Linear Capital Taxation and Tax Smoothing

1 Finite Horizon

1.1 Setup

- 2 periods \( t = 0, 1 \)
- preferences \( U^i(c_0, c_1, l_0) \)
- sequential budget constraints in \( t = 0, 1 \)

\[
c^i_0 + p b^i_1 + k^i_1 \leq w_0(1 − \tau^i) l^i_0 + R_0 k^i_0
\]
\[
c^i_1 \leq R_1 k^i_1 + b^i_1
\]

for all \( i \), where

\[
R_t ≡ 1 + (r_t − \delta)(1 − \tau_t^k)
\]

- idea: firms just rent capital at rental rate \( r_t \). Consumers own capital, get paid the rental rate but incur depreciation costs. But they can deduct this from rental income when computing capital tax liability.

- can combine to period-0 present value budget constraint

\[
c^i_0 + p c^i_1 \leq w_0(1 − \tau^i) l^i_0 + R_0 k^i_0 + (pR_1 − 1)k^i_1
\]

- sequential aggregate resource constraints

\[
k_1 + c_0 + g_0 \leq F^0(k_0, l_0) + (1 − \delta)k_0
\]
\[
c_1 + g_1 \leq F^1(k_1) + (1 − \delta)k_1
\]

- firms:

\[
\max_{k_0, l_0} F^0(k_0, l_0) − r_0 k_0 − w_0 l_0
\]
with FOCs

\[ r_0 = F^0_k(k_0, l_0) \]
\[ w_0 = F^0_l(k_0, l_0) \]

- no arbitrage requires

\[ R_1 = \frac{1}{p'} \]

i.e. the return on bonds and the return on capital must be the same

- consumers’ budget constraint becomes

\[ c_0^i + p c_1^i \leq w_0 (1 - \tau^i) l_0^i + R_0 k_0^i \]

where \( k_1^i \) has vanished, so from the perspective of the consumers, we are left with 3 goods \( c_0, c_1, l_0 \)

### 1.2 Uniform Taxation

- for instance, consider linear technology with

\[ F^0(k_0, l_0) = \bar{r}k_0 + \bar{w}l_0 \]
\[ F^1(k_1) = \bar{r}k_1 \]

so that the aggregate resource constraints can be combined to

\[ c_0 + g_0 + \bar{R}^{-1} (c_1 + g_1) \leq \bar{R}k_0 + \bar{w}l_0 \]

with \( \bar{R} \equiv 1 + \bar{r} - \delta \)

- again, we’re back to a 3 goods economy with \( c_0, c_1, l_0 \)

- since we’ve normalized the tax on \( c_0 \) to zero, the question whether \( c_0 \) and \( c_1 \) should be taxed uniformly comes down to asking under what conditions it’s optimal to set

\[ \frac{1}{p'} = R_1 = \bar{R} \]

and thus equivalently under what conditions

\[ \tau_1^k = 0 \]
• tax on capital income in period 1 is nothing but a distortion on the price between consumption in periods 0 and 1

• hence we can apply the uniform commodity taxation result from the last note, which applies if

\[ U_i(c_0, c_1, l_0) = U^i(G(c_0, c_1), l_0) \]

is separable and \( G(\cdot) \) is homothetic

• then \( R_1 = \bar{R} \) and thus \( \tau^k_1 = 0 \) in any Pareto optimum

• for instance, this would be satisfied if preferences are of the form

\[ U^i(c_0, c_1, l_0) = c_1^{1-\sigma} - \frac{\sigma}{1-\sigma} \beta c_1^{1-\sigma} - v^i(l_0) \]

and e.g. \( v^i(l_0) = v(l_0/\theta^i) \)

• Atkinson-Stiglitz (1976) theorem: if we could tax labor income non-linearly, then homogeneity of \( G(\cdot) \) would not be required, only that

\[ U^i(c_0, c_1, l_0) = U^i(G(c_0, c_1), l_0) \]

for uniform taxation to be Pareto optimal

• note: the sub-utility function \( G(c_1, c_2) \) must be the same for all individuals. See Saez (2002) and Diamond and Spinnewijn (2011) for deviations from this (heterogeneous discount factors \( \beta^i \)). Then capital taxation is generally welfare improving.

• what about taxing initial capital \( k^i_0 \) using \( \tau^k_0 \neq 0 \)? Imitates a lump sum tax. If there is only one representative agent, would want to set \( \tau^k_0 \) as high as possible (not necessarily so with heterogeneity). Time inconsistency problem.

2 Infinite Horizon

2.1 Setup

• CRS technology with aggregate uncertainty

\[ F\left(K(s^{t-1}), L(s^t), s^t, t\right), \]
where \( s^t = (s_0, s_1, ..., s_t) \) captures the history of aggregate uncertainty up to \( t \)

- the capital stock present in period \( t \) is chosen in period \( t - 1 \) and therefore depends on \( s^{t-1} \) only
- representative agent (no heterogeneity) with preferences

\[
\sum_{s^t} \beta^t \Pr(s^t) u(c(s^t), L(s^t))
\]

- Aggregate resource constraint

\[
c(s^t) + g(s^t) + K(s^t) \leq F(K(s^{t-1}), L(s^t), s^t) + (1 - \delta)K(s^{t-1}) \forall s^t (1)
\]

- hence aggregate uncertainty can result from technology shocks or government spending shocks
- Linear taxes on labour income

\[\tau^l(s^t)\]

- Linear taxes on capital income

\[\tau^k(s^t)\]

- (Taxes on consumption are redundant)
- The government has an initial debt equal to \( B_0 \)
- Complete markets where the price of an Arrow-Debreu security is \( p(s^t) \)
- Competitive markets with wages \( w(s^t) \) and rental rate \( r(s^t) \)

2.2 Households, government and firms

- Government budget constraint

\[
\sum_{t,s^t} p(s^t) \left[ g(s^t) - \tau^l(s^t)w(s^t)L(s^t) - \tau^k(s^t)(r(s^t) - \delta)K(s^{t-1}) \right] \leq -B_0
\]
• Household budget constraint

\[
\sum_{t,s^t} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t) \left( 1 - \tau^l(s^t) \right) L(s^t) - R(s^t)K(s^{t-1}) \right] \leq B_0
\]

where

\[
R(s^t) = 1 + (1 - \tau^k(s^t))(r(s^t) - \delta)
\]  

(2)
is the gross after-tax return on capital

• Firm profits:

\[
\pi(K, L, s^t, t) = F(K, L, s^t, t) - w(s^t)L - r(s^t)K
\]

2.3 Equilibrium conditions

Definition 1. A competitive equilibrium is a policy \( \{g(s^t), \tau^k(s^t), \tau^l(s^t)\} \), an allocation \( \{c(s^t), K(s^t), L(s^t)\} \) and prices \( \{w(s^t), r(s^t), p(s^t)\} \), such that households maximize utility s.t. budget constraint, firms maximize profits, the government budget constraint holds and markets clear.

• Firm FOC:

\[
r(s^t) = F_K \left( K(s^{t-1}), L(s^t), s^t, t \right)
\]

(3)

\[
w(s^t) = F_L \left( K(s^{t-1}), L(s^t), s^t, t \right)
\]

(4)

• Consumer FOC

\[
\beta^t \Pr(s^t) u_c \left( c(s^t), L(s^t) \right) - \lambda p(s^t) = 0
\]

\[
\beta^t \Pr(s^t) u_L \left( c(s^t), L(s^t) \right) + \lambda p(s^t) \left( 1 - \tau^l(s^t) \right) w(s^t) = 0
\]

\[
-\lambda p(s^t) + \lambda \sum_{s^{t+1}} p(s^{t+1})R(s^{t+1}) = 0
\]

• No arbitrage:

\[
p(s^t) = \sum_{s^{t+1}} p(s^{t+1})R(s^{t+1})
\]  

(5)
• Intratemporal:

\[ \frac{\beta^t \Pr(s^t) u_c (c(s^t), L(s^t))}{p(s^t)} = -\frac{\beta^t \Pr(s^t) u_L (c(s^t), L(s^t))}{p(s^t) (1 - \tau^t(s^t)) w(s^t)} \]
\[ u_c (c(s^t), L(s^t)) = -\frac{u_L (c(s^t), L(s^t))}{(1 - \tau^t(s^t)) w(s^t)} \]
\[ w(s^t) (1 - \tau^t(s^t)) = -\frac{u_L (c(s^t), L(s^t))}{u_c (c(s^t), L(s^t))} \]  

(6)

• Intertemporal:

\[ \frac{\beta^t \Pr(s^t) u_c (c(s^t), L(s^t))}{p(s^t)} = u_c (c_0, L_0) \]
\[ p(s^t) = \frac{\beta^t \Pr(s^t) u_c (c(s^t), L(s^t))}{u_c (c_0, L_0)} \]  

(7)

• Back to household budget:

\[ \sum_{t,s^t} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t) \left( 1 - \tau^t(s^t) \right) L(s^t) - R(s^t)K(s^{t-1}) \right] \leq B_0 \]
\[ \sum_{t,s^t} p(s^t) \left[ c(s^t) - w(s^t) \left( 1 - \tau^t(s^t) \right) L(s^t) \right] \leq B_0 + R_0K_0 \]
\[ \sum_{t,s^t} \frac{\beta^t \Pr(s^t) u_c (c(s^t), L(s^t))}{u_c (c_0, L_0)} \left[ c(s^t) + \frac{u_L (c(s^t), L(s^t))}{u_c (c(s^t), L(s^t))} L(s^t) \right] \leq B_0 + R_0K_0 \]
\[ \sum_{t,s^t} \frac{\beta^t \Pr(s^t) u_c (c(s^t), L(s^t))}{u_c (c_0, L_0)} \left[ c(s^t) + \frac{u_L (c(s^t), L(s^t))}{u_c (c(s^t), L(s^t))} L(s^t) \right] \leq u_c (c_0, L_0) [B_0 + R_0K_0] \]
\[ \sum_{t,s^t} \beta^t \Pr(s^t) \left[ u_c (c(s^t), L(s^t)) c(s^t) + u_L (c(s^t), L(s^t)) L(s^t) \right] \leq u_c (c_0, L_0) [B_0 + R_0K_0], \]  

(8)

where we used the no arbitrage and the inter- and intratemporal conditions

• This is again the implementability constraint familiar from the last note. Prices and taxes have disappeared (primal approach), except for the initial period.

**Proposition 1.** An allocation \( \{c(s^t), K(s^t), L(s^t)\} \) can be part of a competitive equilibrium iff (1) and (8) hold with equality.

\(^\text{1}\)Note that I use \( K_0 \) to denote initial capital, although to be 100% consistent I should denote this \( K_{-1} \).
Proof.

- Only if: shown above
- If:

1. Construct prices and taxes
   (a) Find \( r(s^t) \) and \( w(s^t) \) from (3) and (4)
   (b) Find \( p(s^t) \) from (7)
   (c) Find \( \tau^l(s^t) \) from (6)
   (d) Find \( \tau^k(s^t) \) from (5) and (2). Note: many solutions. Many patterns of state-contingent capital-income tax/government debt can implement same allocation.

2. Check for equilibrium
   (a) Factor prices imply firm optimization and zero profits by Euler’s theorem
   (b) Prices and taxes imply household FOCs hold
   (c) (8) implies that household budget constraint holds with equality \( \Rightarrow \) household optimization
   (d) Use the budget constraint, constant returns to scale and the resource constraint:

\[
\sum_{t,s^t} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t) \left( 1 - \tau^l(s^t) \right) L(s^t) - R(s^t)K(s^{t-1}) \right] = B_0
\]

\[
\sum_{t,s^t} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t) L(s^t) + w(s^t) \tau^l(s^t) L(s^t) - \left[ 1 + (1 - \tau^k(s^t))(r(s^t) - \delta) \right] K(s^{t-1}) \right] = B_0
\]

\[
\sum_{t,s^t} p(s^t) \left[ c(s^t) + K(s^t) - w(s^t) L(s^t) + w(s^t) \tau^l(s^t) L(s^t) - (1 - \delta) K(s^{t-1}) - r(s^t)K(s^{t-1}) + \tau^k(s^t)(r(s^t) - \delta)K(s^{t-1}) \right] = B_0
\]

\[
\sum_{t,s^t} p(s^t) \left[ c(s^t) + K(s^t) - F(K(s^{t-1}), L(s^t), s^t, t) - (1 - \delta) K(s^{t-1}) + w(s^t) \tau^l(s^t) L(s^t) + \tau^k(s^t)(r(s^t) - \delta)K(s^{t-1}) \right] = B_0
\]

\[
\sum_{t,s^t} p(s^t) \left[ -g(s^t) + w(s^t) \tau^l(s^t) L(s^t) + \tau^k(s^t)(r(s^t) - \delta)K(s^{t-1}) \right] = B_0
\]

so the government budget constraint holds. This is really just Walras’ Law.

\[
\square
\]
2.4 Optimal Taxes

The government’s problem is

\[
\max_{c(s^t), L(s^t), K(s^t), \tau_k^t} \sum_{s^t} \beta^t \Pr(s^t) u(c(s^t), L(s^t))
\]

s.t. \( c(s^t) + g(s_t) + K(s^t) = F\left(K(s^{t-1}), L(s^t), s^t, t\right) + (1 - \delta)K(s^{t-1}) \)

\[
\sum_{t, s^t} \beta^t \Pr(s^t) \left[ u_c(c(s^t), L(s^t)) c(s^t) + u_L(c(s^t), L(s^t)) L(s^t) \right] = u(c_0, L_0) \left[B_0 + R_0K_0 \right]
\]

Suppose \( \tau_k^t \) is fixed for now. Let \( \mu \) be the multiplier on the implementability constraint. Define

\[
W(c, L) \equiv u(c, L) + \mu [u_c(c, L) c + u_L(c, L) L]
\]

Problem becomes

\[
\max_{c(s^t), K(s^t), \tau_k^t} \sum_{s^t} \beta^t \Pr(s^t) W(c(s^t), L(s^t)) - \mu u_c(c_0, L_0) \left[B_0 + R_0K_0 \right]
\]

s.t. \( c(s^t) + g(s_t) + K(s^t) = F\left(K(s^{t-1}), L(s^t), s^t, t\right) + (1 - \delta)K(s^{t-1}) \)

For \( t \neq 0 \), FOCs are:

\[
\beta \Pr(s^t) W_c(c(s^t), L(s^t)) - \gamma(s^t) = 0
\]

\[
\beta \Pr(s^t) W_L(c(s^t), L(s^t)) + \gamma(s^t) F_L\left(K(s^{t-1}), L(s^t), s^t, t\right) = 0
\]

\[
-\gamma(s^t) + \sum_{s_{t+1}} \gamma(s^t, s_{t+1}) \left[F_K\left(K(s^t), L(s^{t+1}), s^{t+1}, t+1\right) + (1 - \delta) \right] = 0
\]

Intratemporal:

\[
-W_L(c(s^t), L(s^t)) W_c(c(s^t), L(s^t)) = F_L\left(K(s^{t-1}), L(s^t), s^t, t\right) \quad (9)
\]

Intertemporal:

\[
W_c(c(s^t), L(s^t)) = \beta \sum_{s_{t+1}} \Pr(s_{t+1} | s^t) W_c\left(c(s^{t+1}), L(s^{t+1})\right) R^*(s^{t+1}) \quad (10)
\]

where

\[
R^*(s^t) \equiv 1 + F_K\left(K(s^{t-1}), L(s^t), s^t, t\right) - \delta
\]
Recall, from household problem, using (6) and (4):

\[ F_L \left( K(s^{t-1}), L(s^t), s^t, t \right) \left( 1 - \tau^t(s^t) \right) = -\frac{u_L \left( c(s^t), L(s^t) \right)}{u_c \left( c(s^t), L(s^t) \right)} \]

so using (9) we solve out for the optimal labor tax

\[ \tau^{t^*}(s^t) = 1 - \frac{u_L \left( c(s^t), L(s^t) \right) W_c \left( c(s^t), L(s^t) \right)}{W_L \left( c(s^t), L(s^t) \right) u_c \left( c(s^t), L(s^t) \right)} \] (11)

Also recall from the household problem, using (5) and (7):

\[ u_c \left( c(s^t), L(s^t) \right) = \beta \sum_{s_{t+1}} \Pr(s_{t+1}|s^t) u_c \left( c(s_{t+1}), L(s_{t+1}) \right) R(s_{t+1}) \] (12)

There are many choices of \( \tau^k(s_{t+1}) \) (or, equivalently, \( R(s_{t+1}) \)) that make (12) compatible with (10). One particular solution is:

\[ R(s_{t+1}) = R^*(s_{t+1}) \frac{W_c \left( c(s_{t+1}), L(s_{t+1}) \right) u_c \left( c(s^t), L(s^t) \right)}{u_c \left( c(s_{t+1}), L(s_{t+1}) \right) W_c \left( c(s^t), L(s^t) \right)} \] (13)

### 2.5 Capital Taxation

**Proposition 2.** Suppose that (i) there is no uncertainty, and (ii) there is a steady state. Then in the steady state \( \tau^k = 0 \) is optimal.

**Proof.** Impose steady state and no uncertainty on (13) to obtain

\[ R(ss) = R^*(ss) \]

\[ \Leftrightarrow 1 + (1 - \tau^k(ss))(F_K(K(ss), L(ss)) - \delta) = 1 + F_K(K(ss), L(ss), ss) - \delta) \]

which is achieved with \( \tau^k(ss) = 0. \)

This result is due to Chamley (1986) and Judd (1985).

- uniform taxation of consumption at different dates in the steady state without conditions on preferences
- steady state capital supply perfectly elastic:

\[ 1 = \beta R(ss) \]

\[ = \beta \left[ 1 + (1 - \tau^k)(F_K(ss) - \delta) \right] \]
from the Euler equation (12) in the steady state

- Special case:
  \[ u(c, L) = \frac{c^{1-\sigma}}{1-\sigma} - v(L) \]

Then

\[ W(c, L) = \frac{c^{1-\sigma}}{1-\sigma} - v(L) + \mu \left[ c^{-\sigma} c - v'(L)L \right] \]

\[ = \left( \frac{1}{1-\sigma} + \mu \right) c^{1-\sigma} - [v(L) + \mu v'(L)L] \]

so

\[ W_c = (1 + \mu (1 - \sigma)) c^{-\sigma} \]

\[ = (1 + \mu (1 - \sigma)) u_c \]

\[ \frac{W_c}{u_c} = 1 + \mu (1 - \sigma) \]

Therefore (13) reduces to

\[ R(s^{t+1}) = R^*(s^{t+1}), \]

so for these preferences \( \tau^k = 0 \) is optimal even with uncertainty and outside of steady state, for every period other than the first one.

- This is similar to the separability and homotheticity requirement for the uniform linear taxation result in the static or finite horizon models

- In general, in steady state, the optimal tax rate fluctuates around zero (Zhu, 1992)

- Atkeson/Chari/Kehoe (1999): Chamley-Judd result holds under much more general conditions than shown here:
  - heterogeneous agents, including a model with workers and capitalists, workers are hand-to-mouth (i.e. they do not hold capital), capitalists do not work, and the planner puts no social welfare weight on capitalists
  - the government cannot issue debt but has to satisfy its budget constraint period by period (see problem set)
  - durable consumption goods
  - endogenous growth
  - open economy (fixed interest rate)
– OLG models (with some caveats, see our discussion of estate taxation later)

### 2.6 Tax smoothing

Special case: iso-elastic labor supply

\[ v(L) = \alpha \frac{L^\gamma}{\gamma} \]

Then

\[ W(c, L) = \left( \frac{1}{1-\sigma} + \mu \right) c^{1-\sigma} - \alpha \left( \frac{1}{\gamma} + \mu \right) L^\gamma \]

Same functional form as \( u(\cdot) \) but with more disutility of labour (as long as \( \gamma > 1 - \sigma \)). Furthermore

\[
\frac{W_c}{u_c} = 1 + \mu (1 - \sigma) \\
\frac{W_L}{u_L} = 1 + \mu \gamma
\]

so (11) becomes

\[
\tau_l(s^t) = 1 - \frac{1 + \mu (1 - \sigma)}{1 + \mu \gamma} \quad \forall s^t
\]

- Tax perfectly smooth over time and states of the world
- Smooth out distortions over time
- Role of debt and state-contingent securities
- What happens if there is an expenditure shock with high \( g \)? State contingent debt. State contingent capital tax.
- This result was first shown by Lucas/Stokey (1983) for an economy without capital, but with state-contingent debt (equivalent to a state-contingent tax to the return on bonds). Outside the special case of iso-elastic labor supply, the optimal labor tax rate follows the same stochastic process as \( g \).
- In contrast to Barro (1979): labor tax should follow a random walk to smooth out expenditure shocks due to convex deadweight loss. Finance shock by debt and pay back through permanently increased tax.
• Chari, Christiano and Kehoe (1994) consider the present model with both capital and state contingent debt and find that the optimal labor income tax has tiny fluctuations (which vanish with iso-elastic labor supply). Labor tax smoothing can be achieved both by using state-contingent debt and the state-contingent capital tax.

• In particular, the optimal policy is to set the ex ante (expected) tax on capital income to zero, but vary the ex post rate. This leaves investment incentives undistorted, while the ex post rate can be used like a lump-sum tax to finance the spending shocks (see handout).

• Implications for estimating effect of taxes on investment and saving: If we find a high variability of ex post capital tax rates but ex ante rates are roughly constant, we expect saving to be roughly constant. In the data, we then observe varying tax rates and constant saving and might falsely conclude that taxes have no effect on saving. Would need to measure ex ante (expected) tax rates.

• Aiyagari et al. (2002) study the case where there is no state-contingent debt and no capital. Ex post capital taxation is not possible. Find that then, labor tax rate (almost) follows random walk, as in Barro (1979)

• Farhi (2010) introduces capital in AMSS-economy, but government is slow and the tax rate on capital can only be changed with some delay after a spending shock. Then agents observe a high shock and expect an increase in the capital tax rate. Hence all the effect of ex post taxation is gone. Similar results to AMSS.

• bottom line: possibility of issuing state-contingent debt or ex post capital taxation is key for tax smoothing results

2.7 Initial period taxation and time inconsistency

Return to government’s problem and assume $\tau_0$ can be chosen freely:

\[
\max \sum_{s_t} \beta_t \Pr(s_t) u \left( c(s_t), L(s_t) \right)
\]

\[
s.t. \quad c(s_t) + g(s_t) + K(s_t) = F (K(s_t^{-1}), L(s_t), s_t, t) + (1 - \delta) K(s_t^{-1})
\]

\[
\sum_{s_t} \beta_t \Pr(s_t) \left[ u_c \left( c(s_t), L(s_t) \right) c(s_t) + u_L \left( c(s_t), L(s_t) \right) L(s_t) \right]
\]

\[
= u_c (c_0, L_0) \left[ B_0 + (1 + (1 - \tau_0^k)(r_0 - \delta)) K_0 \right]
\]
FOC w.r.t. $\tau_0^k$:

$$-\mu u_c(c_0, L_0)(r_0 - \delta)K_0 = 0$$

- problem is linear (and increasing) in $\tau_0^k$ (note $\mu < 0$)
- Tax initial capital until that is enough to pay for all government expenditure
- May require $\tau_0^k > 1$
- Replicable with consumption tax, so no consumption tax is w.l.o.g. only if $\tau_0^k$ is allowed
- Achieve first-best allocation
- Non-distortionary, so no time inconsistency problem arises
- Suppose we impose $\tau^k \leq \bar{\tau}$ and this is not enough to satisfy government budget constraint. Then $\tau_0^k = \bar{\tau}$ is optimal, but optimal plan is not time consistent.
- Plan to tax initial capital highly, but future capital at zero.
- If cannot commit, reoptimize each period, with high taxes and lower welfare
- Werning (2007) introduces heterogeneity, allows for lump-sum tax. Shows that Chamley-Judd and tax smoothing results go through. Initial capital taxation and time inconsistency are more subtle: depends on asset distribution and redistributive motives.