Non-linear Income Taxation

1 Setup

- readings: Mirrlees (REStud 1971), Diamond (AER 1998), Saez (REStud 2001), Werning (working paper 2007), Choné/Laroque (AER 2010)
- preferences
  \[ U(c, Y, \theta), \]
e.g. \( U(c, y/\theta) \) where \( l \equiv y/\theta \) is labor supply and \( \theta \) is productivity
- skill distribution with cdf \( F(\theta) \)
- technology
  \[ G + \int (c(\theta) - Y(\theta))dF(\theta) \leq e \]
  equivalent to
  \[ G - e \leq \int T(Y(\theta))dF(\theta), \]
government budget constraint
- non-linear income taxation: choose a tax schedule \( T(Y) \) such that the budget set becomes
  \[ B \equiv \{(c, Y)|c \leq Y - T(Y)\}, \]
  where we call \( R(Y) \equiv Y - T(Y) \) the retention function
- normative criterion:
  1. social welfare function (Mirrlees, Diamond, Saez)
  2. Pareto efficiency (Werning, Choné/Laroque)

2 Feasibility and Incentive Compatibility

- agents solve
  \[ \max_{c,Y} U(c, Y, \theta) \text{ s.t. } c \leq R(Y) \]
• **Definition:** An allocation \( c(\theta), Y(\theta) \) and a tax function \( T(Y) \) are feasible if (i) (1) holds and (ii) \( c(\theta), Y(\theta) \) solves (2) given \( T(Y) \).

• An allocation \( c(\theta), Y(\theta) \) is feasible if there is a tax function \( T(Y) \) such that \( c(\theta), Y(\theta), T(Y) \) are feasible.

• **figure 1:** whenever we have \( c(\theta), Y(\theta) \), we have \( R(Y) \) and hence \( T(Y) \)

• **Observation:** If \( c(\theta), Y(\theta) \) is feasible, then

\[
U(c(\theta), Y(\theta), \theta) \geq U(c(\theta'), Y(\theta'), \theta) \quad \forall \theta, \theta',
\]

incentive compatibility constraints

• the converse is also true (**Lemma**): If \( c(\theta), Y(\theta) \) satisfies (1) and (3), then it is feasible.

• **idea (figure 2):** can always find a retention function that implements an incentive-compatible allocation, e.g.

\[
R(\tilde{Y}) \equiv \sup_{\tilde{c}} \{ \tilde{c} | U(c(\theta), Y(\theta), \theta) \geq U(\tilde{c}, \tilde{Y}, \theta) \quad \forall \theta \}
\]

lower envelope of indifference curves

• by construction,

\[
U(c(\theta), Y(\theta), \theta) \geq U(R(\tilde{Y}), \tilde{Y}, \theta) \quad \forall \tilde{Y}, \theta,
\]

so that agents faced with retention function \( R(\tilde{Y}) \) choose \( \tilde{Y} = Y(\theta), R(\tilde{Y}) = c(\theta) \)

• I.e. at points chosen by some type, retention function is just consumption in the allocation. At other points, it ‘fills the gaps.’

• Taxation principle or revelation principle (with private information). Here, we did not assume private information about \( \theta \), but just that we are restricted to a budget set \( B \) that has to be the same for all \( \theta \).

• **Marginal taxes (from (2)):** If \( T'(\tilde{Y}) \) exists and \( \tilde{Y} = Y(\theta) \), then

\[
T'(\tilde{Y}) = T'(Y(\theta)) = 1 - MRS(c(\theta), Y(\theta), \theta)
\]

with

\[
MRS(c, Y, \theta) \equiv -\frac{U_Y(c, Y, \theta)}{U_c(c, Y, \theta)}
\]
• slope of 1 in \((Y, c)\)-space means zero marginal tax rate

• adding structure: preferences satisfy single-crossing if

\[
MRS(c, Y, \theta) \text{ is decreasing in } \theta
\]

• Observation (figure 3): If \(c(\theta), Y(\theta)\) is incentive compatible and preferences satisfy single crossing, then \(c(\theta)\) and \(Y(\theta)\) must be increasing in \(\theta\) (monotonicity)

• Define

\[
v(\theta) \equiv U(c(\theta), Y(\theta), \theta) = \max_{\theta'} U(c(\theta'), Y(\theta'), \theta) \forall \theta
\]

by the (global) incentive constraints (3)

• FOC (local incentive constraints)

\[
U_c(c(\theta), Y(\theta), \theta)c'(\theta) + U_Y(c(\theta), Y(\theta), \theta)y'(\theta) = 0, \quad (5)
\]

evaluated at truth-telling

• equivalent envelope condition:

\[
v'(\theta) = U_\theta(c(\theta), Y(\theta), \theta) \quad (6)
\]

or in integral form

\[
v(\theta) = \int_{\tilde{\theta}}^{\theta} U_\theta(c(\tilde{\theta}), Y(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta} + v(\tilde{\theta})
\]

• As we have seen, incentive constraints (3) imply

1. local incentive constraints (6) and

2. monotonicity of \(Y(\theta)\) (and \(c(\theta)\)).

These are also sufficient.

• Lemma: If preferences satisfy single-crossing, then the allocation \(c(\theta), Y(\theta)\) is incentive compatible if and only if (6) is satisfied and \(Y(\theta)\) is non-decreasing for all \(\theta\).
3 Pareto Improvements and Laffer Effects

- start from a tax schedule $T_0(Y)$, which induces incomes $Y_0(\theta)$
- is this Pareto efficient?
- by resource constraint, we must have

$$G - e \leq \int T_0(Y_0(\theta))dF(\theta)$$

(7)

- **Lemma**: If the inequality in (7) is strict, then there exists another tax schedule $T_1 \succ T_0$ (in the Pareto sense)
- Idea: Assume $T^* \equiv \max_\theta T_0(Y_0(\theta)) < \infty$ for simplicity (e.g. bounded type space). Define $T_\varepsilon(Y) = \min\{T_0(Y), T^* - \varepsilon\}$ for some small $\varepsilon > 0$ (figure 4). The lost tax revenue moves continuously with $\varepsilon$ (it may even increase if types with lower $Y$ move up to higher incomes and pay $T^* - \varepsilon$ now). This way, one can close the gap in (7) and clearly $T_\varepsilon \succ T_0$.
- Suppose $T_1 \succ T_0$. Then it must be that

$$G - e = \int T_0(Y_0(\theta))dF(\theta) \leq \int T_1(Y_1(\theta))dF(\theta)$$

(8)

for feasibility, where $Y_1(\theta)$ is induced by $T_1(Y)$.
- **Lemma**: For a Pareto improvement, it must be that

$$T_1(Y_1(\theta)) \leq T_0(Y_1(\theta)) \forall \theta.$$  

(9)

To see this, note that

$$U(Y_1(\theta) - T_1(Y_1(\theta)), Y_1(\theta), \theta) \geq U(Y_0(\theta) - T_0(Y_0(\theta)), Y_0(\theta), \theta)$$

$$\geq U(Y_1(\theta) - T_0(Y_1(\theta)), Y_1(\theta), \theta),$$

where the first inequality follows from the Pareto improvement and the second from the assumption that $T_0(Y)$ induces agents to choose incomes $Y_0(\theta)$.
- Hence, (8) and (9) imply that Pareto-improving tax reforms must take the form of tax reductions that do not reduce tax revenue: (sophisticated) Laffer effects
4 Pareto Efficient Taxation

- change in variables from \( c, Y \) to \( v, Y \) with
  
  \[
  v(\theta) = U(c(\theta), Y(\theta), \theta)
  \]
  
  \[
  c(\theta) = e(v(\theta), Y(\theta), \theta).
  \]

- use dual approach to Pareto problem: maximize resources subject to delivering \( v(\theta) \) or more to all \( \theta \):

  \[
  \max_{v(\theta), Y(\theta)} \int (Y(\theta) - e(v(\theta), Y(\theta), \theta)) dF(\theta)
  \]

  s.t.
  
  \[
  v'(\theta) = U_\theta(e(v(\theta), Y(\theta), \theta), Y(\theta), \theta) \forall \theta
  \]

  \[
  v(\theta) \geq \bar{v}(\theta) \forall \theta
  \]

  and the monotonicity constraint that \( Y(\theta) \) must be increasing.

- Ignore monotonocity constraint and check later whether the solution satisfies it. If not, need to consider bunching/ironing.

- Once we’ve found the solution to this problem, we can find \( c(\theta) = e(v(\theta), Y(\theta), \theta) \) and with \( c(\theta), Y(\theta) \) find the retention function \( R(Y) \) and the optimal tax schedule \( T(Y) \)

- Put some multiplier \( \xi(\theta) = \lambda(\theta)f(\theta)/\eta \) on the last constraints. \( \bar{v}(\theta) \) and \( \xi(\theta) \) will be related at the optimum.

- solve

  \[
  \max_{v(\theta), Y(\theta)} \int (Y(\theta) - e(v(\theta), Y(\theta), \theta)) dF(\theta) + \frac{1}{\eta} \int \lambda(\theta)v(\theta)f(\theta)d\theta
  \]

  s.t.

  \[
  v'(\theta) = U_\theta(e(v(\theta), Y(\theta), \theta), Y(\theta), \theta) \forall \theta
  \]

  or equivalently

  \[
  \frac{1}{\eta} \left\{ \max_{v(\theta), Y(\theta)} \eta \int (Y(\theta) - e(v(\theta), Y(\theta), \theta)) dF(\theta) + \int \lambda(\theta)v(\theta)f(\theta)d\theta \right\},
  \]
which is the Lagrangian that comes out of the dual problem

$$\max_{v(\theta), Y(\theta)} \int \lambda(\theta)v(\theta) f(\theta) d\theta$$

s.t.

$$v'(\theta) = U_\theta(e(v(\theta), Y(\theta), \theta), Y(\theta), \theta) \forall \theta$$

$$\int (Y(\theta) - e(v(\theta), Y(\theta), \theta)) dF(\theta) \geq G - e$$

for some $G - e$ (which will be related to $\eta$ at the optimum)

- Utilitarian social welfare function is then captured by $\lambda(\theta) = 1 \forall \theta$, the Rawlsian case by $\lambda(\theta) = 0$ for all $\theta$ except $\theta$

- Solve using
  - optimal control with $v$ as state and $Y$ as control variable
  - Lagrangian and integration by parts

- form Lagrangian

$$\mathcal{L} = \int (Y(\theta) - e(v(\theta), Y(\theta), \theta)) dF(\theta) + \frac{1}{\eta} \int \lambda(\theta)v(\theta) f(\theta) d\theta$$

$$+ \int \mu(\theta) v'(\theta) d\theta - \int \mu(\theta) U_\theta(e(v(\theta), Y(\theta), \theta), Y(\theta), \theta) d\theta$$

and note that (after integration by parts)

$$\int \mu(\theta) v'(\theta) d\theta = \mu(\bar{\theta}) v(\bar{\theta}) - \mu(\theta) v(\theta) - \int \mu'(\theta) v(\theta) d\theta$$

- let’s take pointwise FOCs for $Y(\theta)$:

$$(1 - e_Y(\theta)) f(\theta) - \mu(\theta) [U_{\theta c}(\theta) e_Y(\theta) + U_{\theta Y}(\theta)] = 0 \forall \theta, \quad (10)$$

where I simplified notation

- note that

$$e_Y(\theta) = \frac{dc(\theta)}{dY(\theta)} = -\frac{U_Y(\theta)}{U_c(\theta)} = \text{MRS}(\theta)$$

and hence

$$1 - e_Y(\theta) = 1 - \text{MRS}(\theta) = T'(Y(\theta)) \equiv \tau(\theta).$$
• also,

\[ -U_{\theta c}(\theta) \frac{U_Y(\theta)}{U_c(\theta)} + U_{\theta Y}(\theta) = U_c(\theta) \frac{U_{Y\theta}(\theta)U_c(\theta) - U_{c\theta}(\theta)U_Y(\theta)}{U_c(\theta)^2} \]

\[ = -U_c(\theta) \frac{\partial}{\partial \theta} \left[ - \frac{U_Y(\theta)}{U_c(\theta)} \right] = -U_c(\theta) \frac{\partial \text{MRS}(\theta)}{\partial \theta}. \]  

(11)

• with this, (10) becomes

\[ \tau(\theta)f(\theta) = -\mu(\theta)U_c(\theta) \frac{\partial \text{MRS}(\theta)}{\partial \theta} \]

or, since \( \text{MRS}(\theta) = 1 - \tau(\theta) \),

\[ \frac{\tau(\theta)}{1 - \tau(\theta)}f(\theta) = -\mu(\theta)U_c(\theta) \frac{\partial \log \text{MRS}(\theta)}{\partial \theta}. \]  

(12)

• the FOC for \( v(\theta) \) (for interior \( \theta \)) can be written as

\[ -e_v(\theta)f(\theta) - \mu'(\theta) - \mu(\theta)U_{\theta c}(\theta)e_v(\theta) + \lambda(\theta)f(\theta)/\eta = 0, \]  

or, noting that \( e_v(\theta) = 1/U_c(\theta) \) and \( \lambda(\theta)f(\theta)/\eta \geq 0 \),

\[ -U_c(\theta)\mu'(\theta) - \mu(\theta)U_{\theta c}(\theta) \leq f(\theta). \]  

(14)

• I.e. any Pareto efficient allocation has to satisfy this inequality, since otherwise we cannot find non-negative multipliers \( \xi(\theta) \) on the Pareto constraints \( v(\theta) \geq \overline{v}(\theta) \). In particular, the Rawlsian allocation would have to satisfy (14) with equality for all (interior) \( \theta \).

• let’s change variables from \( \mu(\theta) \) to \( \hat{\mu}(\theta) \equiv U_c(\theta)\mu(\theta) \), so that (recall that \( U_c(\theta) = U_c(c(\theta), Y(\theta), \theta) \))

\[ \hat{\mu}'(\theta) = U_c(\theta)\mu'(\theta) + \mu(\theta) \left[ U_{\theta c}(\theta) + U_{cc}(\theta)c'(\theta) + U_{cY}(\theta)Y'(\theta) \right] \]

• substituting in (14) yields

\[ -\hat{\mu}'(\theta) + \hat{\mu}(\theta) \frac{U_{cc}(\theta)c'(\theta) + U_{cY}(\theta)Y'(\theta)}{U_c(\theta)} \leq f(\theta) \]
\[ \frac{U_{cc}(\theta)c'(\theta) + U_{cY}(\theta)Y'(\theta)}{U_c(\theta)} = \frac{U_{cc}(\theta)c'(\theta)}{U_c(\theta)} + \frac{U_cY(\theta)}{U_c(\theta)}Y'(\theta) \]

\[ = \frac{-U_{cc}(\theta)U_Y(\theta) + U_{cY}(\theta)U_c(\theta)}{U_c(\theta)^2}Y'(\theta) = -\frac{\partial}{\partial c} \left[ -\frac{U_Y(\theta)}{U_c(\theta)} \right] Y'(\theta) = -\frac{\partial \text{MRS}(\theta)}{\partial c} Y'(\theta) \]

since \( c'(\theta)/Y'(\theta) = -U_Y(\theta)/U_c(\theta) \) by the local incentive constraints (5)

- substituting all this, the two conditions for Pareto efficiency (12) and (14) become

\[ \frac{\tau(\theta)}{1 - \tau(\theta)} f(\theta) = -\hat{\mu}(\theta) \frac{\partial \log \text{MRS}(c(\theta), Y(\theta), \theta)}{\partial \theta} \]

(15)

\[ -\hat{\mu}'(\theta) - \hat{\mu}(\theta) \frac{\partial \text{MRS}(c(\theta), Y(\theta), \theta)}{\partial c} Y'(\theta) \leq f(\theta) \]

(16)

- for any given preferences \( U(c, Y, \theta), \) (differentiable) tax schedule \( T(Y) \) and skill distribution \( F(\theta) \) (and thus the resulting allocation \( c(\theta), Y(\theta) \)), (15) gives \( \hat{\mu}(\theta) \) uniquely

- then (16) is the test for Pareto efficiency

5 An Example and Interpretation

- consider preferences with no income effect and a constant labor supply elasticity (Diamond 1998)

- in this case, \( \partial \text{MRS}(c(\theta), Y(\theta), \theta)/\partial c = 0 \) (figure 5)

- e.g. functional form

\[ U(c, Y, \theta) = c - (Y/\theta)^{\alpha}, \]

implying a (constant) wage elasticity of labor supply \( \varepsilon \equiv 1/(\alpha - 1) \)

- under what conditions would a “flat” income tax (Hall/Rabushka 1995) with a constant marginal tax rate \( \tau \) be Pareto efficient?

- with quasi-linear preferences, the agent’s problem

\[ \max_Y Y - \tau Y - \left( \frac{Y}{\theta} \right)^{\alpha} \]
implies
\[ 1 - \tau = \alpha Y(\theta)^{\alpha-1} \theta^{-\alpha} = MRS(Y(\theta), \theta) \]  \hspace{1cm} (17)

and hence
\[ \frac{\partial \log MRS(c(\theta), Y(\theta), \theta)}{\partial \theta} = -\frac{\alpha}{\theta} = -\frac{1 + \alpha}{\epsilon} \]

• substituting in (15) yields
\[ -\hat{\mu}(\theta) = -\frac{\tau}{1 - \tau} \frac{\epsilon}{1 + \epsilon} \theta f(\theta) \Rightarrow -\hat{\mu}'(\theta) = -\frac{\tau}{1 - \tau} \frac{\epsilon}{1 + \epsilon} \left[ f(\theta) + \theta f'(\theta) \right] \]

• substituting in (16) yields

**Proposition:** Given quasilinear preferences with constant wage elasticity \( \epsilon \) and a skill density \( f(\theta) \), a flat tax with constant marginal tax rate \( \tau \) is Pareto efficient if and only if
\[ \frac{\tau}{1 - \tau} \left[ -1 - \frac{f'(\theta)\theta}{f(\theta)} \right] \leq 1 \hspace{0.5cm} \forall \theta \]  \hspace{1cm} (18)

• Interpretation:

1. Note the role of the skill distribution: For any \( \tau \) and \( \epsilon \), there exists a set of skill densities \( f(\theta) \) such that \( \tau \) is Pareto efficient and a set of skill densities such that it is Pareto inefficient.

2. Also: For any \( \epsilon \) and \( f(\theta) \), there exists a set of flat tax schedules \( \tau \) that are Pareto efficient and a set of \( \tau \)'s that are Pareto inefficient. Hence, without guidance on \( f(\theta) \) (more on that later), “anything goes.”

3. Rawlsian flat tax would satisfy (18) with equality at \( \max_{\theta} \{ -1 - f'(\theta)\theta / f(\theta) \} \)

4. \( \epsilon \) and \( d \log f(\theta) / d \log \theta \) enter intuitively. Notably, the latter captures the (local) Laffer effects that can lead to Pareto inefficiency (figure 6).

5. Problem Set: check condition for different skill distributions

• More generally, suppose preferences are still quasilinear with a constant wage elasticity, but we aim at testing the Pareto efficiency of any non-linear (differentiable) tax schedule

• (15) implies
\[ -\hat{\mu}(\theta) = -\frac{\epsilon}{1 + \epsilon} \frac{\tau(\theta)}{1 - \tau(\theta)} \theta f(\theta) \Rightarrow -\hat{\mu}'(\theta) = -\frac{\epsilon}{1 + \epsilon} \frac{d}{d \theta} \left[ \frac{\tau(\theta)}{1 - \tau(\theta)} \theta f(\theta) \right] \]
• substituting in (16) yields

\[ -\frac{\varepsilon}{1 + \varepsilon} \frac{d}{d\theta} \left[ \frac{\tau(\theta)}{1 - \tau(\theta)} \theta f(\theta) \right] \leq f(\theta) \quad \forall \theta \]

and hence, after integrating,

• **Proposition:** Given quasilinear preferences with constant wage elasticity $\varepsilon$ and a skill distribution $F(\theta)$, a differentiable tax schedule with marginal tax rates $\tau(\theta) = T'(Y(\theta))$ is Pareto efficient if and only if

\[ \frac{1}{1 + 1/\varepsilon} \frac{\tau(\theta)}{1 - \tau(\theta)} f(\theta) \theta + F(\theta) \]

is weakly increasing in $\theta$.

• “integral form” of (18)

• easy to see that flat tax with $\tau(\theta) = \tau$ is special case

• skill distribution again plays crucial role

6 **Identifying the Skill Distribution**

• “anything goes” with Pareto efficiency unless we have restrictions on the skill distribution

• but we don’t observe the skill distribution

• Saez’s (2001) identification step: Suppose we observe an income distribution with cdf $H(Y)$ induced by a tax schedule $T(Y)$. Then we can back out the skill distribution from the relationship

\[ H(Y(\theta)) = F(\theta) \Leftrightarrow h(Y(\theta))Y'(\theta) = f(\theta) \quad \forall \theta. \]

• Moreover, given the tax schedule $T(Y)$ and preferences $U(c, Y, \theta)$, the agents’ utility maximization problem gives $Y(\theta)$ from

\[ Y(\theta) = \arg \max_Y U(Y - T(Y), Y, \theta), \]
where $Y(\theta)$ is implicitly defined by the FOC

$$U_c(Y(\theta) - T(Y(\theta)), Y(\theta), \theta)(1 - T'(Y(\theta))) + U_Y(Y(\theta) - T(Y(\theta)), Y(\theta), \theta) = 0. \quad (19)$$

- That way, we can identify the skill distribution from the distribution of observed incomes and the observed tax schedule, assuming some preferences.
- see Problem Set for an example
- this allows us to formulate a test for Pareto efficiency directly in terms of the observed income distribution
- in particular, (19) implies (simplifying notation)

$$Y'(\theta) = -\frac{U_c(1 - T') + U_Y}{U_{cc}(1 - T')^2 + 2U_{cY}(1 - T') - UC'T'' + U_{YY}} \quad (20)$$

and note that the numerator is

$$U_c(1 - T') + U_Y = -U_c\frac{U_Y}{U_c} + U_Y$$

$$= -U_c \frac{\partial MRS(c(\theta), Y(\theta), \theta)}{\partial \theta}$$

$$= -U_c(1 - T') \frac{\partial \log MRS(c(\theta), Y(\theta), \theta)}{\partial \theta}$$

from (11) and $(1 - T') = MRS$.

- We can thus rewrite (20) as follows

$$Y'(\theta) = -\frac{-\frac{\partial \log MRS}{\partial \theta}}{-\left(\frac{U_c}{U_c(1 - T')} + 2\frac{U_{cY}}{U_c} + \frac{U_{YY}}{U_c(1 - T')}\right) + \frac{T''}{1 - T'}} \quad (21)$$

- (19) also implies that

$$\frac{dY}{d(1 - T')} = -\frac{U_c}{U_{cc}(1 - T')^2 + 2U_{cY}(1 - T') + U_{YY}}$$

and hence the (after-tax) wage elasticity of income is

$$\varepsilon^w_w \equiv \frac{dY}{d(1 - T')} \frac{1 - T'}{Y} = -\frac{U_c/Y}{U_{cc}(1 - T') + 2U_{cY} + U_{YY}/(1 - T')}.$$
• Substituting in the denominator of (21) finally gives

\[
Y'(\theta) = -\frac{\partial \log \text{MRS}(c, Y, \theta)}{\frac{1}{\epsilon_w Y} + \frac{T''}{1 - T'}}.
\]  

(22)

• Defining \( \hat{\mu}(Y) \equiv \hat{\mu}(Y^{-1}(Y)) \), the two conditions for Pareto efficiency (15) and (16) can thus be rewritten exclusively in terms of \( Y \) and \( h(Y) \) as follows

\[
\hat{\mu}(Y) \equiv \frac{T'(Y)}{1 - T'(Y)} \frac{h(Y)}{\epsilon_w^*(Y) Y} + \frac{T''(Y)}{1 - T'(Y)}
\]

(23)

\[
-\hat{\mu}'(Y) - \hat{\mu}(Y) \frac{\partial \text{MRS}}{\partial c} \leq h(Y).
\]

(24)

• Saez (2001) defines the “virtual density” as

\[
h^*(Y) = \frac{h(Y)}{1 + Y \epsilon_w^*(Y) \frac{T''(Y)}{1 - T'(Y)}} = \frac{h(Y)}{\Phi(Y)}
\]

• then (23) becomes

\[
\hat{\mu}(Y) = \frac{T'(Y)}{1 - T'(Y)} h^*(Y) \epsilon_w^*(Y) Y
\]

(25)

• in logs

\[
\log \hat{\mu}(Y) = \log \left( \frac{T'(Y)}{1 - T'(Y)} \right) + \log h^*(Y) + \log \epsilon_w^*(Y) + \log Y
\]

and differentiating w.r.t. \( \log Y \)

\[
\frac{\hat{\mu}'(Y) Y}{\hat{\mu}(Y)} = \frac{d \log \hat{\mu}(Y)}{d \log Y} = \frac{d \log h^*(Y)}{d \log Y} + \frac{d \log \epsilon_w^*(Y)}{d \log Y} + 1
\]

(26)

• returning to (24), let’s substitute the definition of the virtual density \( h^*(Y) \) to obtain

\[
-\hat{\mu}'(Y) - \hat{\mu}(Y) \frac{\partial \text{MRS}}{\partial c} \leq h^*(Y) \Phi(Y)
\]

\footnote{Saez (2001) shows that this is the density of incomes that would occur at \( Y \) if the non-linear tax schedule \( T(Y) \) were replaced by the linear tax schedule that is tangent to \( T(Y) \) at income level \( Y \), i.e. a tax schedule with a constant marginal tax rate \( \tau = T'(Y) \) and intercept \( T = Y - T(Y) - Y(1 - \tau) \).}
and multiplying through by $Y / \hat{\mu}(Y)$

$$- \frac{\hat{\mu}'(Y)Y}{\hat{\mu}(Y)} - \frac{\partial MRS}{\partial c} Y \leq h^*(Y) \Phi(Y) \frac{Y}{\hat{\mu}(Y)}.$$  

- substitute (25) on the RHS and (26) on the LHS to get

$$- \frac{d \log \left( \frac{T'(Y)}{1 - T'(Y)} \right)}{d \log Y} - \frac{d \log h^*(Y)}{d \log Y} - \frac{d \log e^*_w(Y)}{d \log Y} - 1 - \frac{\partial MRS}{\partial c} Y \leq \frac{Yh^*(Y)\Phi(Y)}{1 - T'(Y)h^*(Y)e^*_w(Y)} Y,$$

which simplifies and leads to the following result

- **Proposition:** A tax schedule $T(Y)$ is Pareto efficient if and only if

$$\frac{T'(Y)}{1 - T'(Y)} \frac{e^*_w(Y)}{\Phi(Y)} \left[ - \frac{d \log \left( \frac{T'(Y)}{1 - T'(Y)} \right)}{d \log Y} - \frac{d \log h^*(Y)}{d \log Y} - \frac{d \log e^*_w(Y)}{d \log Y} - 1 - \frac{\partial MRS}{\partial c} Y \right] \leq 1.$$  

(27)

- **Interpretation:**

1. Many terms cancel if $T(Y)$ is flat tax, there are no income effects and the wage elasticity is constant: $T'(Y) = \tau, \Phi(Y) = 1, h^*(Y) = h(Y),$

$$\frac{d \log \left( \frac{T'(Y)}{1 - T'(Y)} \right)}{d \log Y} = 0, \frac{d \log e^*_w(Y)}{d \log Y} = 0, \frac{\partial MRS}{\partial c} = 0,$$

so that we get back to

$$\frac{\tau}{1 - \tau} e^*_w \left( -1 - \frac{d \log h(Y)}{d \log Y} \right) \leq 1,$$

but now the test is in terms of the observable income distribution induced by the flat tax.

2. Otherwise,

- income effects $\partial MRS/\partial c > 0$,
- an increasing wage elasticity $d \log e^*_w(Y) / d \log Y > 0$, and
- progressivity $d \log \left( \frac{T'(Y)}{1 - T'(Y)} \right) / d \log Y > 0$

help with justifying higher marginal tax rates. Intuition: again local Laffer effects.
7 Relation to Utilitarian Social Welfare Function

• How does the test (27) relate to the optimal tax formulas in Mirrlees (1971) and Saez (2001)?

• by the duality of the problem that we pointed out above, the utilitarian social welfare function approach is equivalent to setting $\xi(\theta) = \lambda(\theta) f(\theta) / \eta = f(\theta) / \eta$ in (13)

• (15) and (16) turn into the optimality conditions

$$\frac{\tau(\theta)}{1 - \tau(\theta)} f(\theta) = -\hat{\mu}(\theta) \frac{\partial \log MRS(c(\theta), Y(\theta), \theta)}{\partial \theta}$$

$$-\hat{\mu}'(\theta) - \hat{\mu}(\theta) \frac{\partial MRS(c(\theta), Y(\theta), \theta)}{\partial c} Y'(\theta) = f(\theta) \left( 1 - \frac{U_c(c(\theta), Y(\theta), \theta)}{\eta} \right)$$

with the transversality conditions $\hat{\mu}(\theta) = \hat{\mu}(\theta) = 0$ (if the skill distribution is bounded)

• do transformation to income as above, and use virtual density $\hat{h}(Y) = h(Y) / \Phi(Y)$:

$$\hat{\mu}(Y) = \frac{T'(Y)}{1 - T'(Y)} \epsilon_w^* (Y) Y h^*(Y)$$

$$-\frac{\partial MRS(c(\theta), Y(\theta), \theta)}{\partial c} \hat{\mu}(Y) = h(Y) \left( 1 - \frac{U_c(c(\theta), Y(\theta), \theta)}{\eta} \right) + \hat{\mu}'(Y),$$

• Note that the ODE (31) has an analogy with the valuation of an asset that pays a flow profit $\pi(t)$ in continuous time and is discounted at interest rate $r(t)$, leading to the no arbitrage condition for the value $V(t)$

$$r(t) V(t) = \pi(t) + V'(t),$$

which is the similar to (31). This yields

$$V(s) = \int_s^\infty e^{-\int_s^t r(z) dz} \pi(t) dt.$$

• analogously, integrating (31) in the same way and using (30) yields

$$\frac{T'(Y)}{1 - T'(Y)} \epsilon_w^* (Y) Y h^*(Y) \int_y^\infty \left( 1 - \frac{U_c}{\eta} \right) \exp \left( \int_y^x \frac{\partial MRS}{\partial c} (z) dz \right) \frac{h(x)}{1 - H(Y)} dx,$$
which is the optimal income tax formula given in Saez (2001)

- The marginal tax rate is negatively related to the elasticity and virtual density at $Y$ and positively to income effects, since

$$\exp\left(\int_y^x \frac{\partial MRS}{\partial c}(z)dz\right) > 1 \ \forall x > y$$

if $\partial MRS/\partial c > 0$

8 An Example and Numerical Computation

- How to compute the optimal income tax schedule (given a utilitarian SWF) numerically?

- Consider again the simpler case where preferences exhibit no income effect and the wage elasticity is constant. In particular, assume

$$U(c, Y, \theta) = c - \frac{1}{2} \left(\frac{Y}{\theta}\right)^2$$

and a social welfare function $W(v) = \log(v)$ as in Saez (2001)

- In this case, (28) can be used to solve for $Y(\theta)$ because $\partial \log MRS/\partial \theta = -2/\theta$ as seen above, $MRS = Y\theta^{-2}$ and hence

$$\frac{\tau(\theta)}{1 - \tau(\theta)} f(\theta) = -\hat{\mu}(\theta) \frac{\partial \log MRS(c(\theta), Y(\theta), \theta)}{\partial \theta} \Leftrightarrow \frac{1 - Y(\theta)\theta^{-2}}{Y(\theta)\theta^{-2}} f(\theta) = \frac{2\hat{\mu}(\theta)}{\theta}$$

$$\Leftrightarrow Y(\theta) = \frac{f(\theta)\theta^3}{2\hat{\mu}(\theta) + f(\theta)\theta}$$

(33)

- Moreover, (29) simplifies to

$$-\hat{\mu}'(\theta) = f(\theta) \left(1 - \frac{1}{\eta v(\theta)}\right)$$

(34)

and the local incentive constraint to

$$v'(\theta) = U_{\theta}(c(\theta), Y(\theta), \theta) = \frac{Y(\theta)^2}{\theta^3} = \left(\frac{f(\theta)}{2\hat{\mu}(\theta) + f(\theta)\theta}\right)^2 \theta^3.$$ (35)
• for any given density $f(\theta)$, the optimum can therefore be computed as follows

1. fix some multiplier $\eta$
2. solve the system of ODE (34) and (35) for $v(\theta)$ and $\mu(\theta)$ using the boundary conditions $\mu(\theta) = \mu(\theta) = 0$. In particular
   - start with $\mu(\theta) = 0$ and some (guessed) $v(\theta)$ as initial conditions
   - solve the ODE system to obtain $\mu(\theta)$
   - repeat this while adjusting $v(\theta)$ until $\mu(\theta) = 0$
3. from the solved $\mu(\theta)$ and $v(\theta)$ schedules, compute $Y(\theta)$ and $c(\theta)$ using (33) and
   $$c(\theta) = v(\theta) + \frac{1}{2} \left( \frac{Y(\theta)}{\theta} \right)^2$$
4. repeat this while adjusting $\eta$ until the resource constraint
   $$\int (Y(\theta) - c(\theta))dF(\theta) = 0$$
   is satisfied
5. finally, check whether $Y(\theta)$ is increasing in $\theta$, so that the monotonicity constraint is satisfied. otherwise, an additional iterative procedure is required to determine optimal bunching.

• Problem Set

9 Untouched Issues

• extensive margin, occupational choice
• dynamics, life-cycle
• uncertainty, tax system as part of social insurance system