of the T-product of the corresponding operators; the index 1 in the right-hand side of (14) denotes that the corresponding quantities are computed within the framework of the theory (9).

The right-hand side of (14) can be represented in the form of the following diagrams (i.e., in the form of an expansion with respect to the propagators and vertex functions of the ψ-fields):



(15)

It can be seen that the right-hand side of (14) has only a first-order pole, for which the position coincides with the position of the pole of the propagator D_ψ of the ψ-field. This implies that the position of the pole of the propagator D_φ of the φ-field (the left-hand side of (14)) coincides with the position of the pole of D_ψ. Indeed, if the positions of the poles of D_φ and D_ψ would not coincide, an infinite series of multiple poles would appear in the right-hand side of (14), but this does not happen, as we saw. This important result means that the mass of the field does not change under the substitution (10), but was obtained with fixed bare constants, i.e., without taking into account the need to regularize the theory.

The renormalization constants of the two theories are related by the following equation

$$Z = Z_1(1+\Gamma)^2, \tag{16}$$

where Γ is the value of the vertex function

(17)

on the mass shell of the ψ-field.

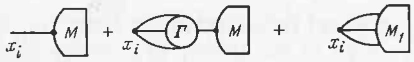
We consider the scattering matrix elements obtained from the Green's functions

$$G_\varphi(x_1, \ldots, x_n) = \langle \varphi(x_1) \ldots \varphi(x_n) \rangle \tag{18}$$

by multiplication (in momentum space) by $(p_1^2 - m^2) \ldots (p_n^2 - m^2)$ and taking the limit $p_i^2 - m^2 \to 0$; p_i is the momentum corresponding to the i-th field operator, m is the physical mass of the field φ (and also of ψ). Eq. (13) yields an expression of G_φ^n in terms of the Green's functions of the field ψ in the following form:

$$G_\varphi^n(x_1, \ldots, x_n) = \langle \varphi(\psi(x_1)) \ldots \varphi(\psi(x_n)) \rangle_1. \tag{19}$$

The right-hand side of Eq. (19) has the following diagrammatic structure:



(20)

In Eq. (20) we have explicitly drawn only those lines which are connected to the field $\varphi(\psi(x_i))$. M represents the n-point Green's function of the ψ-fields without external lines (the amputated Green's function). It is clear that one-particle singularities are possible only for diagrams of the first and second types in (20). The diagram of the third type does not exhibit one-particle poles (in the momentum p_i). Thus, on the mass shell the Green's function (18) [≡ (19)] can be replaced by the following:

$$G_\varphi^n(x_1, \ldots, x_n)|_{\text{M.o.}} = (1+\Gamma)^n \langle \psi(x_1) \ldots \psi(x_n) \rangle_1 = (1+\Gamma)^n G_\psi^n(x_1, \ldots, x_n). \tag{21}$$

It follows from (21) that if we are interested in the expression (13) only as a generating functional for Green's functions on the mass shell in all external lines, the source term may be replaced by

$$Z^{-1/2}\varphi(\psi)J \to Z^{-1/2}(1+\Gamma)\psi J = Z_1^{-1/2}\psi J. \tag{22}$$

Then (13) becomes exactly (11), which is the generating functional for the Green's functions of the field ψ.

Thus, we obtain that the scattering matrix elements of the renormalized fields in both theories, i.e., (1) and (9), coincide, and this is the content of the equivalence theorem.

We note that the proof given above for the equivalence theorem generalizes to the case of arbitrary canonical transformations of the fields φ and π. We indicate the conditions under which this proof would be rigorous, rather than only formal. Consider a theory describing a set of fields φ_i. Some components φ_i have indefinite metric (regulators), so that the theory is free of divergences. Assume that transformations of the type (10) do not violate this property of the theory, i.e., that the theory written in terms of the fields ψ_i also does not contain any divergences. Then as long as the regularization is not removed, the proof given above has a more than formal meaning. The removal of the regularization must be carried out for fixed renormalized charges and physical masses. It remains to be proved that the renormalized charges in both theories may be considered equal (for the masses we have proved it already).

Such conditions are realized in the theory of gauge fields, theory which will be considered in the following two sections.

3. THE GAUGE INVARIANCE OF A MASSLESS YANG-MILLS FIELD

Consider the theory of a massless Yang-Mills field, described by the Lagrangian

$$L = -\tfrac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + \text{reg}. \tag{23}$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \lambda f^{abc} A_\mu^b A_\nu^c, \tag{24}$$

where f^{abc} are the structure constants of the group G, the transformation group with coordinate-dependent parameters which leaves the Lagrangian (23) invariant. The infinitesimal form of such transformations is

$$A_\mu^a(x) \to A_\mu'^a(x) = A_\mu^a(x) + \nabla_\mu^{ab}(x)\xi^b(x), \tag{25}$$

where ∇_μ^{ab} is the covariant derivative:

$$\nabla_\mu^{ab}(x) = \delta^{ab}\partial_\mu + \lambda A_\mu^c(x) f^{acb}. \tag{26}$$

The generating functional for the theory of a massless Yang-Mills field is constructed according to the rules described in[2]

$$W = \int dA_\mu^a \exp\left\{ i \int dx \left(L + \frac{\alpha}{2} t^a t^a + A_\mu^a \bar{J}_\mu^a \right) + \operatorname{Sp} \ln D^{-1} \right\}$$
$$= \int dA_\mu^a \, d\psi_1 \, d\psi_1^+ \exp\left\{ i \int dx \left(L + \frac{\alpha}{2} t^a t^a + \psi_1^{+a} D_{ab}^{-1} \psi_1^b + A_\mu^a \bar{J}_\mu^a \right) \right\}. \tag{27}$$

In (27) ψ_1 is a scalar fermion field (fictitious particle), t^a is the function characterizing the supplementary condition, which is necessary in gauge-invariant theories for the construction of a quantum theory, the function D is related in a certain way to t^a [2]. We consider the following class of supplementary conditions:

$$t_{\beta,\gamma}^a = \beta \partial_\mu A_\mu^a + \gamma n_\mu \partial_\mu n_\nu A_\nu^a, \qquad n^2 = \pm 1, \tag{28}$$

where n_μ is a numerical vector. In this case the D-function is defined as follows:

$$[D_{\beta\gamma}]_{ab}^{-1} = (\beta\partial_\mu + \gamma n_\mu n_\nu \partial_\nu) \nabla_\mu^{ab}. \tag{29}$$

The class of gauges (28) contains all important cases: for $\gamma = 0$ we obtain a generalized Feynman gauge; for $\beta = 0$, $n_\mu = \delta_{\mu 3}$, $\gamma \to \infty$ one obtains the axial gauge $A_3^a = 0$; for $\beta = -\gamma = 1$, $n_\mu = \delta_{\mu 0}$, $\alpha \to \infty$ one obtains the Coulomb gauge $\partial_k A_k^a = 0$. $\bar{J}_\mu^a$ is a source only for the physical components of A_μ^a, which will be defined below. We prove that in the space of physical states the S-matrix does not depend on α, β, γ (and consequently not on the vector n_μ).

We consider the structure of the Green's function of the field A_μ^a:

$$\delta^{ab} D_{\mu\nu}(x-y) = \langle A_\mu^a(x) A_\nu^b(y) \rangle. \tag{30}$$

The tensor structure of $D_{\mu\gamma}$ must be composed of the following quantities:

$$g_{\mu\nu},\ \partial_\mu,\ n_\mu.$$

We separate from $D_{\mu\nu}$ a part corresponding to the tensor $Q_{\mu\nu}$:

$$D_{\mu\nu}(x) = Q_{\mu\nu} d(x) + D'_{\mu\nu}(x), \tag{31}$$

$$Q_{\mu\nu} = g_{\mu\nu} + (\Box n^2 - \hat{\partial}^2)^{-1} (n_\mu \partial_\nu \hat{\partial} + \partial_\mu n_\nu \hat{\partial} - n_\mu n_\nu \Box - \partial_\mu \partial_\nu n^2), \tag{32}$$

$$\hat{\partial} \equiv n_\mu \partial_\mu. \tag{33}$$

The kinematic structure of $D'_{\mu\nu}$ is constructed only from n_μ and ∂_μ. The tensor $Q_{\mu\nu}$ has the following properties:

$$n_\mu Q_{\mu\nu} = \partial_\mu Q_{\mu\nu} = 0, \quad Q_{\mu\nu} Q_{\nu\lambda} = Q_{\mu\lambda}. \tag{34}$$

Thus,

$$Q_{\mu\nu} D_{\nu\lambda}(x) = Q_{\mu\lambda} d(x). \tag{35}$$

In order to prove the gauge invariance of the S-matrix we carry out the following change of integration variables in the generating functional W (in the first line of the expression (27))[2]:

$$\begin{aligned} A_\mu^a(x) \to A_\mu^{\prime a}(x) &= A_\mu^a(x) + \nabla_\mu^{ab}(x) \xi^b(x) \\ &= A_\mu^a(x) + \int dy\, \nabla_\mu^{ab}(x) D_\beta^{bc}(x, y) \Lambda^c(y), \end{aligned} \tag{36}$$

$$\Lambda^c(y) = \left[\left(\frac{1}{2} \frac{\delta\alpha}{\alpha} \beta + \delta\beta \right) \partial_\mu + \frac{1}{2} \frac{\delta\alpha}{\alpha} n_\mu \hat{\partial} \right] A_\mu^c(y). \tag{37}$$

(Without loss of generality we set in the sequel $\gamma = 1$.)

Making use of the Jacobi identity for f^{abc} we obtain the following relation for the Jacobian of the transformation (up to inessential numerical factors) to first order in $\delta\alpha$ and $\delta\beta$:

$$\ln \frac{D(A')}{D(A)} \approx -\delta\, \mathrm{Sp} \ln D_\beta^{-1} + \mathrm{Sp}[\ln D_{\beta+\delta\beta}^{-1} - \ln D_\beta^{-1}]. \tag{38}$$

The first term in (38) is the variation of $\mathrm{Tr} \ln D^{-1}$ under the substitution (36). Since the transformation (36) has the structure of a gauge transformation, L remains invariant under such a substitution. In addition,

$$\frac{\alpha}{2} \int dx\, t_\beta^a t_\beta^a \to \frac{\bar{\alpha}}{2} \int dx\, t_{\bar{\beta}}^a t_{\bar{\beta}}^a, \quad \bar{\alpha} \equiv \alpha + \delta\alpha, \quad \bar{\beta} = \beta + \delta\beta. \tag{39}$$

The final expression for the generating functional transforms to the following form:

$$W_{\bar{\alpha}, \bar{\beta}} \equiv \int dA_\mu^a \exp \left\{ i \int dx \left(L + \frac{\bar{\alpha}}{2} t_{\bar{\beta}}^a t_{\bar{\beta}}^a + \bar{J}_\mu^a (A_\mu^a + \nabla_\mu^{ab} \xi^b) \right) + \mathrm{Sp} \ln D_{\bar{\beta}}^{-1} \right\}. \tag{40}$$

Considering J^a in (40) to be arbitrary we obtain the following transformation law for the Green's functions when the gauge is changed:

$$\begin{aligned} \langle A_\mu^a(x) A_\nu^b(y) \rangle_{\alpha, \beta} = \langle A_\mu^a(x) A_\nu^b(y) \rangle_{\bar{\alpha}, \bar{\beta}} - i \int dz \{ \langle (\partial_\mu \delta^{ac} + \lambda A_\mu^d(x) f^{adc}) \\ \times \psi_1^c(x) \psi_1^{+f}(z) \Lambda^f(z) A_\nu^b(y) \rangle_{\bar{\alpha}, \bar{\beta}} + (a, \mu, x \leftrightarrow b, \nu, y) \}. \end{aligned} \tag{41}$$

In (41) we have used the following relation for the Green's function of fictitious particles:

$$\langle \psi_1^a(x) \psi_1^{+b}(y) P(A) \rangle = i \langle D^{ab}(x, y) P(A) \rangle. \tag{42}$$

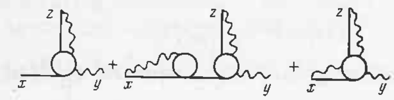
We introduce the notations:

$\langle A_\mu^a A_\nu^b \rangle =$

(43)

$\langle \psi_1^a \psi_1^{+b} \rangle =$

Then the second term in (41) can be represented in diagrammatic language as follows:



(44)

The first two terms in (44) have one-particle singularities (poles) in the x coordinate; the last term has no such singularities and can be written in the form

$$i \int dz\, [\Gamma_{\mu\lambda}^{ac}(x, z) \langle A_\lambda^c(z) A_\nu^b(y) \rangle_{\bar{\alpha}, \bar{\beta}} + (x, a, \mu \leftrightarrow y, b, \nu)], \tag{45}$$

where $\Gamma_{\mu\nu}^{ac}(x, z)$ has no one-particle singularities. From the structure of the expression (41) it follows that the first two terms in (44) have a tensor structure composed only of n_μ and ∂_μ. Therefore we get the following relation (in momentum space):

$$Q_{\mu\nu} d_{\alpha\beta}(p) = Q_{\mu\nu} d_{\bar{\alpha}, \bar{\beta}}(p)(1 + 2\Gamma(p)), \tag{46}$$

where Γ is the invariant function in the expansion

$$\Gamma_{\mu\nu}^{ab} = \delta^{ab}(Q_{\mu\nu}\Gamma + \Gamma'_{\mu\nu});$$

$\Gamma'_{\mu\nu}$ has a tensor structure composed only of n_μ and ∂_μ.

Since $\Gamma(p)$ has no one-particle poles, it follows from (46) that the position of the pole in the invariant amplitude $d_{\alpha,\beta}$ does not depend on α and β. Indeed, if the position of the pole in $d_{\alpha,\beta}$ would depend on the gauge (i.e., on α, β), then the left-hand side of (46) should have a pole of order two. Since in covariant gauges ($\alpha\beta^2 \to \alpha^1 = \mathrm{const.}$, $\alpha\beta \to 0$, $\alpha \to 0$) $D_{\mu\nu}$ has only a pole at the point $p^2 = 0$[8,9,10], we obtain the following important result: in an arbitrary gauge $d_{\alpha,\beta}$ has only one pole at the point $p^2 = 0$. It is also clear from (46) that the residues at these poles (wave function "renormalization constants") in different gauges are related by

$$Z(\mathbf{p})_{\alpha,\beta} = (1 + 2\Gamma(\mathbf{p})) Z(\mathbf{p})_{\bar{\alpha}, \bar{\beta}}, \tag{47}$$

$\Gamma(p)$ is the value of $\Gamma(p)$ at the point $p^2 = 0$.

We define the following components of the field A_μ^a as physical components (cf. also Appendix 1):

$$\bar{A}_\mu^a = Q_{\mu\nu} A_\nu^a. \tag{48}$$

then

$$\langle \bar{A}_\mu^c(x) \bar{A}_\nu^b(y) \rangle_{\alpha,\beta} = \delta^{cd} Q_{\mu\nu} d_{\alpha\beta}(x-y), \tag{49}$$

and the states described by the field $\overline{A_\mu^a}$ are massless particles analogous to photons. The states of the field $\overline{A_\mu^a}$ have two independent polarization vectors ξ_μ^i, $i = 1, 2$, with the properties

$$\xi_\mu^i(\mathbf{p}) \xi_\mu^j(\mathbf{p}) \quad -\delta_{ij}, \qquad p_\mu \xi_\mu^i = n_\mu \xi_\mu^i = 0, \qquad \xi_\mu^i \xi_\nu^i = -Q_{\mu\nu}. \tag{50}$$

These properties of ξ_μ^i allow one to rewrite the scattering matrix element for physical particles (renormalized with respect to the physical particles)

$$\begin{gathered} M_{ij\ldots}^{ab\ldots}(p, q, \ldots) = (Z(\mathbf{p}) Z(\mathbf{q}) \ldots)^{-1/2} (\xi_\mu^i(\mathbf{p}) \xi_\nu^j(\mathbf{p}) \ldots) \Gamma_{\mu\nu\ldots}^{ab\ldots}(p, q \ldots), \\ p^2 = q^2 = \ldots = 0, \end{gathered} \tag{51}$$

in the following way:

$$M_{ij\ldots}^{ab\ldots} = \xi_\mu^i \xi_\nu^j \ldots \bar{\Gamma}_{\mu\nu\ldots}^{ab\ldots}, \tag{51a}$$

where $\Gamma_{\mu\nu\ldots}^{ab\ldots}$ is the Green's function of the appropriate number of A_μ^a fields without external lines, and

$$\overline{\Gamma}^{ab\ldots}_{\mu\nu\ldots}(p, q, \ldots) \equiv (Z(\mathbf{p})Z(\mathbf{q})\ldots)^{-1/2}(Q_{\mu\mu'}Q_{\nu\nu'}\ldots)\Gamma^{ab\ldots}_{\mu'\nu'\ldots}(p, q, \ldots). \quad (52)$$

We define $\overline{J}_\mu^a$ in the form (27) in such a way, as to obtain the scattering matrix elements directly in the form (51a):

$$\overline{J}_\mu^a(x) = Z^{-1/2}(\mathbf{p})Q_{\mu\nu}J_\nu^a(x). \quad (53)$$

We define two other polarization vectors $\xi_\mu^{(1)i}(\mathbf{p})$:

$$\xi_\mu^{(1)i}(\mathbf{p}) = Q^{(1)}_{\mu\nu}\xi_\nu^i(\mathbf{p}),\ \xi_\mu^i = Q_{\mu\nu}\xi_\nu^{(1)i}, \quad (54)$$

where $Q^{(1)}_{\mu\nu}$ is a projection operator constructed according to Eq. (32), but with a different vector $n_\mu^{(1)}$. It is easy to check that the vectors $\xi_\mu^{(1)i}$ satisfy the relation (50) with the vector $n_\mu^{(1)}$.

From the second equation (54) and from (51a)–(53) it follows that in computing the scattering matrix elements (and also in the unitarity condition) one may use as polarization vectors the vectors ξ_μ^i corresponding to an arbitrary vector n_μ, rather than to the one which enters into the gauge condition. In particular, the most convenient are the polarization vectors e_μ^i of the Coulomb gauge, corresponding to the vector $n_\mu = \delta_{\mu 0}$ and having a direct physical interpretation. This result also shows that the physical states defined by us represent indeed massless particles of helicities ±1.

We consider in more detail the structure of the expression (40). An analysis completely analogous to the discussion of (41) shows that if we restrict ourselves to the calculation of the matrix elements for the scattering of physical particles (but do not consider Green's functions), we can make the following substitution in (40)

$$\int dx\overline{J}_\mu^a(A_\mu^a + \nabla_\mu^{ab}\xi^b) \to \int dx\, A_\mu^a(1 + \Gamma(\mathbf{p}))J_\mu^a \quad (55)$$

and making use of (47) we obtain

$$\int dx\overline{J}_\mu^a(A_\mu^a + \nabla_\mu^{ab}\xi^b) \to \int dx A_\mu^a Z^{-1/2}_{\overline{\alpha},\overline{\beta}}(\mathbf{p})\, Q_{\mu\nu}J_\nu^a. \quad (56)$$

The expression (56) represents a source term in the generating functional for the scattering matrix elements for (correctly normalized) physical particles in the gauge $\alpha + \delta\alpha, \beta + \delta\beta$. Thus, it follows from (40) that the renormalized[4)] S-matrix in the subspace of physical states does not depend on the gauge α, β.

In the transition to the covariant gauge $\alpha\beta^2 \to \alpha^1$ = const., $\alpha \to 0, \alpha\beta \to 0$ the scattering matrix element (which does not depend on α, β) takes on the form

$$M^{ab\ldots}_{ij\ldots} = Z_{\alpha'}^{-n/2}(e_\mu^i Q_{\mu\mu'}e_\nu^j Q_{\nu\nu'}\ldots)\Gamma^{ab\ldots}_{\mu'\nu'\ldots;\alpha'}, \quad (57)$$

where $\Gamma^{ab\ldots}_{\mu'\nu'\ldots;\alpha'}$ are calculated in the covariant gauge α', and the projection operators $Q_{\mu\nu}$ have survived, since they do not depend on α and β. However, the transversality properties of the vectors e_μ^i and of the functions $\Gamma^{ab}_{\mu\nu\ldots;\alpha'}$ [9,10] it follows that in the expression (57) the tensors Q can be omitted, so that in covariant gauges the matrix element has the usual form

$$M^{ab\ldots}_{ij\ldots} = Z_{\alpha'}^{-n/2}(e_\mu^i e_\nu^j\ldots)\Gamma^{ab\ldots}_{\mu\nu\ldots;\alpha'}. \quad (58)$$

The results obtained above reduce to the following: in an arbitrary gauge (28) we have defined the physical states (cf. also Appendix 1); for $\gamma \neq 0$ an arbitrary vector n_μ can be used for the construction of the physical states; it was shown that the mass of the physical states does not depend on the gauge. In particular, the S-matrices in the Coulomb, the axial and the generalized Feynman gauges all coincide.

The proof proposed above was in fact obtained for a fixed bare charge λ. Since, however, the theory involves charge divergences, it is necessary to keep the renormalized charge λ_R constant. We prove that the renormalized charge can be considered gauge-independent. In addition, we obtain a relation analogous to the one derived by Taylor[8] between the renormalization constants of the fermion and the vector particle only for the Lorentz gauge, but valid in any gauge for a renormalizable theory. We restrict our attention to covariant gauges $\beta = 1, \gamma = 0$ and give no details of the calculations.

We add to L in (23) a term describing the fermions,

$$\overline{\psi}\left(i\gamma_\mu\left(\delta^{ab}\partial_\mu - i\frac{\lambda}{2}\tau_c^{ab}A_\mu^c\right) - m\delta^{ab}\right)\psi^b \quad (59)$$

and introduce into the generating functional a fermion source term

$$W_\alpha = \int dA_\mu^a d\psi\, d\overline{\psi}\exp\left\{i\int dx\left(L + \frac{\alpha}{2}(\partial_\mu A_\mu^a)^2 + A_\mu J_\mu + \overline{\psi}\eta + \overline{\eta}\psi\right) + \mathrm{Sp}\ln D^{-1}\right\}. \quad (60)$$

The D-function is given by (29) with $\beta = 1, \gamma = 0$.

In order to exhibit the dependence on the gauge of the ψ-field propagator and of the vertex functions we change the integration variables in (60):

$$A_\mu^a \to A_\mu'^a = A_\mu^a + \nabla_\mu^{ab}\xi^b,\ \psi^a \to \psi'^a = \psi^a + i\lambda\xi^c\tau_c^{ab}\psi^b, \quad (61)$$

$$\overline{\psi}^a \to \overline{\psi}'^a = \overline{\psi}^a - i\lambda\xi^c\overline{\psi}^b\tau_c^{ba},$$

$$\xi^a(x) = \frac{\delta\alpha}{2\alpha}\int dz D^{ab}(x, z)\,\partial_\mu A_\mu(z), \quad (62)$$

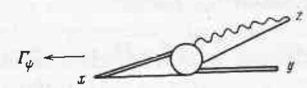
with the result

$$W_\alpha \equiv \int dA_\mu^a d\psi\, d\overline{\psi}\exp\left\{i\int dx\left(L + \frac{\overline{\alpha}}{2}(\partial_\mu A_\mu^a)^2 + J_\mu^a(A_\mu^a + \nabla_\mu^{ab}\xi^b) + \overline{\eta}^a(\psi^a + i\lambda\xi^c\tau_c^{ab}\psi^b) + (\overline{\psi}^a - i\lambda\xi^c\overline{\psi}^b\tau_c^{ba})\eta^a\right) + \mathrm{Sp}\ln D^{-1}\right\}. \quad (63)$$

In complete analogy to the case of the propagator of the vector particle we find from (63) that the renormalized fermion mass is gauge-independent and that the renormalization constants are related by an equation analogous to (47):

$$Z_{\psi,\alpha} = Z_{\psi,\overline{\alpha}}(1 + 2\delta\alpha\Gamma_\psi), \quad (64)$$

where Γ_ψ stems from the diagram (without the right-hand fermion line, which is denoted by two parallel lines)



(65)

The relation (47) can also be rewritten (in the class of gauges under consideration) in the form

$$Z_{A,\alpha} = Z_{A,\overline{\alpha}}(1 + 2\delta\alpha\Gamma_A). \quad (66)$$

Considering the expressions for the three-point functions $\langle AAA\rangle$ and $\langle A\psi\overline{\psi}\rangle$ which follow from (63), and normalizing the corresponding renormalized vertex functions (with all external lines on the mass-shell) in the same manner in all gauges, we obtain

$$Z_\alpha^{A^3} = Z_{\overline{\alpha}}^{A^3}(1 + 3\delta\alpha\Gamma_A), \quad (67)$$

$$Z_\alpha^{A\psi\overline{\psi}} = Z_{\overline{\alpha}}^{A\psi\overline{\psi}}(1 + \delta\alpha\Gamma_A)(1 + 2\delta\alpha\Gamma_\psi). \quad (68)$$

It follows from (64)–(68) that

$$\lambda_{R,\alpha} \equiv \lambda \frac{Z_{A,\alpha}^{3/2}}{Z_\alpha^{A^3}} = \lambda \left[\frac{Z_{A,\alpha}}{1+2\delta\alpha\Gamma_A} \right]^{3/2} \frac{1+3\delta\alpha\Gamma_A}{Z_\alpha^{A^3}} = \lambda_{R,\bar\alpha}, \tag{69}$$

$$\frac{Z_{A,\alpha}}{Z_\alpha^{A^3}} \frac{Z_\alpha^{A\psi\bar\psi}}{Z_{\psi,\alpha}} = \frac{Z_{A,\bar\alpha}}{Z_{\bar\alpha}^{A^3}} \frac{Z_{\bar\alpha}^{A\psi\bar\psi}}{Z_{\psi,\bar\alpha}}. \tag{70}$$

Equation (70) signifies that the ratio in it does not depend on the gauge. Taylor[8] has proved that in the transverse gauge it equals one. Therefore in an arbitrary gauge

$$\frac{Z_{A,\alpha}}{Z_\alpha^{A^3}} = \frac{Z_{\psi,\alpha}}{Z_\alpha^{A\psi\bar\psi}} \tag{71}$$

Equation (69) signifies that the renormalized coupling constant corresponding to the 3A vertex does not depend on the gauge (if all renormalized vertex functions are normalized in the same way). Similarly, one could verify that the renormalized coupling constants corresponding to all other vertices do not depend on the gauge. We shall not do this explicitly, since the results of Slavnov[9] and Taylor[8] as well as (71) imply that they are all equal to λ_R.

In conclusion of this section we wish to note that the application of this method, or a simple consideration of the free propagators, shows that the position of the poles of the Green's functions of the unphysical components of the field A_μ and of the fictitious particles depend, generally speaking, on the gauge.

4. THE GAUGE INVARIANCE OF A THEORY OF A YANG-MILLS FIELD WITH SPONTANEOUS SYMMETRY BREAKDOWN

We shall consider a gauge invariant model of a Yang-Mills field, interacting with a scalar isospinor field φ_i in such a manner that as a result of the spontaneous symmetry breakdown by the scalar field, the vector field acquires a mass[3]. The model is described by the following Lagrangian (for the SU(2) group):

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + \left(\partial_\mu \delta^{ij} + i\frac{\lambda}{2} A_\mu^c \tau_c^{ki}\right) \varphi_k^\pm \left(\partial_\mu \delta^{il} - i\frac{\lambda}{2} A_\mu^{c'} \tau_{c'}^{il}\right) \varphi_l$$
$$- g^2\left(\varphi_i^+ \varphi_i - \frac{a^2}{2}\right) + \text{reg.}, \tag{72}$$

and the field φ suffers spontaneous symmetry breakdown

$$\langle 0|\varphi|0\rangle = b \neq 0. \tag{73}$$

Without loss of generality the constant b can be chosen real and having only a nonzero upper (isospin) component. We rewrite the field φ in the form

$$\varphi = \begin{pmatrix} b+\sigma+iB_3 \\ iB_1 - B_2 \end{pmatrix} \frac{1}{\sqrt{2}}, \quad \sigma + b \equiv \bar\sigma, \tag{74}$$

so that the vacuum expectation values of the fields σ and B_a vanish

$$\langle 0|\sigma|0\rangle = \langle 0|B_a|0\rangle = 0. \tag{75}$$

We do not write the expression of the Lagrangian (72) in terms of the fields σ and B_a. The corresponding expression, as well as the scheme of perturbation theory are given in[3].

For us it is important only that the Lagrangian (72) is invariant with respect to the following gauge transformation:

$$A_\mu^a \to A_\mu^a + \nabla_\mu^{ab}\xi^b, \quad \bar\sigma \to \bar\sigma - \frac{\lambda}{2}\xi^a B_a,$$
$$B_a \to B_a + \frac{\lambda}{2}\bar\sigma\xi^a - \frac{\lambda}{2}\varepsilon^{abc}\xi^b B_c; \tag{76}$$

here ϵ^{abc} is the completely antisymmetric tensor of rank three, which also represents the structure constants of the group SU(2). The transformations (76) for σ and B_a follow from the isospin transformation properties of φ_i.

The generating functional of the model can be constructed according to the rules given in[2]:

$$W = \int dA_\mu^a d\sigma dB_a \exp\left\{ i \int dx \left(L + \frac{\alpha}{2} t^a t^a + J_\mu^a A_\mu^a \sigma\eta + \theta^a B_a \right) + \operatorname{Sp}\ln D^{-1} \right\}. \tag{77}$$

We restrict our attention to covariant gauges

$$t^a = \partial_\mu A_\mu^a + \beta B_a. \tag{78}$$

The corresponding D-function is

$$[D_\beta]_{ab}^{-1} = \partial_\mu \nabla_\mu^{ab} + \beta\left(M_{0;\alpha,\beta} + \frac{\lambda}{2}\sigma\right)\delta^{ab} + \frac{\lambda\beta}{2}\varepsilon^{acb}B_c. \tag{79}$$

In the sequel the indices α and β will indicate that the corresponding quantities are taken or are calculated in the α, β-gauge. Here $M_{0;\alpha,\beta} \equiv (\lambda/2) b_{\alpha,\beta}$ is the bare mass of the vector field.

The class (78) of gauges comprises several interesting cases: for $\beta = 0$, $\alpha \to \infty$ we obtain the transverse gauge, for $\beta = \gamma\alpha^{-1/2}$, 0, $\gamma \to \infty$ we obtain the canonical gauge $B_a = 0$; for $\alpha\beta = -M_0$ we obtain a gauge in which the bare propagator $\langle A_\mu^a B_b \rangle$ vanishes; if in addition $\alpha = -1$, the bare propagator of the vector field is diagonal.

Let us find the connection between Green's functions in different gauges. For this purpose we carry out in the functional integral (77) the substitution (76), where

$$\xi^a(x) = \int dz D_\beta^{ab}(x,z)\Lambda^b(z), \tag{80}$$

$$\Lambda^b(z) = -\frac{1}{2}\frac{\delta\alpha}{\alpha}\partial_\mu A_\mu^b(z) + \left(\frac{1}{2}\frac{\delta\alpha}{\alpha}\beta + \delta\beta\right) B_b(z). \tag{81}$$

A direct calculation of the Jacobian of the transformation yields

$$\ln\frac{D(A', B', \sigma')}{D(A, B, \sigma)} = -\delta \operatorname{Sp}\ln D_\beta^{-1} + \operatorname{Sp}[\ln D_{\bar\beta}^{-1} - \ln D_\beta^{-1}], \tag{82}$$
$$\bar\alpha = \alpha + \delta\alpha, \quad \bar\beta = \beta + \delta\beta.$$

As a result of this the expression for the generating functional takes the form

$$W_{\alpha,\beta} \equiv \int dA_\mu^a dB_a d\sigma \exp\left\{ i\int dx \left(L + \frac{\bar\alpha}{2} t_{\bar\beta}^a t_{\bar\beta}^a + J_\mu^a (A_\mu^a + \nabla_\mu^{ab}\xi^b) + Q\right) + \operatorname{Sp}\ln D_{\bar\beta}^{-1} \right\}; \tag{83}$$

here Q is the source-term for the fields σ and B_a. Since all vacuum expectation values will be computed by means of (83) in the gauge $\bar\alpha, \bar\beta$, we separate from the field $\bar\sigma$ the field for which the vacuum expectation value vanishes in the gauge $\bar\alpha, \bar\beta$ (and we denote it again by σ). Then Q takes the form

$$Q = \eta\left(b_{\bar\alpha,\bar\beta} - b_{\alpha,\beta} + \sigma - \frac{\lambda}{2}B_a\xi^a\right)$$
$$+ \theta^a\left(B_a + \frac{\lambda b_{\bar\alpha,\bar\beta}}{2}\xi^a + \frac{\lambda}{2}\sigma\xi^a - \frac{\lambda}{2}\varepsilon^{abc}\xi^b B_c\right). \tag{84}$$

Since $W_{\alpha,\beta}$ does not change when the integration variables are changed the Green's functions can be expressed in terms of (83) by means of the corresponding number of functional derivatives with respect to the sources, as well as in terms of (77).

A.

$$0 = \frac{\delta}{\delta\eta(x)} W_{\alpha,\beta}\Big|_{J_\mu^a = \eta = \theta_a = 0} = -ib_{\alpha,\beta} + \int dz \langle B_a(x)\psi_1^a(x)\psi_1^{+b}(z)\Lambda^b(z)\rangle_{\bar\alpha,\bar\beta}.$$

This relation shows that $\langle 0 | \varphi | 0 \rangle$ does depend on the gauge.

B. $$\langle \sigma(x)\sigma(y)\rangle_{\alpha,\beta} = \langle \sigma(x)\sigma(y)\rangle_{\bar\alpha,\bar\beta} + i\frac{\lambda}{2}\int dz\,\{\langle \sigma(x)\psi_1^a(y)B_a(y) \times \psi_1^{+b}(z)\Lambda^b(z)\rangle_{\bar\alpha,\bar\beta} + (x \leftrightarrow y)\}. \tag{85}$$

We see that in the right-hand side of Eq. (85) there is only the pole related to the σ-particle. The poles related to ψ_1^a, B_a and $\partial_\mu A_\mu^a$ are absent, owing to the structure of the internal space, We obtain that the right-hand side of (85) has only one first-order pole and that the mass of the σ-particle is gauge-independent (since there are no second order poles). In different gauges the renormalization constants of the σ-field are related by

$$Z_{\sigma;\alpha,\beta} = Z_{\sigma;\bar\alpha,\bar\beta}(1 + 2\Gamma_\sigma), \tag{86}$$

where $Z_{\sigma;\overline{\alpha,\beta}}\Gamma_\sigma$ is the residue at the σ-pole in the second term of (85).

C. $$\langle A_\mu^a(x)A_\nu^b(y)\rangle_{\alpha,\beta} = \langle A_\mu^a(x)A_\nu^b(y)\rangle_{\bar\alpha,\bar\beta} - i\int dz\,\{\langle \nabla_\mu^{ac}(x)\psi_1^c(x)\psi_1^{+f}(z)\Lambda^f(z) \times A_\nu^b(y)\rangle_{\bar\alpha,\bar\beta} + (x, a, \mu \leftrightarrow y, b, \nu)\}. \tag{87}$$

The second term of (87) has one-particle poles for A_μ^a and ψ_1^a. It is easy to check that the pole for ψ_1^a appears in the longitudinal part of the tensor structure and does not occur in the transverse part. If we introduce the expansion

$$\delta^{ab}D_{\mu\nu;\alpha,\beta} \equiv \langle A_\mu A_\nu^b\rangle_{\alpha,\beta} = \delta^{ab}[P_{\mu\nu}d_{\alpha,\beta}^t + L_{\mu\nu}d_{\alpha,\beta}^l], \tag{88}$$
$$P_{\mu\nu} = g_{\mu\nu} - \partial_\mu\partial_\nu/\Box, \quad L_{\mu\nu} = \partial_\mu\partial_\nu/\Box;$$

we obtain the following relation (in momentum space)

$$d_{\alpha,\beta}^t(p) = d_{\bar\alpha,\bar\beta}^t(p)(1 + 2\Gamma_A(p)), \tag{89}$$

where $\Gamma_A(p)$ has no poles. The structure of $\Gamma_A(p)$ is clear from (87).

Thus, we find that the mass of the transverse part of the field A_μ^a is gauge-independent and that the renormalization constants of the transverse parts of the field A_μ^a in different gauges are related by

$$Z_{A;\alpha,\beta} = Z_{A;\bar\alpha,\bar\beta}(1 + 2\Gamma_A), \tag{90}$$

where Γ_A is the value of $\Gamma_A(p)$ at the pole of the vector particle.

D. We have seen that the longitudinal part of the right-hand side of (87) contains the poles of the longitudinal parts of A_μ^a and ψ_1^a. It follows from the Ward identity[5]

$$\alpha\langle t_\mu^a(x)A_\nu^b(y)\rangle_{\alpha,\beta} + \langle \nabla_\nu^{bd}(y)\psi_1^d(y)\psi_1^{+a}(x)\rangle_{\alpha,\beta} = 0 \tag{91}$$

that the poles of the longitudinal parts of A_μ^a and ψ_1^a coincide. Thus, the longitudinal tensor structure of the right-hand side of (87) has, in the general case, a pole of higher order than the longitudinal tensor structure of the left-hand side of (87); this means that the position of the pole of the longitudinal component of the field A_μ^a (as well as of ψ_1^a) depends on the gauge.

A similar consideration shows that the position of the pole of the B_a-field also depends on the gauge.

These results are confirmed by a calculation in the zeroth approximation of the positions of the poles for the propagators of the longitudinal parts of the fields A_μ^a, B_a and ψ_1^a, which are all situated at the point $\beta\lambda b_{\alpha,\beta}/2$ and explicitly depend on the gauge (i.e., on β), since to lowest order $b_{\alpha,\beta}$ does not depend on α or β.

E. We make in the generating functional (77) and (83) the substitution

$$J_\mu^a \to Z_{A;\alpha\beta}^{-1/2}J_\mu^{a\perp}(\overrightarrow{\Box} + M_A^2), \quad (\Box + M_A^2)J_\mu^{a\perp} = \partial_\mu J_\mu^{a\perp} = 0, \tag{92}$$
$$\eta \to Z_{\sigma;\alpha,\beta}^{-1/2}\eta(\overrightarrow{\Box} + M_\sigma^2), \quad (\Box + M_\sigma^2)\eta = 0, \tag{93}$$
$$\theta^a \to 0. \tag{94}$$

Here M_σ and M_A are the physical masses of the field σ and of the transverse part of A_μ^a (which are gauge-independent). Then $W_{\alpha,\beta}$ becomes the generating functional of the renormalized matrix elements for the scattering of transverse massive vector mesons and of the field σ.

A discussion analogous to that carried out in subsections B and C shows that after the substitution (92)–(94) the source-term in (83) becomes (taking into account (86) and (90))

$$\int dx\,[Z_{A;\alpha,\beta}^{-1/2}(1 + \Gamma_A)J_\mu^{a\perp}(\overrightarrow{\Box} + M_A^2)A_\mu^a + Z_{\sigma;\alpha,\beta}^{-1/2}(1 + \Gamma_\sigma)\eta(\overrightarrow{\Box} + M_\sigma^2)\sigma]$$
$$= \int dx\,[Z_{A;\bar\alpha,\bar\beta}^{-1/2}J_\mu^{a\perp}(\overrightarrow{\Box} + M_A^2)A_\mu^a + Z_{\sigma;\bar\alpha,\bar\beta}^{-1/2}\eta(\overrightarrow{\Box} + M_\sigma^2)\sigma]. \tag{95}$$

Thus, the right-hand side of (83) becomes the generating functional of the renormalized matrix elements for the scattering of the fields σ and of the transverse part of A_μ^a in the gauge $\bar\alpha, \bar\beta$. This means that the renormalized S-matrix does not depend on the gauge α, β in the subspace of physical states of the massive vector field and of the massive scalar particle. Since in the gauge where $B_a = 0$ ($\beta = \alpha^{-1/2}\gamma$; $\alpha \to 0$, $\gamma \to \infty$) there are only physical states present (at least, within the framework of perturbation theory), and the S-matrix is unitary, the results obtained show that the S-matrix is unitary in the physical subspace in any gauge α, β.

F. The preceding discussion was in fact valid for the case when at least one of the bare charges λ or g was fixed (the theory is characterized by three parameters; M_A and M_σ can be chosen as two of these). Since the theory has charge divergences, we have to fix the renormalized charges. We show that the renormalized charge λ_R may be considered to be gauge-independent. Indeed, studying the relation between the vertex functions in different gauges by means of (83), we obtain

$$Z_{\alpha,\beta}^{A^3} = Z_{\bar\alpha,\bar\beta}^{A^3}(1 + 3\Gamma_A), \tag{96}$$

where the normalization of $\Gamma_{\mu\nu\lambda}^{A^3}$ is carried out on the mass shell of the transverse part of A_μ^a in all three external lines. It follows from (96) and (90) that

$$\lambda_{R;\alpha,\beta} \equiv \lambda\frac{Z_{A;\alpha,\beta}^{3/2}}{Z_{\alpha,\beta}^{A^3}} = \lambda\frac{Z_{A;\bar\alpha,\bar\beta}^{3/2}}{Z_{\bar\alpha,\bar\beta}^{A^3}} \equiv \lambda_{R;\bar\alpha,\bar\beta}. \tag{97}$$

One can also check that the constants g_R and $b_R \equiv b_{\alpha,\beta}Z_{\sigma;\alpha,\beta}^{-1/2}$ do not depend on the gauge (and that b_R) is finite, as was done in a paper by Fradkin and one of the present authors[5] for the case of a neutral gauge field. However, this requires a study of the relations between different renormalization constants and will not be done here. In any case, we already have the necessary number of finite parameters which completely characterize the theory and do not depend on the gauge.

5. CONCLUSION

The proof of the equivalence theorem for non-Abelian gauge theories is at the same time a proof of the unitarity of the theory. In distinction from the Abelian versions of gauge theory[5], a direct proof of unitarity in "renormalizable" gauges become difficult. The Hamiltonian is non-Hermitian in all gauges, except the Lorentz gauge. Therefore the use of the Cutkosky rule in the proof of compensation of fictitious particles and unphysical states of polarization of the vector particles[10,11] is justified only in the Lorentz gauge. But even in this case the proof becomes cumbersome if the number of intermediate states is larger than three.

When the present paper was finished, we became aware of the paper[12], where the equivalence theorem is formulated, but not proved. In particular no relation is obtained there between the wave function renormalization constants in different gauges, and no proof is given of the gauge invariance of the masses of physical particles and of the renormalized coupling constants.

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After this paper was submitted, we have received the preprint of 'tHooft and Veltman[13]. In that paper the equivalence theorem for renormalizable Yang-Mills theories was also proved.

APPENDIX 1

We carry out the quantization of the free electromagnetic field A_μ^a in the gauge (28). We shall see that the definition of the physical components of the massless vector field given in Sec. 3 (Eq. (48)), also follows from the discussion of the free field.

The Lagrangian and the equations of motion of the free field in the gauge (28) have the form

$$L = -\frac{1}{4} F_{\mu\nu}^2 + \frac{\alpha}{2} [\beta \partial_\mu A_\mu + \hat{\partial} n_\mu A_\mu]^2, \tag{A.1}$$

$$\Lambda_{\mu\nu} A_\nu = 0, \tag{A.2}$$

$$\Lambda_{\mu\nu} = \Box g_{\mu\nu} - \partial_\mu \partial_\nu (1 + \alpha\beta^2) - \alpha n_\mu n_\nu \hat{\partial}^2 - \alpha\beta\hat{\partial}(n_\mu \partial_\nu + n_\nu \partial_\mu), \tag{A.3}$$

$$\det \Lambda_{\mu\nu} = \alpha \Box^2 (\beta \Box + \hat{\partial}^2)^2. \tag{A.4}$$

It is clear from (A.4) that the Fourier transform $A_\mu(k)$ has support (is concentrated) at the points $k^2 = 0$ and $\beta k^2 + \hat{k}^2 = 0$ and can be expanded in terms of $\delta(k^2)$ and $\delta(\beta k^2 + \hat{k}^2)$ and their first derivatives.

We introduce four linearly independent polarization vectors

$$n_\mu,\ k_\mu,\ \xi_\mu^i(k),\ i = 1, 2, \tag{A.5}$$

$$\xi_\mu^i(k)\, \xi_\mu^j(k) = -\delta_{ij},\quad k_\mu \xi_\mu^i(k) = n_\mu \xi_\mu^i(k) = 0. \tag{A.6}$$

Writing down the most general form of the expansion of A in terms of the polarization vectors and delta-functions and their derivatives, and substituting this expansion into the equation of motion (A.2), we obtain[6)]

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^{3/2}} \sqrt{2|\mathbf{k}|}\, \theta(k_0)\, e^{-ikx} \Big\{ \xi_\mu^i(k)\, a_k^i\, \delta(k^2) + n_\mu a_k^{(3)} \delta(\bar{k}^2) + k_\mu a_k^{(4)} \delta(\bar{k}^2) + k_\mu \frac{1 + \alpha\beta^2 + \alpha\beta n^2}{\alpha\beta^2} a_k^{(3)} \delta'(\bar{k}^2) \Big\} \tag{A.7}$$

$$+\text{h.c.},\ \bar{k}^2 = k^2 + \frac{1}{\beta} \hat{k}^2;$$

$a_{\mathbf{k}}^i$, $a_{\mathbf{k}}^{(3)}$ and $a_{\mathbf{k}}^{(4)}$ are arbitrary functions of the 3-momentum $\mathbf{k}$. In quantum theory one must, of course, consider them as operators.

Comparing with the canonical commutation relations which follow from (A.1), we have

$$[a_k^i, a_q^{+j}] = \delta_{ij} \delta(\mathbf{k} - \mathbf{q}). \tag{A.8}$$

It is clear from (A.7) that it is natural to consider as physical the states corresponding to $a_{\mathbf{k}}^i$ and as unphysical, the states corresponding to $a_{\mathbf{k}}^{(3)}$, $a_{\mathbf{k}}^{(4)}$. The spectrum of the latter depends on the gauge $a_{\mathbf{k}}^{(4)}$ is related to purely logitudinal states, and the presence of δ' in front of $a_{\mathbf{k}}^{(3)}$ bears witness of the indefinite metric (in fact one can show that $[a_{\mathbf{k}}^{(3)}, a_{\mathbf{k}}^{(3)+}] = 0$).

The gauge-dependence of the states created by the operators $a_{\mathbf{k}}^{+i}$ stems only from the dependence of the polarization vectors $\xi_\mu^i(\mathbf{k})$ on the vector n_μ. From (54) we obtain the following relation between the polarization vectors for two different vectors n_μ and $n_\mu^{(1)}$:

$$\xi_\mu^{(1)i}(k) = \xi_\mu^i(k) - k_\mu \frac{1}{k_\lambda n_\lambda^{(1)}} n_\nu^{(1)} \xi_\nu^i(k). \tag{A.9}$$

Thus, the difference between the two polarization vectors for different n_μ is proportional to k_μ and, as expected, such terms should not contribute to the physical quantities. This fact was proved in Section 3.

The free propagator $D_{\mu\nu}^{(0)}$ of the field A_μ (defined as $\Lambda_{\mu\nu}^{-1}$) is

$$iD_{\mu\nu}^{(0)}(k) = Q_{\mu\nu} \frac{1}{k^2} - (n_\mu k_\nu + k_\mu n_\nu) \frac{\hat{k}(\beta + n^2)}{\beta (k^2 n^2 - \hat{k}^2) \bar{k}^2} + n_\mu n_\nu \frac{1}{k^2 n^2 - \hat{k}^2} + k_\mu k_\nu \frac{1}{\beta^2 (\bar{k}^2)^2} \left(\frac{\hat{k}^2 (\beta + n^2)^2}{k^2 n^2 - k^2} - \frac{1}{\alpha} \right) \tag{A.10}$$

$$= \left(g_{\mu\nu} - \frac{(k_\mu n_\nu + n_\mu k_\nu)\hat{k}}{\beta \bar{k}^2} + \frac{k_\mu k_\nu (n^2 \hat{k}^2 - \beta^2 k^2)}{\beta^2 (\bar{k}^2)^2} \right) \frac{1}{k^2} - \frac{1}{\alpha} k_\mu k_\nu \frac{1}{\beta^2 (\bar{k}^2)^2}.$$

APPENDIX 2

We consider here the canonical quantization of a massless Yang-Mills theory with the regularization of[6]

$$L = -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a - \frac{1}{4\Lambda^2} \nabla_\mu^{ab} G_{\nu\lambda}^b \nabla_\mu^{ac} G_{\nu\lambda}^c. \tag{A.11}$$

We show that the canonical quantization of the theory (A.11) in the axial gauge leads to Feynman rules corresponding to

$$W = \int dA_\mu^a \delta(A_3^a) \exp\left\{ i \int dx (L + J_\mu^a A_\mu^a) \right\}, \tag{A.12}$$

i.e., the introduction of the regularization does not change the form of the generating functional. The Lagrangian (A.12) contains higher-order derivatives, therefore the quantization is carried out by means of the Ostrogradsky method.

In the axial gauge the Hamiltonian has the form

$$H(P_{2i}^a, Q_{2i}^a;\ P_{10}^a, Q_{10}^a;\ P_{1i}^a, Q_{1i}^a) = P_{2i}^a \dot{Q}_{2i}^a + P_{10}^a \dot{Q}_{10}^a + P_{1i}^a \dot{Q}_{1i}^a - L, \tag{A.13}$$

where

$$P_{2i}^a = \frac{\delta L}{\delta \ddot{A}_i^a},\quad P_{10}^a = \frac{\delta L}{\delta \dot{A}_0^a},\quad P_{1i}^a = \frac{\delta L}{\delta \dot{A}_i^a} - \partial_0 \frac{\delta L}{\delta \ddot{A}_i^a},$$

$$Q_{2i}^a = \dot{A}_i^a,\quad Q_{10}^a = A_0^a,\quad Q_{1i}^a = A_i^a,\quad i = 1, 2. \tag{A.14}$$

The generating functional of the canonical theory has the form

$$W = \int dP\, dQ \exp\left\{ i \int dx (P_{1i}^a \dot{Q}_{1i}^a + P_{10}^a \dot{Q}_{10}^a + P_{2i}^a \dot{Q}_{2i}^a - H + J_0^a Q_{10}^a - J_i^a Q_{1i}^a) \right\}. \tag{A.15}$$

In order to arrive at the form (A.12) one must carry out the integrations with respect to P_{1i}^a, P_{2i}^a, P_{10}^a, Q_{2i}^a:

$$P_{2i}^a = \frac{1}{\Lambda^2} \nabla_0^{ab} G_{0i}^b, \tag{A.16}$$

$$\ddot{A}_i^a = -\Lambda^2 P_{2i}^a + \nabla_i^{ab} \dot{A}_0^b + b_i^a,$$

where

$$b_i = \lambda f^{abc} A_i^b A_0^c - \lambda f^{abc} A_0^b G_{0i}^c,$$

$$P_{10}^a = -\frac{1}{\Lambda^2} (\nabla_i^{ab} \nabla_0^{bc} G_{0i}^c + \nabla_3^{ab} \nabla_0^{bc} G_{03}^c)$$

$$= -\frac{1}{\Lambda^2}\nabla_3^{ab}\nabla_0^{bc}G_{03}^c - \nabla_i^{ab}P_{2i}^b, \quad \dot{A}_0^a = -\Lambda^2\partial_3^{-2}(P_{10}^a + \nabla_i^{ab}P_{2i}^b) + a^a, \tag{A.17}$$

$$a^a = \lambda f^{abc}\partial_3^{-1}(A_0^b\partial_3 A_0^c),$$

$$H = P_{1i}^a Q_{2i}^a + \frac{1}{2}\Lambda^2(\partial_3^{-1}(P_{10}^a + \nabla_i^{ab}P_{2i}^b))^2 + a^a(P_{10}^a + \nabla_i^{ab}P_{2i}^b)$$

$$-\frac{1}{2}\Lambda^2(P_{2i}^a)^2 + P_{2i}^a b_i^a + f(Q_{1i}, Q_{2i}, Q_{10}). \tag{A.18}$$

In Eq. (A.15) the integration with respect to P_{1i}^a yields $\delta(\dot{Q}_{1i} - Q_{2i})$, i.e., one can do away with the integration with respect to Q_{2i}.

For the integration with respect to P_{10} we effect the shift

$$P_{10}^a = P_{10}^{\prime a} - \nabla_i^{ab}P_{2i}^b. \tag{A.19}$$

The corresponding integral equals

$$\int dP_{10}' \exp\left\{i\int dx\left(P_{10}^{\prime a}(\dot{A}_0^a - a^a) + \frac{1}{2}\Lambda^2 P_{10}^{\prime a}\partial_3^{-2}P_{10}^{\prime a} + \ldots\right)\right\}$$

$$= \exp\left\{i\int dx\left(\frac{1}{2\Lambda^2}[\partial_3(\dot{A}_0^a - a^a)]^2 + \ldots\right)\right\}. \tag{A.20}$$

One may neglect $\mathrm{Tr}\ln(\Lambda^2\partial_3^{-2})$ in taking the integral in (A.20), since there is no functional dependence on the fields.

The last integration, with respect to P_{2i}, yields

$$\int dP_{2i}\exp\left\{i\int dx\left(-\frac{1}{2}\Lambda^2(P_{2i}^a)^2 + (\ddot{Q}_{1i}^a - \nabla_i^{ab}Q_{10}^b - b_i^a)P_{2i}^a\right)\right\}$$

$$= \exp\left\{-i\int dx\frac{1}{2\Lambda^2}(\ddot{Q}_{1i}^a - \nabla_i^{ab}Q_{10}^b - b_i^a)^2\right\}. \tag{A.21}$$

Taking into account the results of the integrations in (A.20) and (A.21), it is easy to check that the canonical form of the generating functional (A.15) was reduced to the form (A.12).

A completely analogous consideration is applicable to a theory with spontaneous symmetry breakdown, taking into account the regularization.

In the case of a nonsingular gauge condition an improvement of the asymptotic behavior of the longitudinal part of the propagator of the field A_μ^a is achieved by means of the substitution $\alpha \to \alpha(\Box)$. Then the addition to the Lagrangian should be interpreted in the following manner:

$$\frac{\alpha}{2}\int dx t^a(x)t^a(x) \to \frac{1}{2}\int dx t^a(x)\alpha(\Box)t^a(x). \tag{A.22}$$

The modification of all other equations of Secs. 3 and 4 reduces to the substitution $\alpha \to \alpha(\Box)$, $\delta\alpha \to \delta\alpha(\Box)$; the final results remain, of course, unchanged.

APPENDIX 3

Here we derive the Ward identities for the theory (72). We shall see that the Ward identities have, in fact, the same form in the symmetric theory and in the theory with spontaneous symmetry breakdown.[7)]

We carry out in the functional integral (77) the substitution (76), (80), but with $\Lambda^\alpha(x)$ an arbitrary function of the coordinates. Then

$$\ln\frac{D(A', B', \sigma')}{D(A, B, \sigma)} = -\delta\,\mathrm{Sp}\ln D_\beta,$$

$$\frac{\alpha}{2}\int dx t_\beta^a t_\beta^a \to \frac{\alpha}{2}\int dx t_\beta^a t_\beta^a + \alpha\int dx t_\beta^a\Lambda^a. \tag{A.23}$$

Since the generating functional does not change under a change of integration variables, we obtain the Ward identities by setting equal to zero the first variation of $W_{\alpha,\beta}$ with respect to $\Lambda^a(x)$:

$$\left\{\alpha t^a(x) + \int dz\left[J_\mu^b(z)\nabla_\mu^{bc}(z) + \frac{\lambda}{2}(b_{\alpha,\beta} + \sigma(z))\theta^c(z)\right.\right.$$

$$\left.\left.-\frac{\lambda}{2}(\eta(z)\delta^{bc} + \varepsilon^{bdc}\theta^d(z))B_b(z)\right]D_\beta^{ca}(z,x)\right\}W_{\alpha,\beta} = 0, \tag{A.24}$$

where in $\{\ldots\}$ in (A.24) one must make the substitution

$$A_\mu^a \to \frac{\delta}{i\delta J_\mu^a}, \quad \sigma \to \frac{\delta}{i\delta\eta}, \quad B_a \to \frac{\delta}{i\delta\theta^a}.$$

It is clear that in terms of the original field $\bar{\sigma} = b_{\alpha,\beta} + \sigma$ the Ward identities have the same form in the symmetric theory ($b_{\alpha,\beta} = 0$) and in the theory with spontaneous symmetry breakdown

The identity (91) is obtained by differentiating (A.24) with respect to J_μ^a once and then taking the limit $J_\mu^a = \eta = \theta^a = 0$.

1)The problem of the gauge-dependence of the Green's functions has been considered for the case of quantum electrodynamics in the paper of Fradkin (Trudy FIAN, **29**, 7 (1965) and for the massless Yang-Mills field in the paper by Faddeev and Popov [2].

2)Here one takes into account the fact that the renormalization of the particle wave functions has been carried out.

3)A similar expression for W for the construction of the correct Feynman rules was first used by Fradkin [7].

4)We have introduced explicitly only the physically necessary renormalization of the wave functions of real particles.

5)Cf. Appendix 3.

6)The function $\delta'(\bar{k}^2)$ has to be redefined ("regularized"). One can, e.g., interpret it as

$$\delta'(\bar{k}^2) = \frac{1}{2}\frac{1}{k_0 + n_0\hat{k}/\beta}\frac{\partial}{\partial k_0}\delta(\bar{k}^2).$$

We shall not discuss this question here.

7)The Ward identities for a symmetric theory of the Yang-Mills field without scalar particles were derived in [8,9]. We give here the derivation of the Ward identities simply because we have mentioned them in Sec. 4.

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