# Quantum Stabilizer Codes From Graph Embeddings on Manifolds


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Joint work with **Ted Yoder** (IBM T.J. Watson Research Center)

# Basics of quantum error correction

Operations on qubits can be noisy. For example suppose you have the quantum state $|\psi\rangle = |0\rangle$, and you apply the $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gate to it.

**Desired output:** $X|\psi\rangle = |1\rangle$ when there is no error in the gate application and measurement process.

**Typical situation due to quantum errors:** $X|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$.
How much error you have in the final output from a quantum circuit depends on the **individual gate errors**, and **depth of the circuit**.

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How much error you have in the final output from a quantum circuit depends on the **individual gate errors**, and **depth of the circuit**.

- Encode $K$ logical qubits in a subspace of $N > K$ physical qubits.
- Such that any $(D - 1)$ errors can be detected.
- In principle, any $\lfloor (D-1)/2 \rfloor$ errors can be corrected, but it may not be efficient.
- $D$ is called the distance of the quantum code.

# Basics of quantum error correction: Stabilizer Codes

Describe a code using a stabilizer group $\mathcal{S} \leq \mathcal{P}_N = \{I, X, Y, Z\}^{\otimes N}$.

- E.g. $\mathcal{S} = \langle ZZZZ, XXXX \rangle$, with $\mathcal{C}(\mathcal{S}) = \langle \mathcal{S}, XXII, IXXI, ZZII, IZZI \rangle$.
- $|\mathcal{C}(\mathcal{S})| = 2^{N+2K}$, $D = min\{|p|: p \in \mathcal{C}(\mathcal{S}) - \mathcal{S}\}$.
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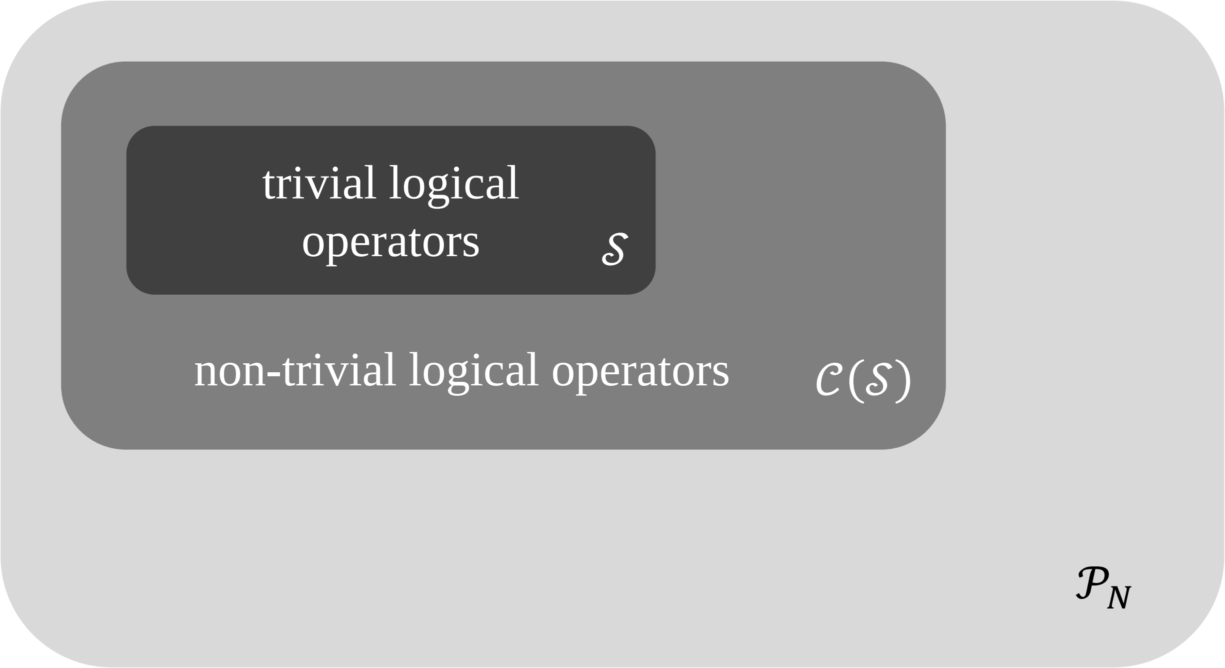
Relations between $\mathcal{S}$, $\mathcal{C}(\mathcal{S})$ and $\mathcal{P}_N$:

- $\mathcal{P}_N$ is the ***N* qubit Pauli group**. $|\mathcal{P}_N| = 4^{N+1}$ . If you ignore phases, $|\mathcal{P}_N| = 4^N$ .
- $\mathcal{S}$ is the **stabilizer group**. It is a subgroup of $\mathcal{P}_N$ where all elements commute.
- $\mathcal{C}(\mathcal{S})$ is the **centralizer group** of $\mathcal{S}$. It is a subgroup of $\mathcal{P}_N$ and consist of all elements that commute with each element in $\mathcal{S}$. Thus $\mathcal{S}$ is also a subgroup of $\mathcal{C}(\mathcal{S})$.
- $K = N - \dim(\mathcal{S})$.

# Relationships summarized via a diagram

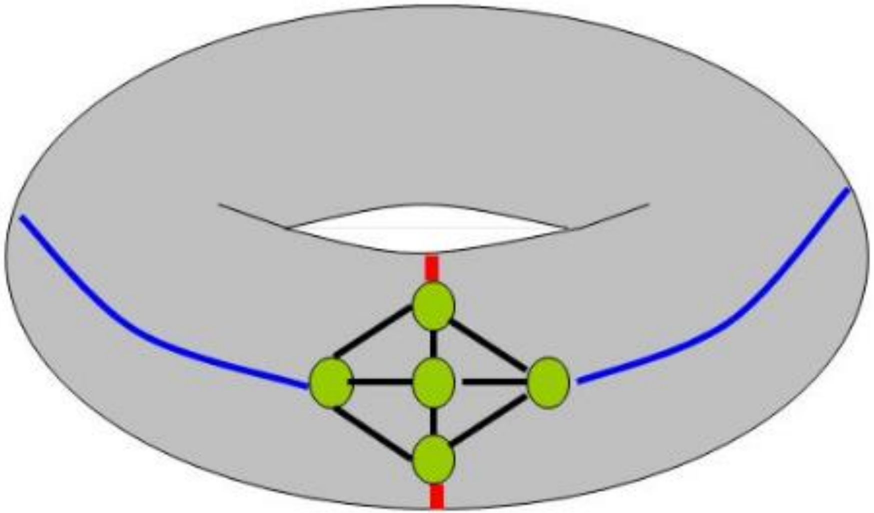



- $D$ is the minimum weight of non-trivial logical operators.
- E.g. of weight calculation.
  - $|XYXY| = 4$
  - $|XZZI| = 3$
  - $|IIIX| = 1$

# Graph Embeddings on Manifolds (2- Manifolds, closed)

A **graph embedding** of a graph $G(V,E)$ in a manifold $M$ is a “drawing” of the graph on $M$ such that it has some nice properties:

- Faces are **homeomorphic** to open discs.
- Edges don’t intersect except at vertices.



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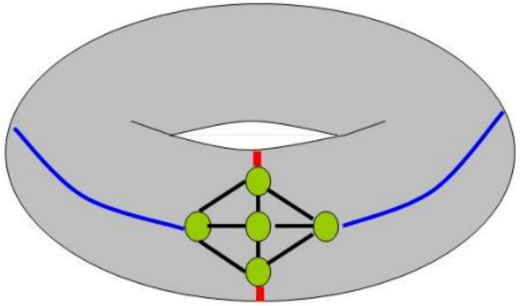

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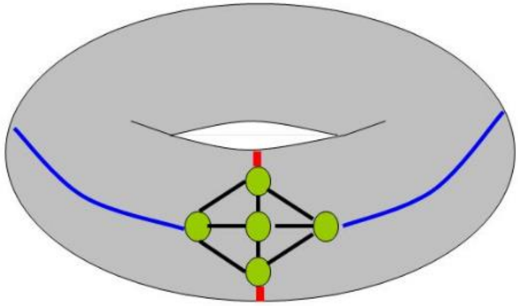

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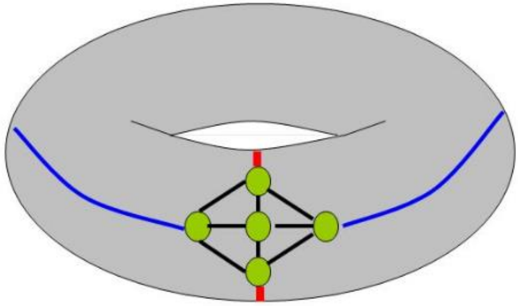

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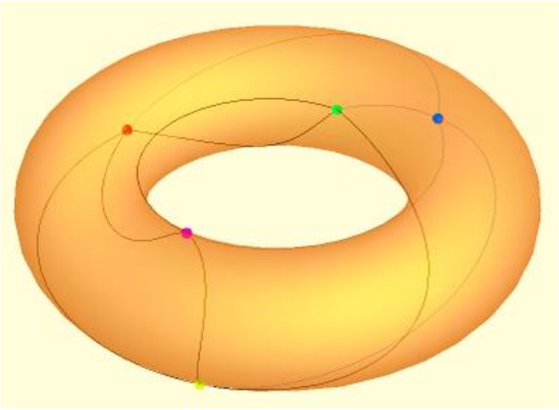
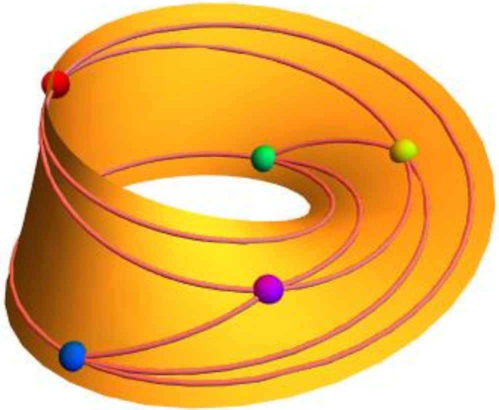
**Manifolds** for us will be 2-manifolds (surfaces), which are closed (meaning does not have a boundary and are compact). Can be orientable or non-orientable.

- Sphere, Torus are orientable manifolds.
- Real projective plane is a non-orientable manifold (these can be difficult to imagine if you have not encountered them before).

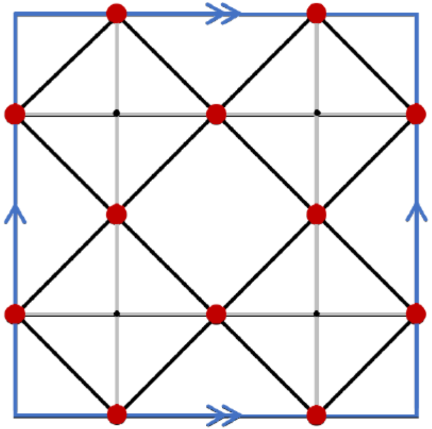
# Examples of graph embeddings



**Torus**



**Mobius band**
(just for illustration,
not a closed manifold)



A different way to
represent a **Torus**

Source: Wolfram

# Outline of the construction given a closed 2-cell graph embedding

**Stabilizer codes from graph embedding:**

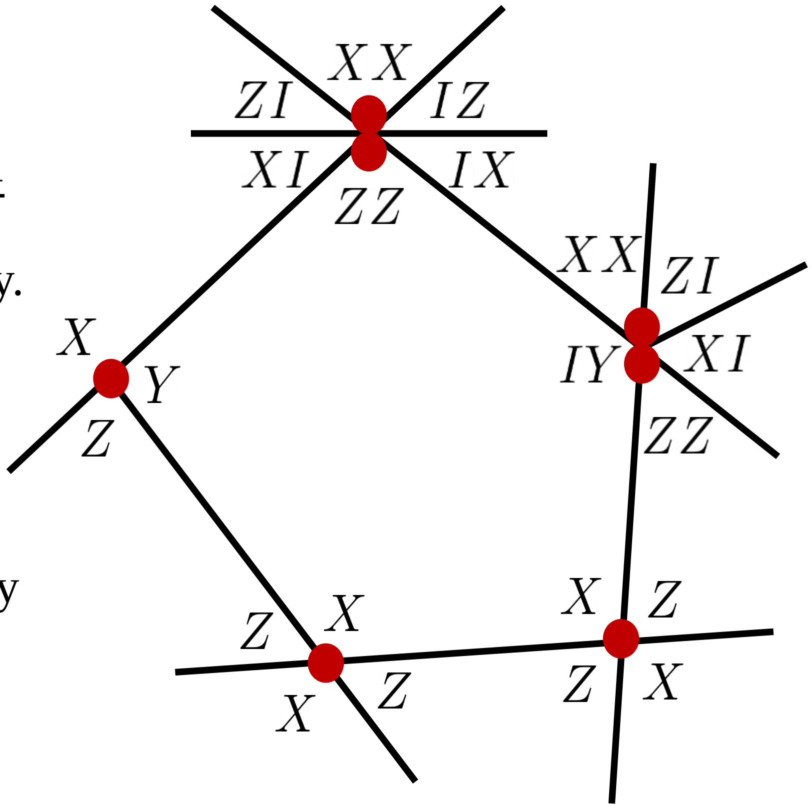
- Write a cyclically anticommuting list of Paulis around each vertex.
- Number of qubits to place at a vertex $v$ is $\frac{\deg(v)-1}{2}$ or $\frac{\deg(v)-2}{2}$ if $\deg(v) \geq 3$ odd or even respectively.
- The **tensor product of Paulis within a face** is a stabilizer.

**Cyclically anticommuting lists (CAL):**

An ordered list of Paulis $\{p_0, p_1, \ldots, p_{l-1}\}$ is cyclically anticommuting if

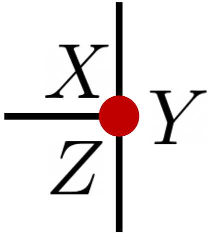
1) $\{p_i, p_{i+1\ (mod\ l)}\} = 0$
2) $[p_i, p_j] = 0$ when $j \neq i \pm 1\ (mod\ l)$

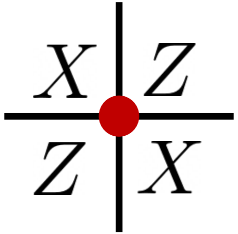
Eg. $\{X, Y, Z\}$ and $\{X, Z, X, Z\}$ are CALs



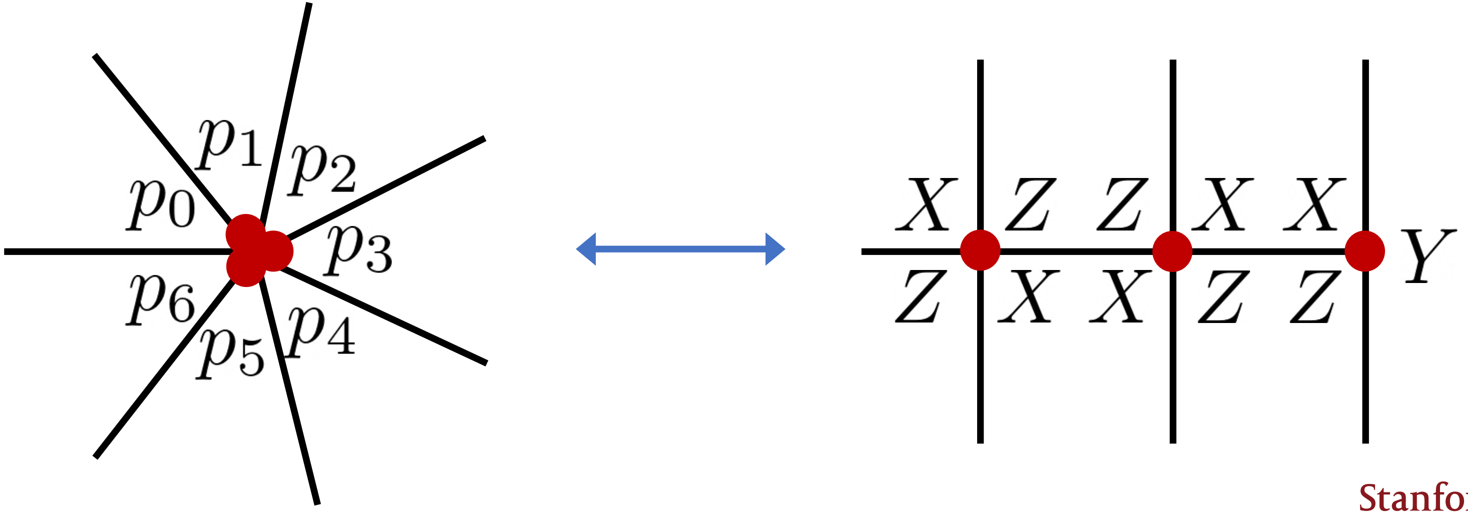
# Reduction to graphs with degree between 3 and 4

Clearly, for $l = 3, 4$




So for $l = 7$



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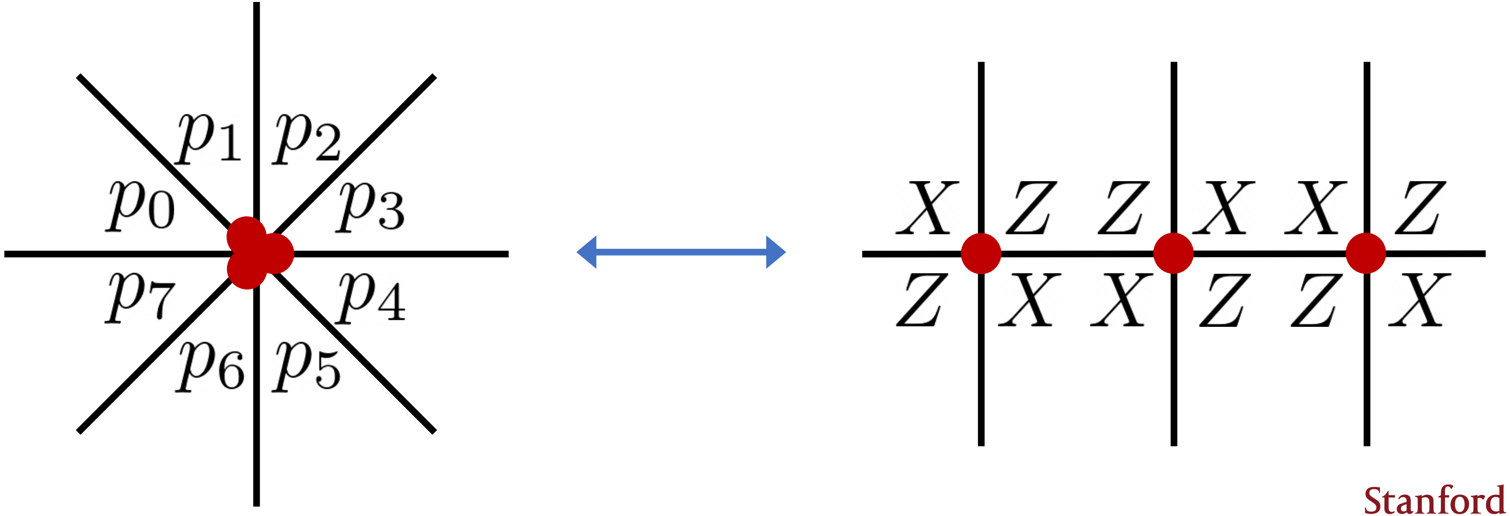

# Number of encoded qubits

**We now have a set of Paulis defined by each face of the graph embedding.**

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**How many qubits does this code encode?**

**Theorem:** A surface code on a genus $g$ manifold with $M$ odd degree vertices encodes $K$ logical qubits given by

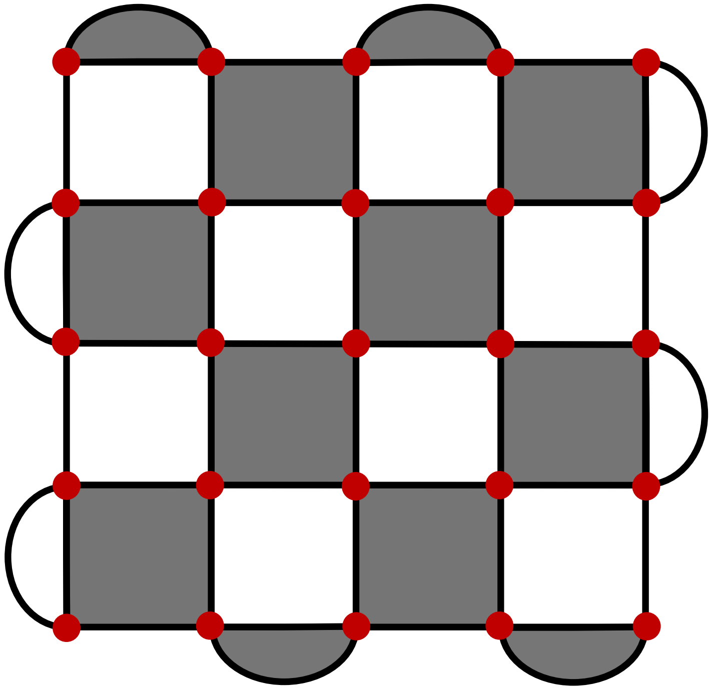
$$K = \begin{cases} 2g & , \ \mathit{orientable} \\ g & , \ \mathit{non\text{-}orientable} \end{cases} + \begin{cases} 0 & , \ \mathit{checkerboardable} \\ (M-2)/2 & , \ \mathit{not\ checkerboardable} \end{cases}$$

Here $g$ is the orientable / non-orientable genus of the manifold $M$.

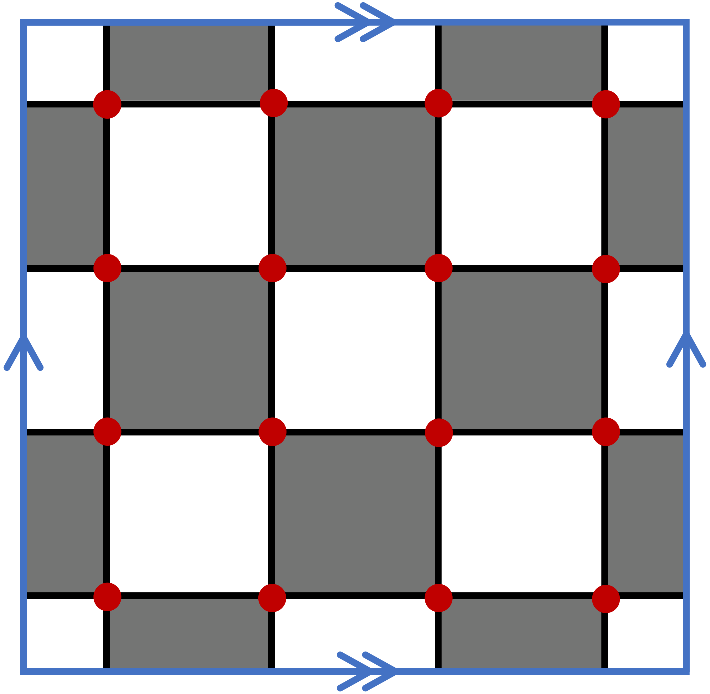
A graph is **checkerboardable** if its faces can be two-colored, with adjacent faces colored differently.

# Checkerboarding examples

$$K = \begin{cases} 2g & , \textit{orientable} \\ g & , \textit{non-orientable} \end{cases} + \begin{cases} 0 & , \textit{checkerboardable} \\ (M-2)/2 & , \textit{not checkerboardable} \end{cases}$$



NOT Checkerboardable, $K = 1$

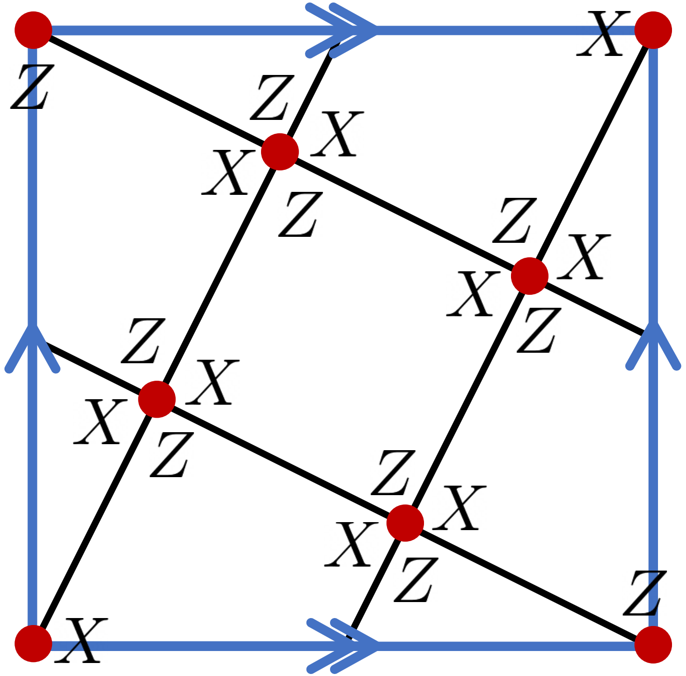


Checkerboardable, $K = 2$

# Example: Cyclic Toric Code

- Parameterize by relatively prime, positive integers $a, b$, with $b > a \geq 1$.
- Draw lines $y = \left(\frac{b}{a}\right)x$ & $y = \left(-\frac{a}{b}\right)x$.

An example on the right with $(a, b) = (1, 2)$.



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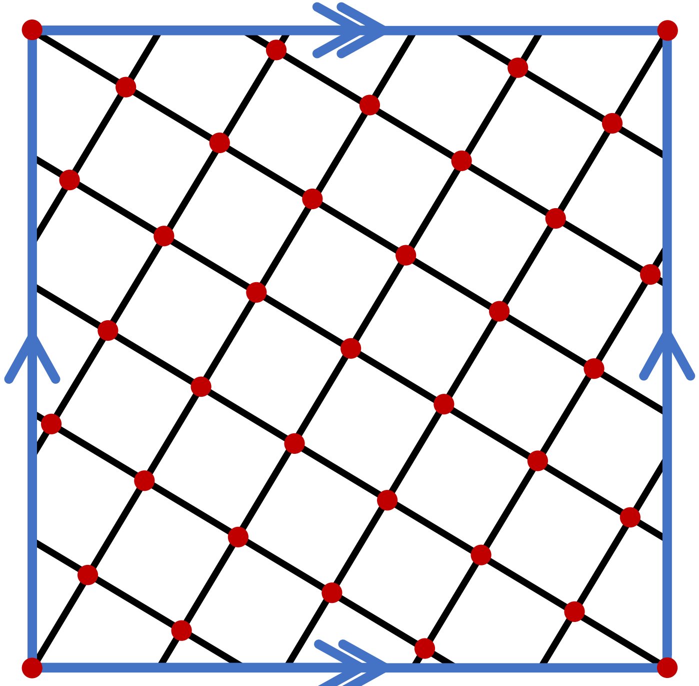
**Code Parameters** $[\![N, K, D]\!]$

- $N = a^2 + b^2$
- If $N$ is odd, $K = 1$ and $D = a + b$
- If $N$ is even, $K = 2$ and $D = \max(a, b)$

This code achieves $N = KD^2/2$ in two regimes, when $a = b - 1$ or when $a = 1$ and $b$ is odd.

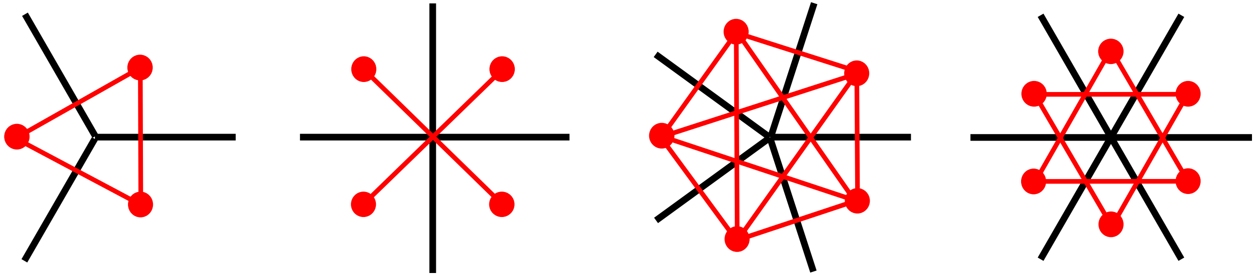
****Proving the distance is non-trivial.**

$(a, b) = (3, 5)$



# General bounds on distance $D$

We need to use a construction called the decoding graph, obtained from the original graph $G(V,E)$.




**Definition 4.2.** The decoding graph $G_{\text{dec}} = (V_{\text{dec}}, E_{\text{dec}})$ of a closed 2-cell embedded graph $G = (V, E, F)$ is constructed so that $V_{\text{dec}} = F$, while edges $E_{\text{dec}}$ are associated to vertices $V$ in a many-to-one fashion. For each vertex $v \in V$, draw an edge between vertices $v_1', v_2' \in V_{\text{dec}}$ (associated to faces $f_1, f_2 \in F$) if there is a Pauli supported on qubits at $v$ that anticommutes with $S_{f_1}$ and $S_{f_2}$ but commutes with all other stabilizers $S_f$.

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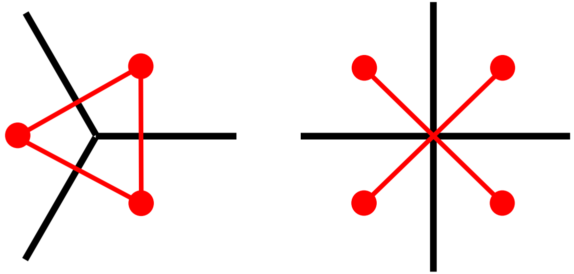


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**Without loss of generality, we only need the case for degree 3 and 4 vertices.**

# Decoding graph properties

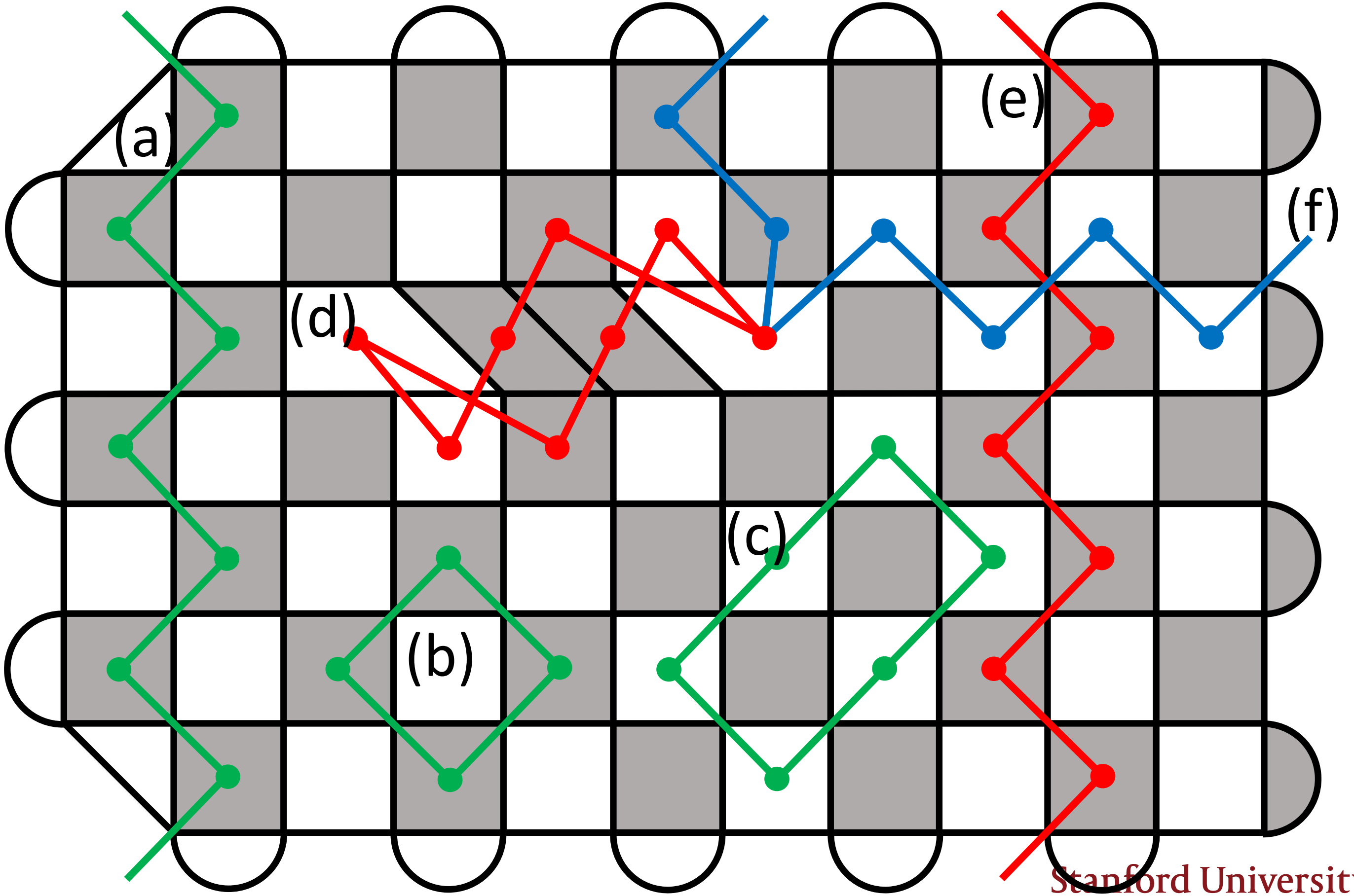
- Each edge represents a Pauli that anticommutes with exactly two faces.
- Any Pauli at a vertex can be represented by taking some subset of edges.
- **Logical operators are cycles in the decoding graph!**
- These are consequences of the CAL construction.

# Decoding graph example

Stabilizers: a,b,c

Logical X: d,e

Logical Z: f



# Efficient algorithm to get distance bounds

1. Given graph $G$, create its decoding graph $G_{dec}$.

2. Find a minimum cycle basis (MCB) of $G_{dec}$. A MCB is a basis of the cycle space $\{c_1, c_2, \ldots, c_b\}$ such that $\Sigma_i |c_i|$ is minimized. This can be done with Horton's algorithm or more efficient, more recent alternatives.

3. Convert each $c_i$ to a Pauli $P_i$. Find nontrivial $c_i$, those for which $P_i$ anticommutes with some other $P_j$.

4. Let $W$ be the length of the shortest nontrivial $c_i$.

**Theorem:** If the graph is checkerboardable, $D = W$. If the graph is not checkerboardable, $W/2 \leq D \leq W$.

# Questions?

For questions, you can email me at rsarkar@stanford.edu

# Acknowledgments

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