Commodity Taxation

1 Setup

- representative consumer, no heterogeneity
- linear taxation, no lump-sum tax to finance exogenous government expenditure (the "Ramsey problem")
- numeraire good labor $l$, untaxed
- $n$ consumption goods $c_1, \ldots, c_n$, prices $p_i$, taxed at linear rate $t_i$
- consumers:

  $$\max_{c_1, \ldots, c_n, l} U(c_1, \ldots, c_n, l) \quad \text{s.t.} \quad \sum_i p_i (1 + t_i) c_i \leq l. \quad (1)$$

- firms: CRS technology

  $$F(x_1, \ldots, x_n, l) \leq 0,$$

  e.g.

  $$\sum_i \bar{p}_i x_i - l \leq 0,$$

  where $1/\bar{p}_i$ is the productivity of labor in good $i$

- profit maximization

  $$\max_{x_1, \ldots, x_n, l} \sum_i p_i x_i - l \quad \text{s.t.} \quad F(x_1, \ldots, x_n, l) \leq 0 \quad (2)$$

- government: exogenous government expenditures $\{g_i\}$, budget constraint

  $$\sum_i p_i g_i \leq \sum_i p_i t_i c_i \quad (3)$$

- note: since $\{g_i\}$ is fixed, we could easily allow for preferences

  $$U(c_1, \ldots, c_n, l; g_1, \ldots, g_n)$$

  without changing results
2 Competitive Equilibrium

A competitive equilibrium with taxes \( \{t_i\} \) and government expenditures \( \{g_i\} \) is an allocation \( \{c_i, x_i, l\} \) and prices \( \{p_i\} \) such that

1. \( \{c_i\} \) solves the consumers’ problem (1) given prices \( \{p_i\} \) and taxes \( \{t_i\} \)
2. \( \{x_i\} \) solves (2) given \( \{p_i\} \) and firms make zero profits
3. \( \{c_i, g_i\} \) and \( \{p_i, t_i\} \) satisfy the government budget constraint (3)
4. all markets clear, i.e.
   \[
   c_i + g_i = x_i \quad \forall i = 1, ..., n
   \]

**Lemma 1.** \( \{c_i, l\} \) and \( \{p_i\} \) is part of a competitive equilibrium with \( \{t_i\} \) and \( \{g_i\} \) if and only if

\[
F(c_1 + g_1, ..., c_n + g_n, l) = 0, \tag{5}
\]

\[
p_i = \frac{F_i(c_1 + g_1, ..., c_n + g_n, l)}{F_l(c_1 + g_1, ..., c_n + g_n, l)}
\]

and \( \{c_i, l\} \) solves (1) given \( \{p_i, t_i\} \).

**Proof.** • only if: clear

• set \( x_i = c_i + g_i \), so 4. is satisfied

• necessary conditions for (2) are

\[
p_i = \gamma F_i
\]

for some \( \gamma \) and

\[
-1 = \gamma F_l
\]

• if \( p_i = -F_i/F_l \), then these conditions are satisfied and profits are

\[
\sum_i p_i x_i - l = -\sum_i \frac{F_i}{F_l} x_i - l = -\frac{1}{F_l} \left( \sum_i F_i x_i + F_l l \right) = 0
\]

by CRS and Euler’s theorem, so 2. is satisfied
as for 3., note

\[ \sum_i p_i g_i = \sum_i t_i p_i c_i \iff \sum_i p_i g_i = \left( \sum_i p_i (1 + t_i) c_i - l \right) - \left( \sum_i p_i c_i - l \right) = - \left( \sum_i p_i c_i - l \right) \]

by the consumers’ budget constraint,

\[ \iff \sum_i p_i (g_i + c_i) - l = \sum_i p_i x_i - l = 0, \]

since profits are zero as shown above. Thus, 3. is satisfied.

\[
3 \text{ Ramsey Problem}
\]

3.1 First Best

\[
\begin{align*}
\max_{c_1, \ldots, c_n, l} & \quad U(c_1, \ldots, c_n, l) \\
\text{s.t.} & \quad F(c_1 + g_1, \ldots, c_n + g_n, l) = 0
\end{align*}
\]

- \[ \frac{U_i}{U_l} = \frac{F_i}{F_l} \quad \forall i \]
- \[ \tau_i = 0 \text{ and lump-sum taxation} \]

3.2 Second Best

\[
\begin{align*}
\max_{c, l, t, p} & \quad U(c_1, \ldots, c_n, l) \\
\text{s.t.} & \quad F(c_1 + g_1, \ldots, c_n + g_n, l) = 0
\end{align*}
\]

and

\[ \{c_1, \ldots, c_n, l\} \in \arg \max_{c_1, \ldots, c_n, l} U(c_1, \ldots, c_n, l) \quad \text{s.t.} \sum_i c_i (1 + t_i) p_i = l \]

- we optimize over quantities \( \{c_i, l\} \) and prices/taxes \( \{p_i, t_i\} \), but the two are related through the last condition
- two approaches:
1. dual: solve quantities as a function of prices and optimize over prices
2. primal: solve prices as a function of quantities and optimize over quantities

- we pursue the second approach as it will be very useful for dynamic taxation later

4 Primal Approach

- by convexity of the consumers’ problem, FOCs are necessary and sufficient:
  \[ U_i = \lambda (1 + t_i) p_i \]
  \[ U_l = -\lambda \]

- solve for prices
  \[ (1 + t_i) p_i = -\frac{U_i}{U_l} \]

- substitute in budget constraint
  \[ \sum_i U_i c_i + U_l l = 0 \] (6)

- “implementability constraint,” no prices left

Proposition 1. Consider any allocation \( \{c_i^*, l^*\} \) that satisfies the implementability constraint (6) and the feasibility constraint (5). Then there exist prices and taxes \( \{p_i^*, t_i^*\} \) such that \( \{c_i^*, l^*\} \) and \( \{t_i^*, p_i^*\} \) is part of a competitive equilibrium with taxes.

- note: many solutions since 2 constraints, but \( n + 1 \) variables

Proof. 

- set
  \[ p_i^* = -\frac{F_i(c_i^* + g_1, ..., c_n^* + g_n, l^*)}{F_i(c_1^* + g_1, ..., c_n^* + g_n, l^*)} \]
  so 2. is satisfied

- given this, FOC for consumers
  \[ p_i^*(1 + t_i^*) = -\frac{U_i(c_1^*, ..., c_n^*, l^*)}{U_l(c_1^*, ..., c_n^*, l^*)} \]
thus set
\[ 1 + t_i^* = \frac{U_i^* F_i^*}{U_i^* F_i^*} \]  

(7)

• moreover, consumers’ budget constraint is satisfied since
\[ \sum_i p_i^*(1 + t_i^*)c_i^* = l^* \]

is equivalent to (substituting \( \{p_i^*, t_i^*\} \) from above)
\[ -\sum_i U_i^* c_i^* = U_i^* l^*, \]

which is the imposed implementability constraint (6). Hence, 1. is satisfied.

• market clearing was guaranteed when deriving \( p_i^* \)

• government budget constraint is satisfied by Walras’ law: consumers’ budget con-

straint
\[ \sum_i p_i^*(1 + t_i^*)c_i^* = l^* \]

zero profits
\[ \sum_i p_i^*(c_i^* + g_i^*) = l^* \]

and subtracting
\[ \sum_i p_i^* t_i^* c_i^* = \sum_i p_i^* g_i \]

\[ \square \]

5 Optimal Tax Rules

• Ramsey problem
\[ \max_{c_1, ..., c_n, l} U(c_1, ..., c_n, l) \]

s.t. (5) and (6)

• Lagrangian
\[ \mathcal{L} = U(c_1, ..., c_n, l) + \mu \left( \sum_i U_i c_i + U_i l \right) - \gamma F(c_1 + g_1, ..., c_n + g_n, l) \]
• FOCs for good $c_j$ and $l$

\[(1 + \mu)u_j + \mu \left( \sum_i u_{ij}c_i + u_{jl}l \right) = \gamma f_j\]

\[(1 + \mu)u_i + \mu \left( \sum_i u_{il}c_i + u_{ll}l \right) = \gamma f_i\]

or

\[
\frac{u_j}{u_i} \frac{1 + \mu + \mu \sum_i u_{ij}c_i + u_{jl}l}{1 + \mu + \mu \sum_i u_{il}c_i + u_{ll}l} = \frac{f_j}{f_i}
\]

• we know from (7) that

\[1 + t_j = \frac{u_j f_j}{u_i f_i} = \frac{1 + \mu + \mu \sum_i u_{ij}c_i + u_{jl}l}{1 + \mu + \mu \sum_i u_{il}c_i + u_{ll}l} \equiv 1 + \mu - \mu H_l \]

\[\mu H_l \]

5.1 Uniform Taxation Rule

• suppose $U$ is separable such that

\[U(c_1, \ldots, c_n, l) \equiv U(G(c_1, \ldots, c_n), l),\]

where $U_{Gl} = 0$ and $G(.)$ is homogeneous of degree one

• if double total spending on all consumption goods, then double demand for each individual good, and demand for consumption goods does not depend on $l$

• then

\[u_j = u_{Gj}\]

and

\[u_{ij} = u_{Gj}G_i + u_{Gij}\]

and

\[u_{ij} = u_{Gj}G_i = 0\]
thus
\[ \sum_i U_{ij} c_i + U_{ij} l = \sum_i \left( U_{GG} G_i G_j c_i + U_G G_i c_i \right) / U_G G_j \]

now use
\[ \sum_i U_{GG} G_i G_j c_i = U_{GG} G_j \sum_i G_i c_i = U_{GG} G_j G \]

and
\[ \sum_i U_G G_i c_i = U_G \sum_i G_i c_i = 0 \]

by Euler’s theorem and since \( G \) is homogeneous of degree one and \( G_j \) is homogeneous of degree zero

hence
\[ \frac{\sum_i U_{ij} c_i + U_{ij} l}{U_j} = \frac{U_{GG} G}{U_G} \]

therefore
\[ 1 + t_j = \frac{1 + \mu + \mu \frac{U_{ij} l}{U_l}}{1 + \mu + \mu \frac{U_{GG} G}{U_G}} \]

independent of \( j \)

exercise: show that the same result more generally goes through when \( G(\cdot) \) is homothetic, i.e.
\[ G(c_1, ..., c_n) \equiv k(K(c_1, ..., c_n)) \]

where \( k'(K) \neq 0 \) and \( K(c_1, ..., c_n) \) is homogeneous of degree \( \rho \).

can also be generalized to heterogeneous agents with preferences \( U^k(G(c_1, ..., c_n), l) \), where \( k \) is the household index. Then the uniform commodity taxation rule applies in any Pareto optimum if (i) \( U^k(\cdot) \) is weakly separable for all \( k \), (ii) the sub-utility \( G(\cdot) \) is the same for all \( k \) and (iii) \( G(\cdot) \) is homothetic. Note the difference to the Atkinson-Stiglitz (1976) theorem: Non-linear taxation of labor, no homotheticity of \( G(\cdot) \) required.

5.2 Inverse Income Elasticities Rule

rearrange (8) to
\[ \frac{t_j}{1 + t_j} = \mu \frac{H_j - H_l}{1 + \mu - \mu H_l} \]
• thus $t_j > t_i$ if $H_j > H_i$

• suppose now that $U$ is separable across all arguments, so that

$$H_j = - \frac{U_{jj} c_j}{U_j} \quad \text{and} \quad H_l = - \frac{U_{ll} l}{U_l}$$

• to consider income effects, suppose consumers are endowed with some exogenous non-labor income $I$, so that the FOC for their utility maximization problem is

$$U_i(c_i(q, I)) = \lambda(q, I) q_i,$$

where

$$q = [p_1(1 + t_1), ..., p_n(1 + t_n)]$$

• differentiate w.r.t. $I$

$$\frac{\partial c_i}{\partial I} = q_i \frac{\partial \lambda}{\partial I} = \frac{U_i \partial \lambda}{\lambda \partial I}$$

so that $H_i$ becomes

$$H_i = - \frac{U_{ii} c_i}{U_i} = - \frac{c_i \partial \lambda / \partial I}{\lambda \partial c_i / \partial I}$$

• define the income elasticity

$$\eta_i \equiv \frac{\partial c_i}{\partial I} \frac{I}{c_i}$$

to get

$$H_i = - \frac{\partial \lambda / \partial I \lambda}{\eta_i}$$

• numerator is positive, so $H_i$ is negatively related to income elasticity $\eta_i$

• hence goods with a higher income elasticity are taxed at a lower rate: tax necessities at a higher rate than luxury goods