Quantum Stabilizer Codes From Graph Embeddings on Manifolds

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Joint work with **Ted Yoder** (IBM T.J. Watson Research Center)

United nations sustainable development goals

Zero hunger

Table 1. Percentage and number of people affected by severe food insecurity in 2016

	Percentage	Millions
World	9.3 (± 0.4)	688.5 (± 27.6)
Africa	27.4 (±0.7)	333.2 (±8.6)
Asia	7.0 (± 0.6)	309.9 (± 26)
Latin America	6.4 (± 0.3)	38.3 (± 2.0)
Northern America and Europe	1.2 (±0.1)	13.0 (±1.3)

Source: www.worldhunger.org

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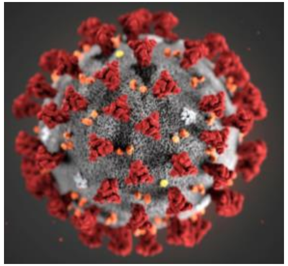
Good health and well being



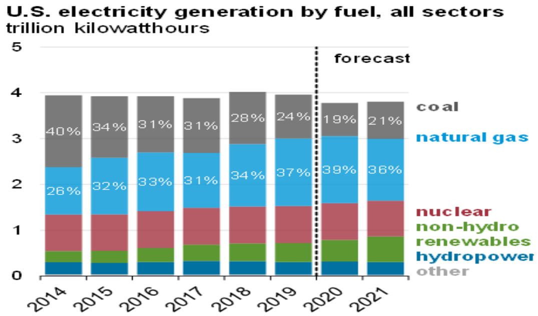
Source: pbs.org

We are in the middle of an epidemic crisis, where more than ever we realize the importance of faster drug discovery!

Enormous social and economic impact.

United nations sustainable development goals

Affordable and clean energy



Source: US energy information administration

United nations sustainable development goals

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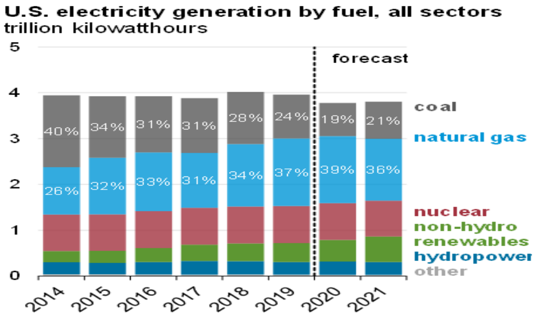

Source: US energy information administration

Climate action

Aftermath of ***Cyclone Amphan*** from last week. (Left: Kolkata airport after the storm).

- Estimated **13 bn USD** in damages.

Hurricane Harvey (2017)

- Estimated **125 bn USD** in damages.

Potential use cases for a quantum computer

The best application of a quantum computer is to simulate quantum phenomena, such as molecules, chemical reactions etc.

Nitrogen fixing fertilizers:

- The Haber-Bosch process uses around 1.5-2% of global energy.
- There are bacteria in nature that can perform the same chemical reaction, and understanding how they do it requires one to be able to simulate the enzymes and proteins forming such bacteria.

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CO_2 capture and catalyst design:

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High Temperature Superconductors / Better Batteries:

- Reduce power transmission losses, increased energy storage capabilities from intermittent sources such as wind and solar.

Facts about quantum computers

Classical computers manipulate bits of **0**s and **1**s.

A quantum computer manipulates **quantum states**, consisting of superpositions of qubit states, for e.g. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

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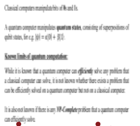
<u>**Known limits of quantum computation:**</u>

While it is known that a quantum computer can ***efficiently*** solve any problem that a classical computer can solve, it is not known whether there exists a problem that can be efficiently solved on a quantum computer but not on a classical computer.

It is also not known if there is any ***NP-Complete*** problem that a quantum computer can efficiently solve.



Basics of quantum error correction

Operations on qubits can be noisy. For example suppose you have the quantum state $|\psi\rangle = |0\rangle$, and you apply the $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gate to it.

Desired output: $X|\psi\rangle = |1\rangle$ when there is no error in the gate application and measurement process.

Typical situation due to quantum errors: $X|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$.
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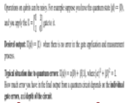
How much error you have in the final output from a quantum circuit depends on the **individual gate errors**, and **depth of the circuit**.

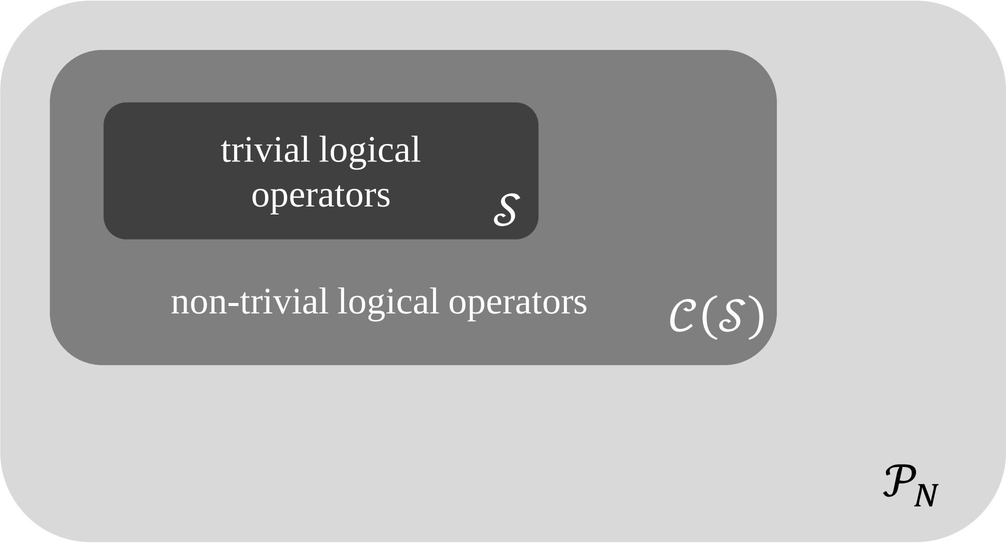
- Encode K logical qubits in a subspace of $N > K$ physical qubits.
- Such that any $(D-1)$ errors can be detected.
- In principle, any $\lfloor (D-1)/2 \rfloor$ errors can be corrected, but it may not be efficient.
- D is called the distance of the quantum code.



Basics of quantum error correction: Stabilizer Codes

Describe a code using a stabilizer group $\mathcal{S} \leq \mathcal{P}_N = \{I, X, Y, Z\}^{\otimes N}$.

- E.g. $\mathcal{S} = \langle ZZZZ, XXXX \rangle$, with $\mathcal{C}(\mathcal{S}) = \langle \mathcal{S}, XXII, IXXI, ZZII, IZZI \rangle$.
- $|\mathcal{C}(\mathcal{S})| = 2^{N+2K}$, $D = min\{|p| : p \in \mathcal{C}(\mathcal{S}) - \mathcal{S}\}$.
- D is called the **distance of the stabilizer code**.

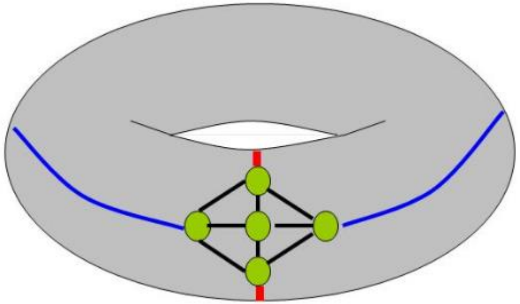


- D is the minimum weight of non-trivial logical operators.
- E.g. of weight calculation.
  - $|XYXY| = 4$
  - $|XZZI| = 3$
  - $|IIIX| = 1$

Graph Embeddings on Manifolds (2- Manifolds, closed)

A **graph embedding** of a graph $G(V, E)$ in a manifold M is a "drawing" of the graph on M such that it has some nice properties:

- Faces are **homeomorphic** to open discs.
- Edges don't intersect except at vertices.

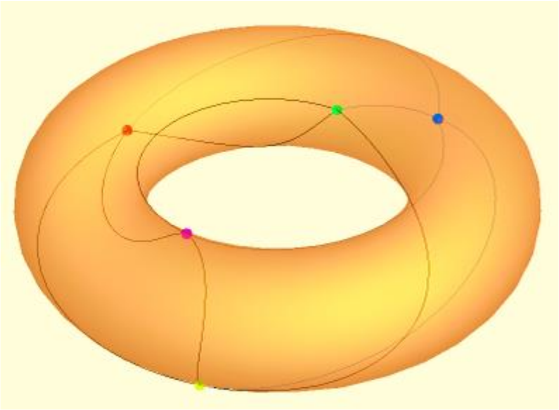


E.g. The graph on the right is K_5 embedded on the torus.

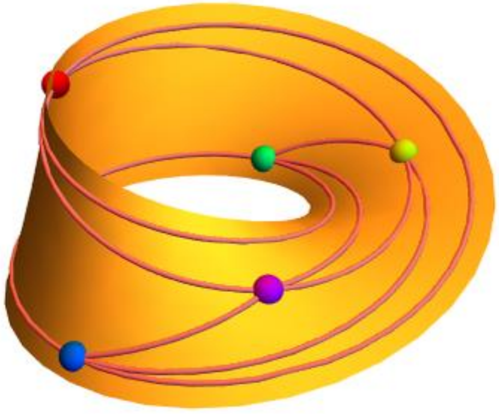
Manifolds for us will be 2-manifolds (surfaces), which are closed (meaning does not have a boundary and are compact). Can be orientable or non-orientable.

- Sphere, Torus are orientable manifolds.
- Real projective plane is a non-orientable manifold (these can be difficult to imagine if you have not encountered them before).

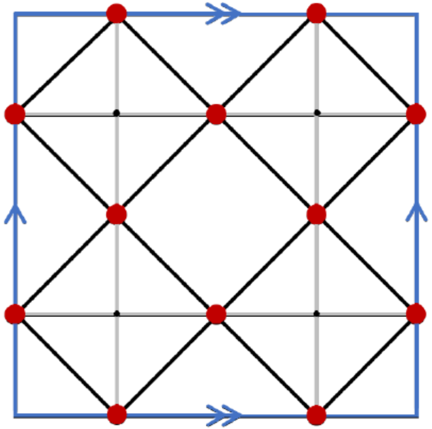
Examples of graph embeddings



Torus



Mobius band
(just for illustration, not a closed manifold)



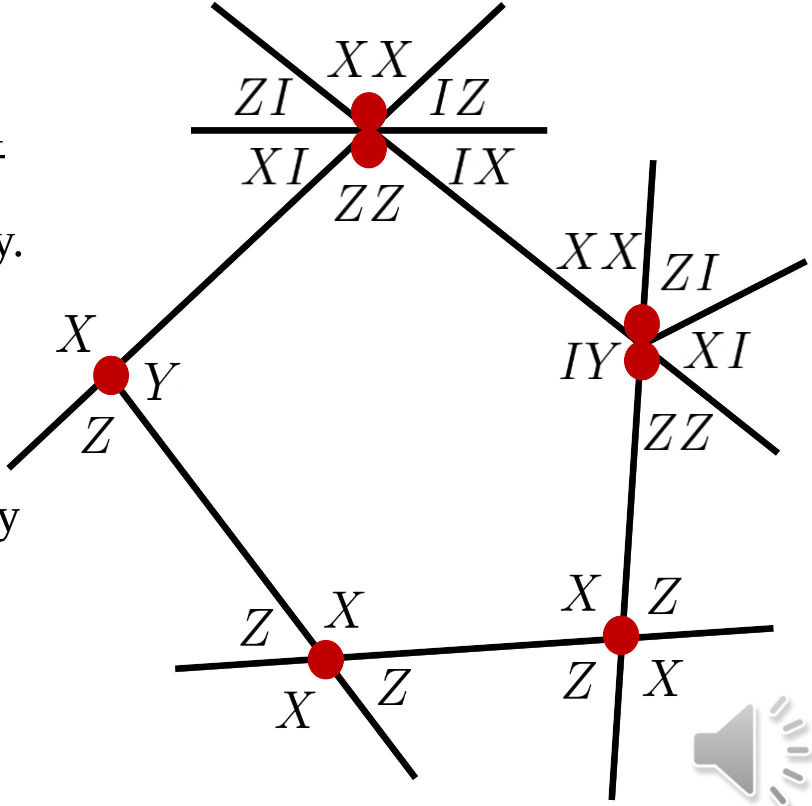
A different way to represent a **Torus**

Source: Wolfram

Outline of the construction given a closed 2-cell graph embedding

Stabilizer codes from graph embedding:

- Write a cyclically anticommuting list of Paulis around each vertex.
- Number of qubits to place at a vertex v is $\frac{\deg(v)-1}{2}$ or $\frac{\deg(v)-2}{2}$ if $\deg(v) \geq 3$ odd or even respectively.
- The **tensor product of Paulis within a face** is a stabilizer.



Cyclically anticommuting lists (CAL):

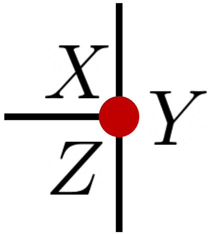
An ordered list of Paulis $\{p_0, p_1, \dots, p_{l-1}\}$ is cyclically anticommuting if

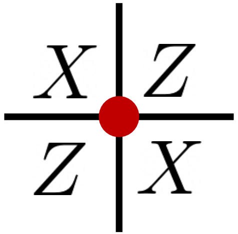
1) $\{p_i, p_{i+1\ (mod\ l)}\} = 0$
2) $[p_i, p_j] = 0$ when $j \neq i \pm 1\ (mod\ l)$

Eg. $\{X, Y, Z\}$ and $\{X, Z, X, Z\}$ are CALs

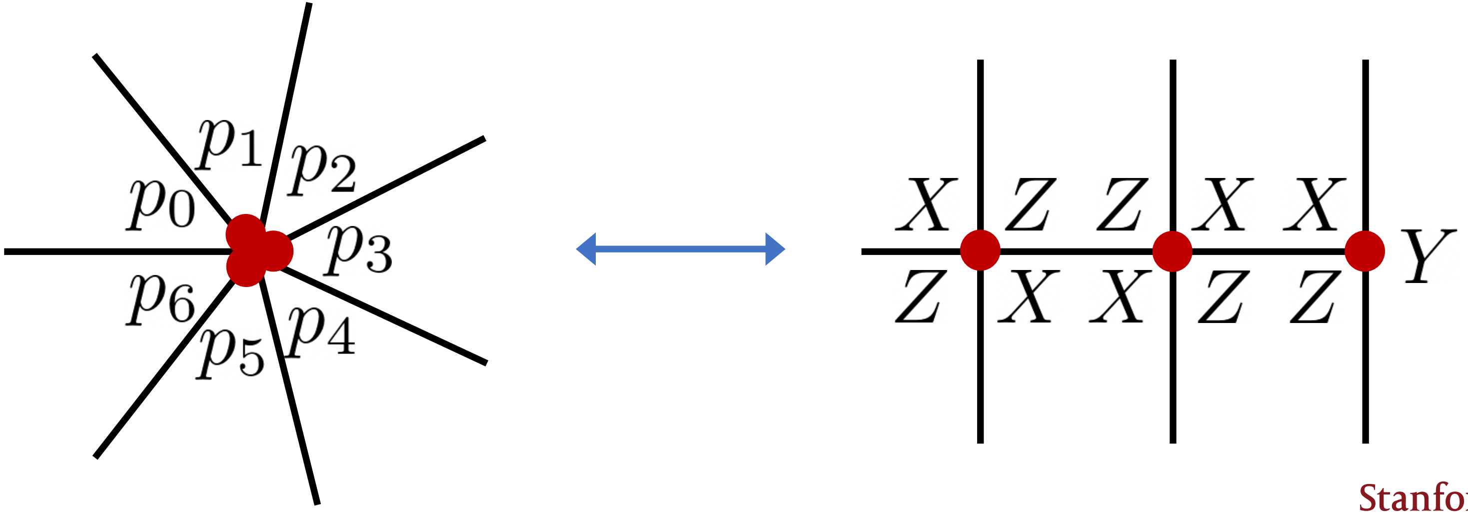
Reduction to graphs with degree between 3 and 4

Clearly, for $l = 3, 4$





So for $l = 7$



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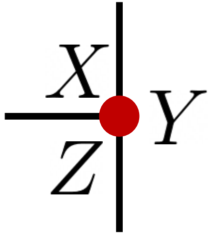
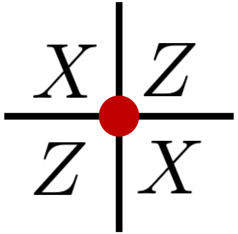
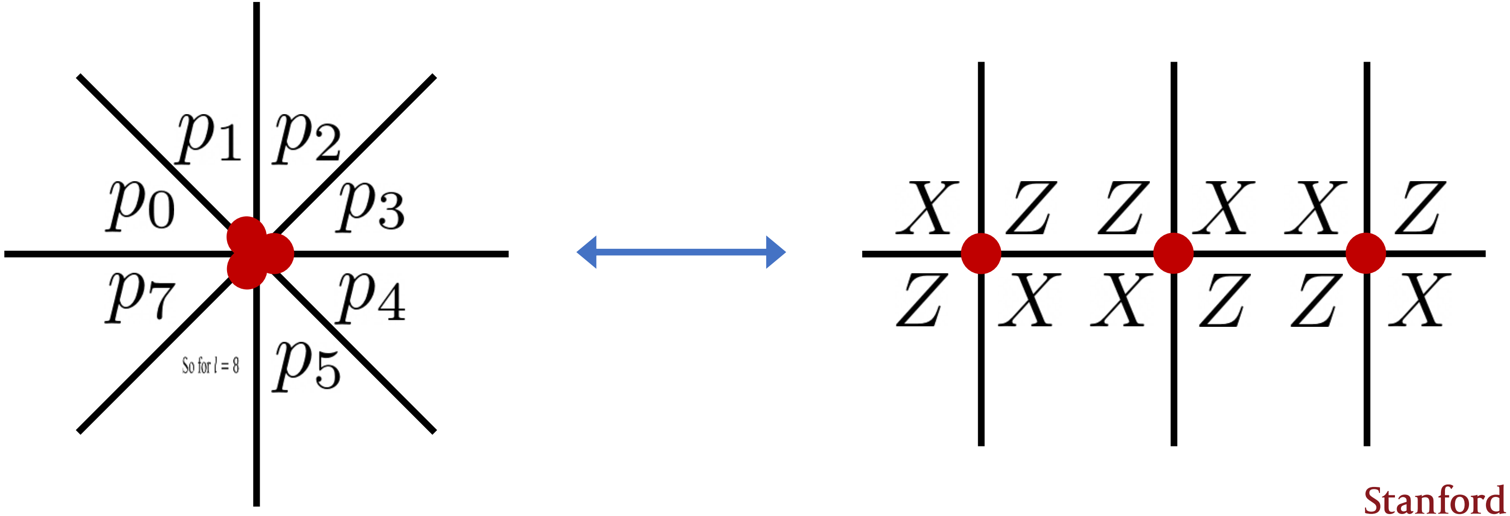




So for $l = 8$



Number of encoded qubits

We now have a set of Paulis defined by each face of the graph embedding.

- It turns out that because of the CAL property, this set of Paulis **commute** with each other.
- Hence the group generated by them is a stabilizer group.

How many qubits does this code encode?

Theorem: A surface code on a genus g manifold with M odd degree vertices encodes K logical qubits given by

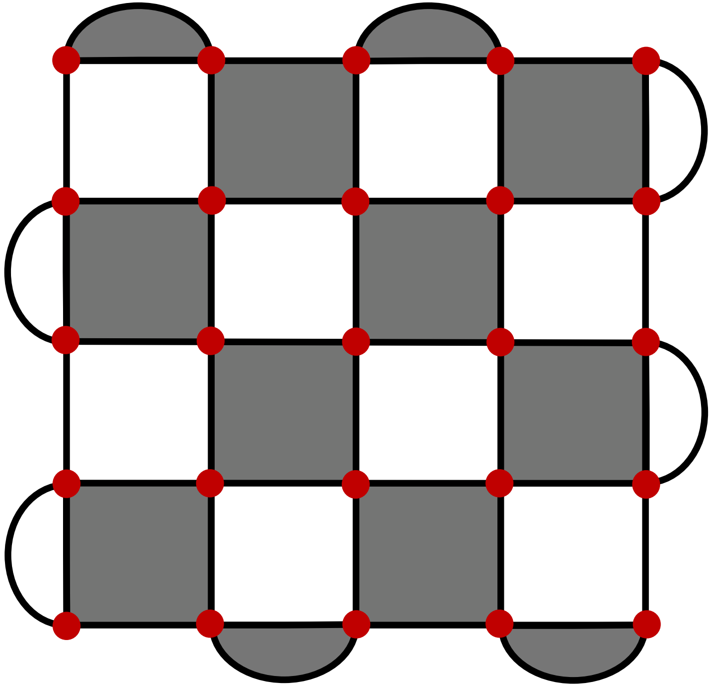
$$K = \left\{ \begin{array}{ll} 2g & , \textit{orientable} \\ g & , \textit{non-orientable} \end{array} \right\} + \left\{ \begin{array}{ll} 0 & , \textit{checkerboardable} \\ (M-2)/2 & , \textit{not checkerboardable} \end{array} \right\}$$

Here g is the orientable / non-orientable genus of the manifold M.

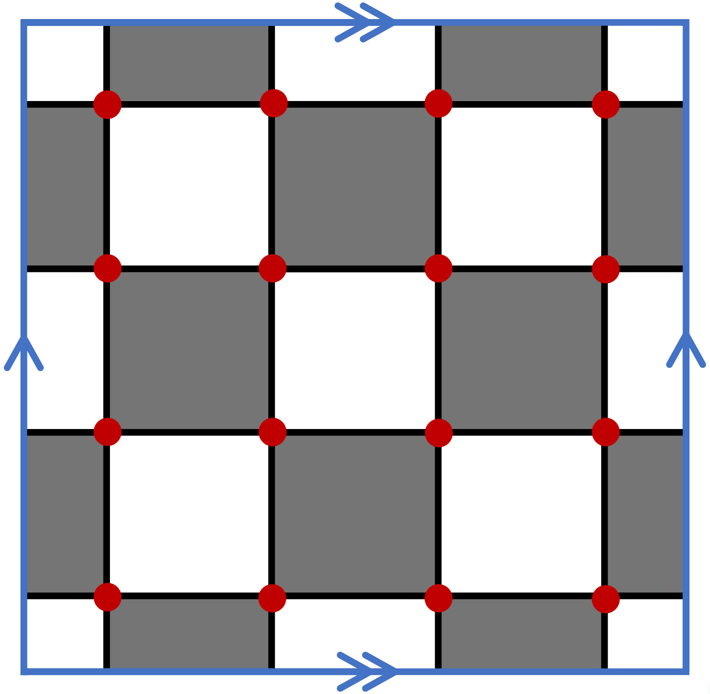
A graph is **checkerboardable** if its faces can be two-colored, with adjacent faces colored differently.

Checkerboarding examples

$$K = \begin{cases} 2g & , \textit{orientable} \\ g & , \textit{non-orientable} \end{cases} + \begin{cases} 0 & , \textit{checkerboardable} \\ (M-2)/2 & , \textit{not checkerboardable} \end{cases}$$



NOT Checkerboardable, $K = 1$

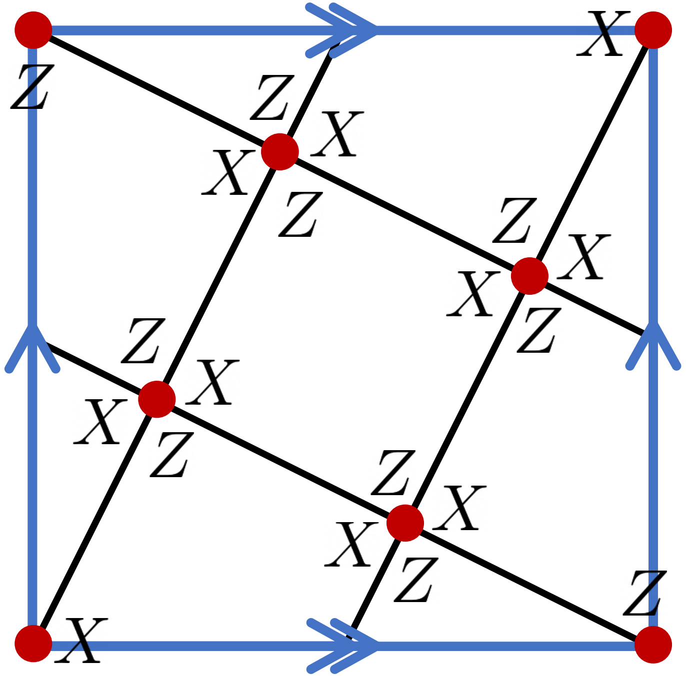


Checkerboardable, $K = 2$

Example: Cyclic Toric Code

- Parameterize by relatively prime, positive integers a, b, with $b > a \geq 1$.
- Draw lines $y = \left(\frac{b}{a}\right)x$ & $y = \left(-\frac{a}{b}\right)x$.

An example on the right with $(a, b) = (1, 2)$.



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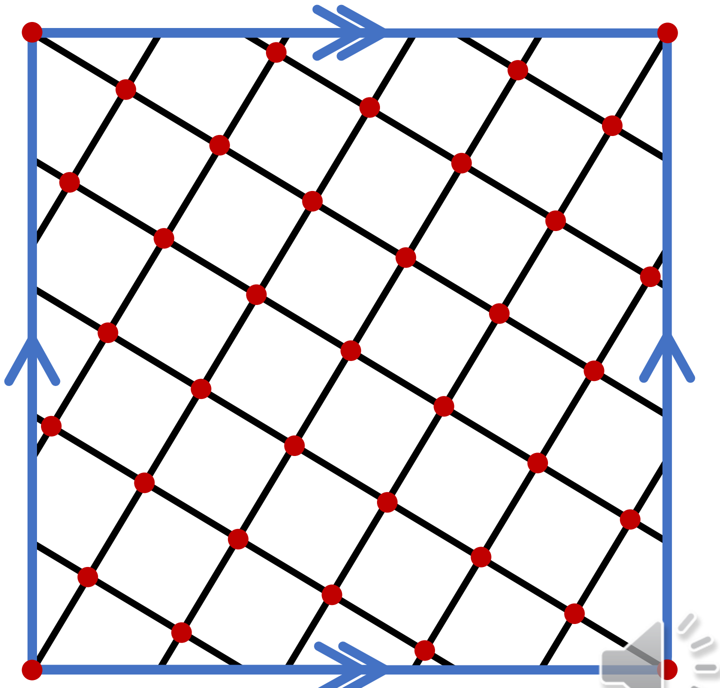
Code Parameters $[\![N, K, D]\!]$

- $N = a^2 + b^2$
- If N is odd, $K = 1$ and $D = a + b$
- If N is even, $K = 2$ and $D = \max(a, b)$

This code achieves $N = KD^2/2$ in two regimes, when $a = b - 1$ or when $a = 1$ and b is odd.

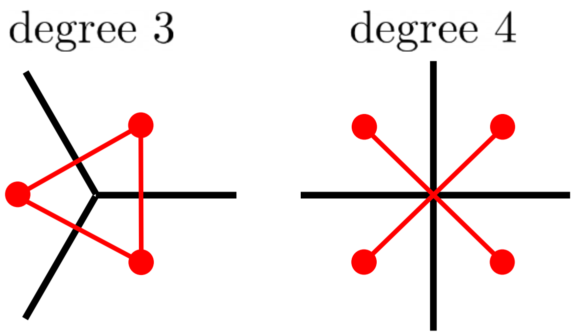
****Proving the distance is non-trivial.**

$(a, b) = (3, 5)$



General bounds on distance D

We need to use a construction called the decoding graph, obtained from the original graph $G(V,E)$.



Definition 4.2. The decoding graph $G_{\text{dec}} = (V_{\text{dec}}, E_{\text{dec}})$ of a closed 2-cell embedded graph $G = (V, E, F)$ is constructed so that $V_{\text{dec}} = F$, while edges E_{dec} are associated to vertices V in a many-to-one fashion. For each vertex $v \in V$, draw an edge between vertices $v_1', v_2' \in V_{\text{dec}}$ (associated to faces $f_1, f_2 \in F$) if there is a Pauli supported on qubits at v that anticommutes with S_{f_1} and S_{f_2} but commutes with all other stabilizers S_f.

Without loss of generality, we only need the case for degree 3 and 4 vertices.

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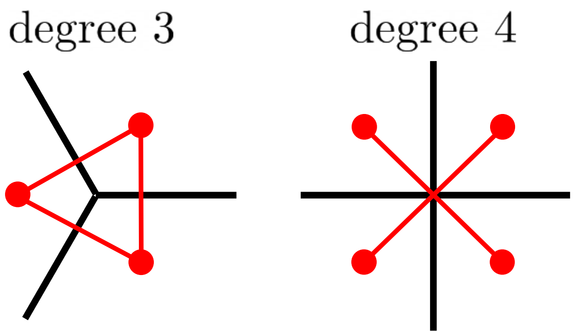

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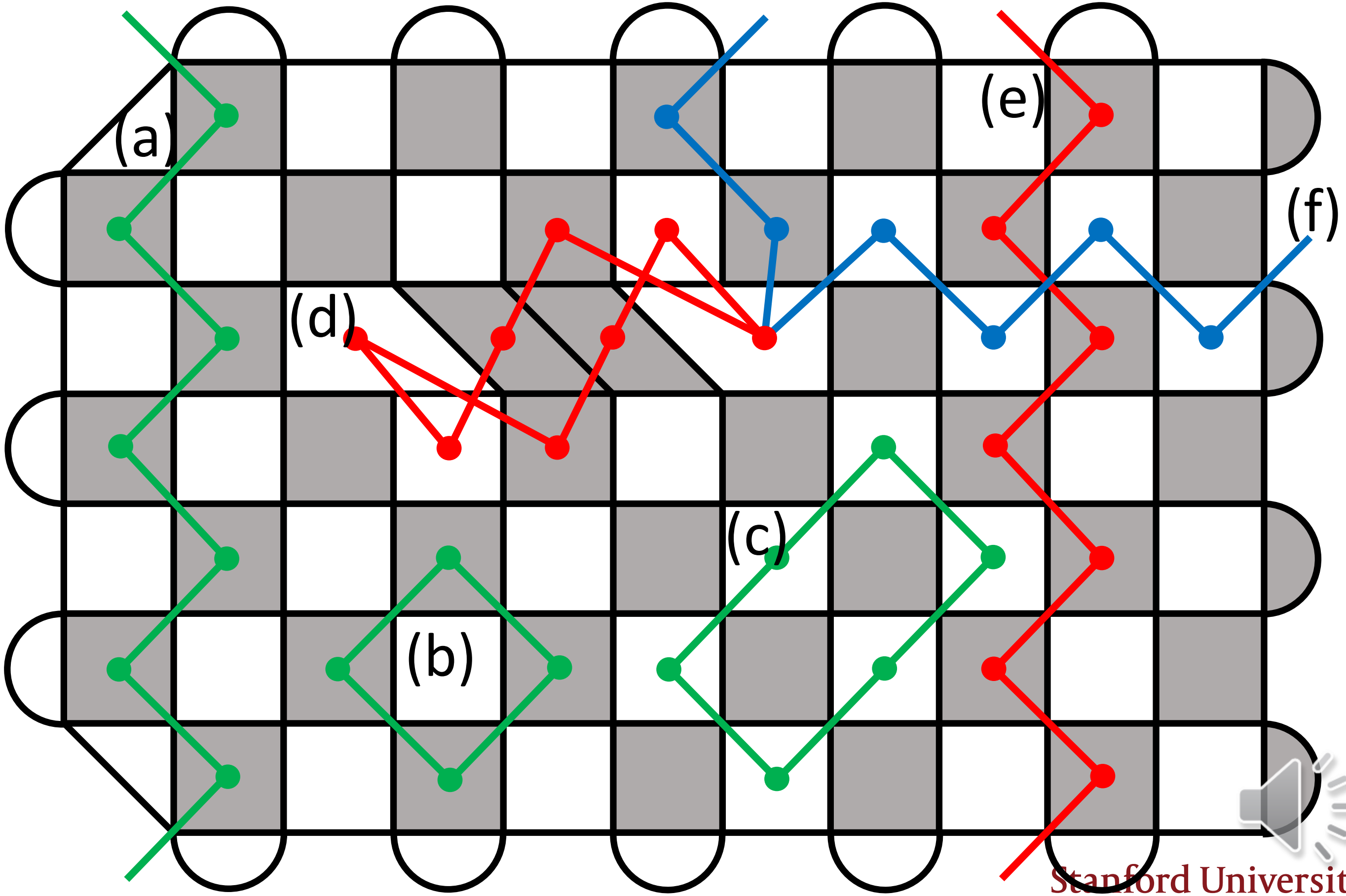
Decoding graph example

Stabilizers: a,b,c

Logical X: d,e

Logical Z: f



Efficient algorithm to get distance bounds

1. Given graph G, create its decoding graph G_{dec}.

2. Find a minimum cycle basis (MCB) of G_{dec}. A MCB is a basis of the cycle space $\{c_1, c_2, \ldots, c_b\}$ such that $\Sigma_i |c_i|$ is minimized. This can be done with Horton's algorithm or more efficient, more recent alternatives.

3. Convert each c_i to a Pauli P_i. Find nontrivial c_i, those for which P_i anticommutes with some other P_j.

4. Let W be the length of the shortest nontrivial c_i.

Theorem: If the graph is checkerboardable, $D = W$. If the graph is not checkerboardable, $W/2 \leq D \leq W$.

Acknowledgments

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Questions?

For questions, you can email me at rsarkar@stanford.edu